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INTERMEDIATE SUMS ON POLYHEDRA: COMPUTATION AND REAL EHRHART THEORY

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Abstract. We study intermediate sums, interpolating between integrals and discrete sums, which were introduced by A. Barvinok in [Computing the Ehrhart quasi-polynomial of a rational simplex. *Math. Comp.* **75** (2006), 1449–1466]. For a given polytope $\mathfrak{p}$ with facets parallel to rational hyperplanes and a rational subspace L, we integrate a given polynomial function h over all lattice slices of the polytope $\mathfrak{p}$ parallel to the subspace L and sum up the integrals. We first develop an algorithmic theory of parametric intermediate generating functions. Then we study the Ehrhart theory of these intermediate sums, that is, the dependence of the result as a function of a dilation of the polytope. We provide an algorithm to compute the resulting Ehrhart quasi-polynomials in the form of explicit step-polynomials. These formulas are naturally valid for real (not just integer) dilations and thus provide a direct approach to real Ehrhart theory.

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§1. *Introduction.* Let $\mathfrak{p}$ be a rational polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on V. A classical problem is to compute the sum of values of $h(x)$ over the set of integral points of $\mathfrak{p}$:

$$S(\mathfrak{p}, h) = \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^d} h(x).$$

The sum $S(\mathfrak{p}, h)$ has generalizations, the *intermediate* sums $S^L(\mathfrak{p}, h)$, where $L \subseteq V$ is a rational vector subspace; these were introduced by Barvinok in [6].

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They interpolate between the discrete sum $S(\mathfrak{p}, h)$ and the integral $\int_{\mathfrak{p}} h(x)\,dx$. For a polytope $\mathfrak{p} \subset V$ and a polynomial $h(x)$,

$$S^L(\mathfrak{p}, h) = \sum_x \int_{\mathfrak{p}\cap(x+L)} h(y)\,dy,$$

where the summation index x runs over the projected lattice in V/L. In other words, the polytope $\mathfrak{p}$ is sliced along affine subspaces parallel to L through lattice points, and the integrals of h over the slices are added up. For $L = V$, there is only one term and $S^V(\mathfrak{p}, h)$ is just the integral of $h(y)$ over $\mathfrak{p}$; at the other extreme, for $L = \{0\}$, we recover the discrete sum $S(\mathfrak{p}, h)$. As in the discrete case, a powerful method is to consider the *intermediate generating function*

$$S^L(\mathfrak{p})(\xi) = \sum_x \int_{\mathfrak{p}\cap(x+L)} \mathrm{e}^{\langle \xi, y\rangle}\,dy \tag{1}$$

for $\xi \in V^*$. If $\mathfrak{p}$ is a polyhedron, not necessarily compact, the generating function still makes sense as a meromorphic function. By writing a polyhedron as the sum of its cones at vertices (Brion's theorem), we need only study the case where $\mathfrak{p}$ is an affine cone.

It is then natural to turn to the Ehrhart theory of the intermediate sums S^L, that is, the study of the intermediate sum $S^L(t\mathfrak{p}, h)$ of a dilated polytope $t\mathfrak{p}$ as a function of the dilation parameter t. It turns out that, just like in the classical case, the Ehrhart function $t \mapsto S^L(t\mathfrak{p}, h)$ is a *quasi-polynomial*, that is, a function of the form

$$S^L(t\mathfrak{p}, h) = \sum_{m=0}^{d+M} E_m^L(\mathfrak{p}, h; t)t^m, \tag{2}$$

with the coefficients $E_m^L(\mathfrak{p}, h; t)$ depending only on $t \bmod q$, where q is an integer such that the vertices of $q\mathfrak{p}$ are lattice points.

Algorithmic results. The main results of this article are of an algorithmic nature. The efficient computation of intermediate generating functions and intermediate Ehrhart quasi-polynomials has two important applications. First, the present paper together with the article [**2**] provide the foundation for future work in which the *weighted Barvinok patched sums* $\sum_L \lambda(L)S^L(\mathfrak{p}, h)$ for a simple polytope, in particular their approximation properties, will be studied. These patched sums provide asymptotics for the number of integer points (or discrete sums of polynomial functions over lattice points) when the polytope is dilated. Second, intermediate generating functions also need to be efficiently computed to obtain an approximation scheme for certain mixed-integer polynomial optimization problems [**8**]. Specifically, we develop:

(1) a polynomial-time algorithm for the computation of the intermediate generating function of a simplicial affine cone when the slicing space has fixed codimension (Theorem 24);
(2) a polynomial-time algorithm for the computation of the weighted intermediate sum $S^L(\mathfrak{p}, h)$ of a simple polytope $\mathfrak{p}$ (given by its vertices)

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and the corresponding Ehrhart quasi-polynomial $t \mapsto S^L(t\mathfrak{p}, h)$ when the slicing space has fixed codimension and the weight depends only on a fixed number of variables or has fixed degree (Theorem 28).

Techniques. The main new feature of this article is the use of a decomposition of any simplicial cone into simplicial cones which have a face parallel to the given space L, the decomposition being given by a closed formula; see Theorem 19. This decomposition is borrowed from [**10**], where it plays a key role in the study of partition functions. We prove that this decomposition is achieved by a polynomial algorithm, if the codimension of L is fixed.

Thus, the computation of the intermediate generating function of a simplicial affine cone is reduced to the case where L is parallel to a face of the cone, and thus the generating function factors into a discrete sum and an integral. This case has already been studied in [**2**]. The output of the algorithm is a "short formula" where both ξ and the vertex s of the cone appear as symbolic variables (see Examples 7, 8 and 26). Once a short formula for the generating function is available, we follow the method of [**2**] for computing the weighted Ehrhart quasi-polynomial of a simple polytope. The method uses the fact that the intermediate generating function decomposes into meromorphic terms of homogeneous ξ-degree. We obtain the Ehrhart coefficient functions $E_m^L(\mathfrak{p}, h; t)$ in a *closed form* as *step-polynomials*, by which we mean sums of functions of the form

$$f((\zeta_1 t) \bmod q_1, \ldots, (\zeta_k t) \bmod q_k),$$

where f is a polynomial and $\zeta_1, \ldots, \zeta_k \in \mathbb{Z}$, $q_1, \ldots, q_k \in \mathbb{N}$; see Example 30.

A real Ehrhart theorem. The results remain valid for non-negative real values of t. Thus our method provides a direct proof of a real Ehrhart theorem (Theorem 31). This extension of Ehrhart theory to real dilation factors has been studied by Linke [**14**] for the classical case. It is even more natural for the intermediate valuations, as one of the cases is $L = V$. Then $S^L(t\mathfrak{p}, h)$ is the integral of h over the $t\mathfrak{p}$, the result being clearly a polynomial formula valid for any non-negative real value of t.

§2. *Notation and basic facts.*

2.1. *Rational and semi-rational polyhedra.* We consider a *rational vector space* V of dimension d, that is to say, a finite-dimensional real vector space with a lattice denoted by Λ. We will need to consider subspaces and quotient spaces of V; this is why we cannot just take $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. An element $v \in V$ is said to be *rational* if $nv \in \Lambda$ for some non-zero integer n. The set of rational points in V is denoted by $V_{\mathbb{Q}}$. A subspace L of V is said to be *rational* if $L \cap \Lambda$ is a lattice in L. If L is a rational subspace, then the image of Λ in V/L is a lattice in V/L, so that V/L is a rational vector space. The image of Λ in V/L is called the *projected lattice*. It is denoted by $\Lambda_{V/L}$. A rational space V with lattice Λ has a canonical Lebesgue measure $dx = dm_\Lambda(x)$, for which V/Λ has measure 1.

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A *convex polyhedron* (or, simply, a *polyhedron*) $\mathfrak{p}$ in V is, by definition, the intersection of a finite number of closed half-spaces bounded by affine hyperplanes. If the hyperplanes are rational, we say that the polyhedron is *rational*. If the hyperplanes have rational directions, we say that the polyhedron is *semi-rational*. For instance, if $\mathfrak{p} \subset V$ is a rational polyhedron, t is a real number and s is any point in V, then the dilated polyhedron $t\mathfrak{p}$ and the translated polyhedron $s + \mathfrak{p}$ are semi-rational. Unless stated otherwise, a polyhedron will be assumed to be semi-rational.

In this article, a cone will mean a polyhedral cone (with vertex 0) and an affine cone a translated set $s + \mathfrak{c}$ of a cone $\mathfrak{c}$. A cone $\mathfrak{c}$ is said to be *simplicial* if it is generated by linearly independent vectors of V. A simplicial cone $\mathfrak{c}$ is said to be *unimodular* if it is generated by linearly independent lattice vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ can be completed to a basis of Λ. An affine cone $\mathfrak{a}$ is said to be *simplicial* (respectively, *simplicial unimodular*) if the associated cone is.

A *polytope* $\mathfrak{p}$ is a compact polyhedron. The set of vertices of $\mathfrak{p}$ is denoted by $\mathcal{V}(\mathfrak{p})$. For each vertex s, the cone of feasible directions at s, i.e. $\{x \in V : s + \lambda x \in \mathfrak{p} \text{ for some } \lambda > 0\}$, is denoted by $\mathfrak{c}_s$.

We denote by $\mathcal{L}(V)$ the space of functions on V that is generated by indicator functions $[\mathfrak{q}]$ of polyhedra $\mathfrak{q}$ which contain lines.

2.2. *Valuations and generating functions.*

Definition 1. Let M be a vector space. An M-valued *valuation* Φ is a map from the set of polyhedra $\mathfrak{p} \subset V$ to the vector space M such that whenever the indicator functions $[\mathfrak{p}_i]$ of a family of polyhedra $\mathfrak{p}_i$ satisfy a linear relation $\sum_i r_i [\mathfrak{p}_i] = 0$, the elements $\Phi(\mathfrak{p}_i)$ satisfy the same relation

$$\sum_i r_i \Phi(\mathfrak{p}_i) = 0.$$

Definition 2. We denote by $\mathcal{H}(V^*)$ the ring of holomorphic functions defined around $0 \in V^*$. We denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0 \in V^*$ and by $\mathcal{M}_\ell(V^*) \subset \mathcal{M}(V^*)$ the subring consisting of meromorphic functions $\phi(\xi)$ which can be written as a quotient of a holomorphic function by a product of linear forms.

The intermediate generating functions of polyhedra which are studied here are important examples of functions in $\mathcal{M}_\ell(V^*)$.

PROPOSITION 3. *Let $L \subseteq V$ be a rational subspace. There exists a unique valuation S^L which associates to every semi-rational polyhedron $\mathfrak{p} \subset V$ a meromorphic function $S^L(\mathfrak{p}) \in \mathcal{M}(V^*)$ such that the following properties hold.*

(a) *If $\mathfrak{p}$ contains a line, then $S^L(\mathfrak{p}) = 0$.*

(b) *We have*

$$S^L(\mathfrak{p})(\xi) = \sum_{x \in \Lambda_{V/L}} \int_{\mathfrak{p} \cap (x+L)} \mathrm{e}^{\langle \xi, y \rangle} \, dm_L(y) \tag{3}$$

for every $\xi \in V^*$ such that the above sum converges.

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(c) *For every point $s \in \Lambda + L$, we have*

$$S^L(s + \mathfrak{p})(\xi) = \mathrm{e}^{\langle \xi, s\rangle} S^L(\mathfrak{p})(\xi).$$

When necessary, we will indicate the lattice and use the notation $S^L(\mathfrak{p}, \Lambda)$.

Remark 4. The fact that $S^L(\mathfrak{p})$ is actually an element of $\mathcal{M}_\ell(V^*)$ will be a consequence of the explicit computations in the next section.

For $L = \{0\}$, we recover the valuation S given by

$$S(\mathfrak{p})(\xi) = \sum_{x \in \mathfrak{p} \cap \Lambda} \mathrm{e}^{\langle \xi, s\rangle},$$

provided that this sum is convergent. For $L = V$, $S^V(\mathfrak{p})(\xi)$ is the integral

$$I(\mathfrak{p})(\xi) = \int_{\mathfrak{p}} \mathrm{e}^{\langle \xi, x\rangle}\, dx$$

if $\mathfrak{p}$ is full-dimensional, and $S^V(\mathfrak{p}) = 0$ otherwise. If $\mathfrak{p}$ is compact, the meromorphic function $S^L(\mathfrak{p})(\xi)$ is actually regular at $\xi = 0$, and its value for $\xi = 0$ is the valuation $E_{L^\perp}(\mathfrak{p})$ considered by Barvinok [6]. The proof is entirely analogous to that for the $L = \{0\}$ and $L = V$ cases (see, for instance, [7, Theorem 3.1]), and so we omit it.

Remark 5 (Rational versus semi-rational polyhedra). Although [7] and other classical texts deal only with rational polyhedra, their proofs extend easily to semi-rational ones. For instance, by summing geometric series, one obtains immediately the generating function of a unimodular cone with a real vertex.

Here are some simple examples. For $t \in \mathbb{R}$, we denote by $\{t\}$ the fractional part of t, i.e. $0 \leqslant \{t\} < 1$ and $t - \{t\} \in \mathbb{Z}$, and by $\lceil t \rceil$ the ceiling of t, i.e. $\lceil t \rceil = t + \{-t\}$. We also introduce the following analytic function.

Definition 6. Let

$$T(\tau, x) = \mathrm{e}^{\tau x} \frac{x}{1 - \mathrm{e}^x} = -\sum_{n=0}^{\infty} B_n(\tau) \frac{x^n}{n!}, \tag{4}$$

where $B_n(\tau)$ are the Bernoulli polynomials.

Example 7. For any $t \in \mathbb{R}$, we have

$$S(t + \mathbb{R}_{\geqslant 0})(\xi) = \frac{\mathrm{e}^{\lceil t \rceil \xi}}{1 - \mathrm{e}^{\xi}} = \mathrm{e}^{t\xi} \frac{\mathrm{e}^{\{-t\}\xi}}{1 - \mathrm{e}^{\xi}} = \mathrm{e}^{t\xi} T\big(\{-t\}, \xi\big) \frac{1}{\xi},$$

where the last form, involving the analytic function $T(\tau, \xi)$, illustrates that $S(t + \mathbb{R}_{\geqslant 0})$ indeed lies in $\mathcal{M}_\ell(V^*)$.

More generally, let $\mathfrak{c} \subset \mathbb{R}^d$ be a unimodular cone with primitive edge generators $v_1, \ldots, v_d$. Let s be any point in $\mathbb{R}^d$, with $s = \sum_{i=1}^d s_i v_i$. Then

$$S(s + \mathfrak{c})(\xi) = \prod_{i=1}^{d} \frac{\mathrm{e}^{\lceil s_i \rceil \langle \xi, v_i\rangle}}{1 - \mathrm{e}^{\langle \xi, v_i\rangle}} = \mathrm{e}^{\langle s, \xi\rangle} \prod_{i=1}^{d} \frac{\mathrm{e}^{\{-s_i\}\langle \xi, v_i\rangle}}{1 - \mathrm{e}^{\langle \xi, v_i\rangle}}.$$

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Example 8. Let $\mathfrak{c} \subseteq \mathbb{R}^2$ be the first quadrant and let $L = \mathbb{R}e_1$. We write $s = (a, b) \in \mathbb{R}^2$. Then we have

$$S^L(s+\mathfrak{c})(\xi_1,\xi_2) = \sum_{n=\lceil b\rceil}^{\infty} \int_a^{+\infty} \mathrm{e}^{n\xi_2 + x\xi_1}\, dx = \left(\sum_{n=\lceil b\rceil}^{\infty} \mathrm{e}^{n\xi_2}\right) \cdot \frac{-\mathrm{e}^{a\xi_1}}{\xi_1}$$
$$= \frac{\mathrm{e}^{\lceil b\rceil \xi_2}}{1-\mathrm{e}^{\xi_2}} \cdot \frac{-\mathrm{e}^{a\xi_1}}{\xi_1} = \mathrm{e}^{\langle s,\xi\rangle} \frac{\mathrm{e}^{\{-b\}\xi_2}}{1-\mathrm{e}^{\xi_2}} \cdot \frac{-1}{\xi_1}.$$

Example 9. Let $\mathfrak{p} \subset \mathbb{R}^2$ be the triangle with vertices (0, 0), (1, 0) and (0, 1). Let $L = \mathbb{R}(1, 0)$. Straightforward calculations give

$$S^L(\mathfrak{p}) = \frac{1-\mathrm{e}^{\xi_1}}{\xi_1}, \qquad I(\mathfrak{p})(\xi) = \frac{1}{\xi_1-\xi_2}\left(\frac{1-\mathrm{e}^{\xi_1}}{\xi_1} - \frac{1-\mathrm{e}^{\xi_2}}{\xi_2}\right),$$
$$S(\mathfrak{p})(\xi) = 1 + \mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2}.$$

These three functions are indeed analytic.

A consequence of the valuation property is the following fundamental theorem. It follows from the Brion–Lawrence–Varchenko decomposition of a polytope into the supporting cones at its vertices [9].

THEOREM 10 (Brion's theorem). *Let $\mathfrak{p}$ be a polytope with set of vertices $\mathcal{V}(\mathfrak{p})$. For each vertex s, let $\mathfrak{c}_s$ be the cone of feasible directions at s. Let $L \subseteq V$ be a rational subspace. Then*

$$S^L(\mathfrak{p}) = \sum_{s\in\mathcal{V}(\mathfrak{p})} S^L(s+\mathfrak{c}_s).$$

§3. *Intermediate generating function for a simplicial affine cone.*

3.1. *Case where the slicing space is parallel to a face.* Let $\mathfrak{c} \subset V$ be a simplicial cone with integral generators $(v_j,\ j = 1, \ldots, d)$, and let $s \in V$. Let us recall the expression for the intermediate generating function $S^L(s+\mathfrak{c})(\xi)$, when the slicing subspace L is generated by some of the v_j.

For $I \subseteq \{1, \ldots, d\}$, we denote by L_I the linear span of the vectors $(v_i,\ i \in I)$. We denote the complement of I in $\{1, \ldots, d\}$ by I^c. We have $V = L_I \oplus L_{I^c}$. For $x \in V$, we write it in component form as

$$x = x_I + x_{I^c}.$$

Thus we identify the quotient V/L_I with L_{I^c} and we denote the projected lattice by $\Lambda_{I^c} \subset L_{I^c}$. Note that $L_{I^c} \cap \Lambda \subseteq \Lambda_{I^c}$, but the inclusion is strict in general.

Example 11. Let $v_1 = (1, 0)$, $v_2 = (1, 2)$ and $I = \{2\}$. The projected lattice Λ_{I^c} on $L_{I^c} = \mathbb{R}v_1$ is $\mathbb{Z}(v_1/2)$. See Figure 1.

We denote by $\mathfrak{c}_I$ the cone generated by the vectors v_j, $j \in I$, and by $\mathfrak{b}_I$ the parallelepiped $\mathfrak{b}_I = \sum_{i\in I}[0, 1[\, v_i$. Similarly, we denote by $\mathfrak{c}_{I^c}$ the cone

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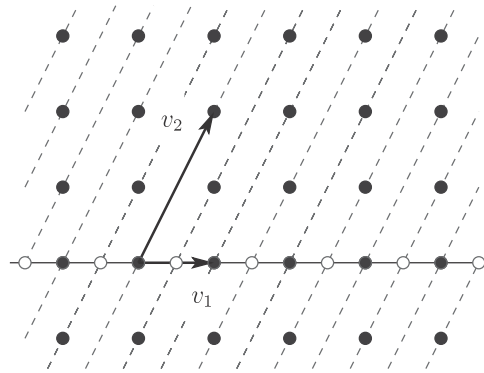


Figure 1: The projected lattice $\Lambda_{\{1\}}$.

generated by the vectors v_j, $j \in I^c$, and let $\mathfrak{b}_{I^c} = \sum_{i \in I^c} [0, 1[\, v_i$. The projection of the cone $\mathfrak{c}$ on $V/L_I = L_{I^c}$ identifies with $\mathfrak{c}_{I^c}$. Note that a generator v_i, $i \in I^c$, may be non-primitive for the projected lattice Λ_{I^c}, even if it is primitive for Λ, as we saw in the previous example. We write $s = s_I + s_{I^c}$.

First, let us recall from [2] how the intermediate generating function $S^{L_I}(s + \mathfrak{c}, \Lambda)$ decomposes as a product. This already appeared in Example 8.

PROPOSITION 12 [2]. *The intermediate sum for the full cone $s + \mathfrak{c}$ breaks up into the product*

$$S^{L_I}(s + \mathfrak{c}, \Lambda)(\xi) = S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)\, I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi). \qquad (5)$$

We remark that in the above equation, the function $S(s_{I^c} + \mathfrak{c}_{I^c}, \Lambda_{I^c})(\xi)$ is a meromorphic function on the space $(L_{I^c})^*$. The integral $I(s_I + \mathfrak{c}_I, L_I \cap \Lambda)(\xi)$ is a meromorphic function on the space $(L_I)^*$. We consider both as functions on V^* through the decomposition $V = L_I \oplus L_{I^c}$.

3.2. *Cone decomposition with respect to a linear subspace.* Let L be a linear subspace of V. Let $\mathfrak{c}$ be a full-dimensional simplicial cone in V with generators $w_1, \ldots, w_d$. The main theorem of this section, Theorem 19, gives a signed decomposition of $\mathfrak{c}$ into a family of full-dimensional simplicial cones $\mathfrak{c}_\sigma$, where each $\mathfrak{c}_\sigma$ has a face parallel to L. More precisely, this decomposition is a relation between indicator functions of cones, modulo the space $\mathcal{L}(V)$ of functions on V that is generated by indicator functions $[\mathfrak{q}]$ of polyhedra $\mathfrak{q}$ which contain lines. This decomposition is borrowed from [10]. The proof relies on the following simple lemma, which is of interest in its own right. The extreme cases, $\dim L = 1$ or $\operatorname{codim} L = 1$, of Theorem 19 follow easily from this lemma (Proposition 15). The general case is proven by induction on either $\dim L$ or $\operatorname{codim} L$.

Given vectors $u_1, \ldots, u_p \in \mathbb{R}^d$, we denote the cone $\sum_{i=1}^p \mathbb{R}_{\geqslant 0} u_i$ by $\mathfrak{c}(u_1, \ldots, u_p)$ and its relative interior $\sum_{i=1}^p \mathbb{R}_{>0} u_i$ by $\tilde{\mathfrak{c}}(u_1, \ldots, u_p)$. For $p = 0$, both $\mathfrak{c}$ and $\tilde{\mathfrak{c}}$ are $\{0\}$.

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LEMMA 13. *Let $u_1, \dots, u_d$ be a basis of V and let $u_{d+1} = -\sum_{k=1}^{d} u_k$. For $i = 1, \dots, d+1$, define semi-open cones*

$$\mathfrak{k}_i = \mathfrak{c}(u_1, \dots, u_{i-1}) + \tilde{\mathfrak{c}}(u_{i+1}, \dots, u_{d+1}).$$

Then V is the disjoint union of the $\mathfrak{k}_i$ for $i = 1, \dots, d+1$.

Proof. Let $x = t_1 u_1 + \cdots + t_d u_d$. If $t_k \geqslant 0$ for all k, then $x \in \mathfrak{k}_{d+1}$. Otherwise, let i be the largest index with $1 \leqslant i \leqslant d$ for which the minimum $\min_{1 \leqslant k \leqslant d} t_k$ is attained. Writing $u_i = -\sum_{1 \leqslant k \leqslant d+1, k \neq i} u_k$, we obtain

$$x = \sum_{1 \leqslant k \leqslant d, k \neq i} (t_k - t_i) u_k - t_i u_{d+1}.$$

As $t_k - t_i \geqslant 0$ if $1 \leqslant k \leqslant d$ and $t_k - t_i > 0$ if $i < k \leqslant d$, and since $t_i < 0$, we see that $x \in \mathfrak{k}_i$.

Let us show that $\mathfrak{k}_i \cap \mathfrak{k}_j$ is empty if $i < j$. Let $x \in \mathfrak{k}_i$; then $x = \sum_{1 \leqslant k \leqslant d+1, k \neq i} t_k u_k$ where $t_k \geqslant 0$ for all k and $t_k > 0$ for $k > i$. Writing $u_j = -\sum_{1 \leqslant k \leqslant d+1, k \neq j} u_k$, we obtain

$$x = \sum_{1 \leqslant k \leqslant d+1,\ k \notin \{i,j\}} (t_k - t_j) u_k - t_j u_i.$$

Noting that u_j no longer appears and that u_i has the coefficient $-t_j < 0$, it follows that $x \notin \mathfrak{k}_j$. □

From the lemma, we deduce other interesting cone decompositions. The first one is the well-known stellar decomposition.

PROPOSITION 14 (Stellar decomposition). *Let $w_1, \dots, w_d$ be a basis of V. Let $r \leqslant s$ and*

$$v = w_1 + \cdots + w_r - (w_{s+1} + \cdots + w_d).$$

Let $\mathfrak{c}$ be the cone generated by the w_i with $1 \leqslant i \leqslant d$, and let $\mathfrak{k}$ be the cone generated by $\mathfrak{c}$ and the vector v.

(a) *Define the semi-open cones $\mathfrak{k}_i$ by*

$$\mathfrak{k}_i = \tilde{\mathfrak{c}}(w_1, \dots, w_{i-1}) + \mathfrak{c}(w_{i+1}, \dots, w_d, v) \quad \text{for } 1 \leqslant i \leqslant r \text{ if } r \geqslant 1,$$
$$\mathfrak{k}_i = \mathfrak{c}(w_1, \dots, w_{i-1}) + \tilde{\mathfrak{c}}(w_{i+1}, \dots, w_d, v) \quad \text{for } s+1 \leqslant i \leqslant d \text{ if } s < d.$$

Then:

(a.1) *if $r \geqslant 1$, $\mathfrak{k}$ is the disjoint union of the $\mathfrak{k}_i$ for $1 \leqslant i \leqslant r$;*

(a.2) *$\mathfrak{k} \backslash \mathfrak{c}$ is the disjoint union of the $\mathfrak{k}_i$ for $s+1 \leqslant i \leqslant d$. (If $s = d$, then $\mathfrak{k} \backslash \mathfrak{c} = \emptyset$.)*

(b) *For $i \in \{1, \dots, r\} \cup \{s+1, \dots, d\}$, let*

$$\mathfrak{c}_i = \mathfrak{c}(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_d, v).$$

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Then

$$[\mathfrak{c}] - \sum_{i=1}^{r} [\mathfrak{c}_i] + \sum_{i=s+1}^{d} [\mathfrak{c}_i] \equiv 0$$

modulo indicator functions of lower-dimensional cones.

Proof. Let us prove (a.2) first. By quotienting out by the subspace generated by $w_1, \dots, w_s$, we can assume that $s = 0$, and thus $v = -(w_1 + \cdots + w_d)$. In this case, we just apply Lemma 13 with $u_k = w_k$ and $u_{d+1} = v$. As the cone $\mathfrak{k}_{d+1}$ of Lemma 13 is just $\mathfrak{c}$, we obtain (a.2).

Let us now assume that $r \geqslant 1$. We write

$$w_1 = v + w_d + w_{d-1} + \cdots + w_{s+2} + w_{s+1} - (w_r + w_{r-1} + \cdots + w_3 + w_2).$$

Upon renaming all vectors appearing in this equation, we obtain

$$\bar{v} = \bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_{\bar{r}-1} + \bar{w}_{\bar{r}} - (\bar{w}_{\bar{s}+1} + \bar{w}_{\bar{s}+2} + \cdots + \bar{w}_{d-1} + \bar{w}_d), \quad (6)$$

where $\bar{w}_1 = v$, $\bar{w}_2 = w_d$ etc, and $\bar{v} = w_1$. We now apply (a.2) to equation (6). The corresponding semi-open cones are

$$\begin{aligned} \bar{\mathfrak{k}}_1 &= \mathfrak{c}(v, w_d, \dots, w_{r+1}) + \tilde{\mathfrak{c}}(w_{r-1}, \dots, w_2, w_1) = \mathfrak{k}_r, \\ &\vdots \\ \bar{\mathfrak{k}}_r &= \mathfrak{c}(v, w_d, \dots, w_2) = \mathfrak{k}_1, \end{aligned}$$

and thus we obtain (a.1).

For $i \in \{1, \dots, r\} \cup \{s+1, \dots, d\}$, the (closed) cone $\mathfrak{c}_i$ differs from the semi-open cone $\mathfrak{k}_i$ by the union of some faces of lower dimension, therefore (b) follows from (a) and inclusion–exclusion relations. □

Next, we obtain decompositions of any simplicial cone into full-dimensional cones which have a given edge or a given facet. Recall that $\mathcal{L}(V)$ is the linear span of indicator functions of cones which contain straight lines.

PROPOSITION 15. *Let $w_1, \dots, w_d$ be a basis of V and let $\mathfrak{c}$ be the cone generated by the w_i with $1 \leqslant i \leqslant d$. Let $0 \leqslant r \leqslant s \leqslant d+1$.*

(a) *Let*

$$v = w_1 + \cdots + w_r - (w_{s+1} + \cdots + w_d). \quad (7)$$

Then we have the following relation between indicator functions of cones:

$$\begin{aligned} [\mathfrak{c}] \equiv &\sum_{i=1}^{r} (-1)^{i-1} [\mathfrak{c}(-w_1, \dots, -w_{i-1}, w_{i+1}, \dots, w_d, v)] \\ &+ \sum_{i=s+1}^{d} (-1)^{d-i} [\mathfrak{c}(w_1, \dots, w_{i-1}, -w_{i+1}, \dots, -w_d, -v)] \\ &\mod \mathcal{L}(V). \end{aligned} \quad (8)$$

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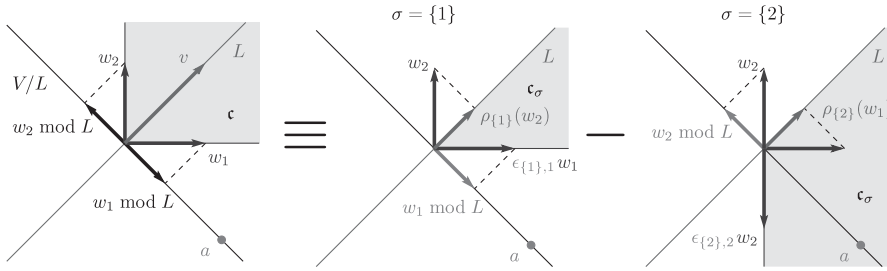


Figure 2: Brion–Vergne decomposition of a cone $\mathfrak{c}$ into cones with a face parallel to the subspace L, modulo cones with lines. The vectors in the quotient V/L determine the signs $\epsilon_{\sigma,j}$.

(b) *Let $L \subset V$ be a hyperplane. Assume that $w_i \in L$ if and only if $r+1 \leqslant i \leqslant s$ and that the w_i lie on one side of L for $1 \leqslant i \leqslant r$ and on the other side for $s+1 \leqslant i \leqslant d$. For $i \in \{1, \ldots, r\} \cup \{s+1, \ldots, d\}$, denote by ρ_i the projection $V \to L$ parallel to w_i. Then we have the following relation between indicator functions of cones:*

$$[\mathfrak{c}] \equiv \sum_{i=1}^{r} [\mathbb{R}_+ w_i + \rho_i(\mathfrak{c})] - \sum_{i=s+1}^{d} [\mathbb{R}_+(-w_i) + \rho_i(\mathfrak{c})] \mod \mathcal{L}(V). \quad (9)$$

Example 16. Let $\mathfrak{c} = \mathfrak{c}(w_1, w_2)$ and $v = w_1 + w_2$. Then

$$[\mathfrak{c}] = -[\mathfrak{c}(-w_2, v)] + [\mathfrak{c}(w_1, v)] + [\mathfrak{c}(w_2, -w_2, w_1)].$$

Indeed, $[\mathfrak{c}(w_2, -w_2, w_1)] - [\mathfrak{c}]$ is the indicator function of the quadrant $\mathfrak{c}(w_1, -w_2)$ minus that of the line $\mathbb{R}w_1$. This is the case for $[\mathfrak{c}(-w_2, v)] + [\mathfrak{c}(w_1, v)]$ as well. See also Figure 2.

Example 17. We consider the three-dimensional cone

$$\mathfrak{c} = \mathfrak{c}((-1, 0, 0), (-1, 2, 0), (-1, 0, 3)).$$

(i) First, let L be the subspace with basis $(0, 1, 1)$. Formula (8) gives

$$\begin{aligned}[\mathfrak{c}] \equiv\ & [\mathfrak{c}((-1, 0, 0), (-1, 2, 0), (0, 1, 1))] \\ & - [\mathfrak{c}((-1, 0, 0), (1, 0, -3), (0, 1, 1))] \\ & + [\mathfrak{c}((-1, 2, 0), (-1, 0, 3), (0, -1, -1))].\end{aligned}$$

(ii) Now, let L be the subspace with basis $(1, 0, 0)$, $(0, 1, 1)$. Formula (9) gives

$$\begin{aligned}[\mathfrak{c}] \equiv\ & [\mathfrak{c}((-1, 2, 0), (-1, 0, 0), (-5, 6, 6))] \\ & - [\mathfrak{c}((1, 0, -3), (-1, 0, 0), (-5, 6, 6))].\end{aligned}$$

This example will be continued in Example 26.

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Proof of Proposition 15. Let us prove (a). By quotienting out by the subspace generated by $w_{r+1}, \dots, w_s$, we can assume that $r = s$. We write (7) as

$$w_r + \cdots + w_1 - v - w_d - \cdots - w_{r+1} = 0.$$

By applying Lemma 13 with

$$u_1 = -w_r, \dots, u_{r+1} = -v, \dots, u_{d+1} = w_{r+1},$$

we see that the whole space V is the disjoint union of the semi-open cones

$$\begin{aligned}
\mathfrak{k}_1 &= \tilde{\mathfrak{c}}(w_{r-1}, \dots, -v, -w_d, \dots, -w_{r+1}),\\
&\vdots\\
\mathfrak{k}_r &= \mathfrak{c}(w_r, \dots, w_2) + \tilde{\mathfrak{c}}(-v, -w_d, \dots, -w_{r+1}),\\
\mathfrak{k}_{r+1} &= \mathfrak{c}(w_r, \dots, w_1) + \tilde{\mathfrak{c}}(-w_d, \dots, -w_{r+1}),\\
\mathfrak{k}_{r+2} &= \mathfrak{c}(w_r, \dots, w_1, -v) + \tilde{\mathfrak{c}}(-w_{d-1}, \dots, -w_{r+1}),\\
&\vdots\\
\mathfrak{k}_{d+1} &= \mathfrak{c}(w_r, \dots, w_1, -v, -w_d, \dots, -w_{r+2}).
\end{aligned}$$

For any basis $u_1, \dots, u_d$, we have

$$\begin{aligned}
&[(\mathfrak{c}(u_1, \dots, u_k) + \tilde{\mathfrak{c}}(u_{k+1}, \dots, u_d))]\\
&\quad \equiv (-1)^{d-k}[\mathfrak{c}(u_1, \dots, u_k, -u_{k+1}, \dots, -u_d)] \mod \mathcal{L}(V).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
[\mathfrak{k}_1] &\equiv (-1)^d[\mathfrak{c}(-w_{r-1}, \dots, -w_1, v, w_d, \dots, w_{r+1})],\\
&\vdots\\
[\mathfrak{k}_r] &\equiv (-1)^{d-r}[\mathfrak{c}(w_r, \dots, w_2, v, w_d, \dots, w_{r+1})],\\
[\mathfrak{k}_{r+1}] &\equiv (-1)^{d-r-1}[\mathfrak{c}(w_r, \dots, w_1, w_d, \dots, w_{r+1})],\\
&\vdots\\
[\mathfrak{k}_{d+1}] &= [\mathfrak{c}(w_r, \dots, w_1, -v, -w_d, \dots, -w_{r+2})].
\end{aligned}$$

Moreover, these indicator functions add up to $[V] \equiv 0 \mod \mathcal{L}(V)$. Thus we have proven (a).

Let us show that (b) is just the dual of a stellar decomposition. Let $\alpha \in V^*$ be such that $\ker \alpha = L$ and $\alpha(w_i) > 0$ for $1 \leqslant i \leqslant r$ while $\alpha(w_i) < 0$ for $s + 1 \leqslant i \leqslant d$. Denote the dual cone by $\mathfrak{c}^\circ \subset V^*$, that is,

$$\mathfrak{c}^\circ = \{\xi \in V^* : \langle \xi, x \rangle \geqslant 0 \text{ for all } x \in \mathfrak{c}\}.$$

Let $(\gamma_1, \dots, \gamma_d)$ be the dual basis of $(v_1, \dots, v_d)$. Thus $\mathfrak{c}^\circ = \mathfrak{c}(\gamma_1, \dots, \gamma_d)$. From the stellar decomposition of the cone $\mathfrak{c}^\circ$ with respect to the extra edge α, we see that the expression

$$[\mathfrak{c}^\circ] - \sum_{i \in \{1,\dots,r\} \cup \{s+1,\dots,d\}} [\mathfrak{c}(\gamma_1, \dots, \hat{\gamma_i}, \dots, \gamma_d, \alpha)]$$

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is a linear combination of indicator functions of lower-dimensional cones. If a linear identity holds for indicator functions of cones, the same identity holds for the indicator functions of their duals (see, for instance, [7, Corollary 2.8]). Moreover, the dual of a lower-dimensional cone contains a line; therefore, by taking indicator functions of dual cones, we obtain an equality mod $\mathcal{L}(V)$. (This "duality trick" goes back to [9].) It remains to compute the dual cone $(\mathfrak{c}(\gamma_1, \ldots, \hat{\gamma_i}, \ldots, \gamma_d, \alpha))^\circ$. We see easily that if $\langle \alpha, w_i \rangle > 0$, then

$$(\mathfrak{c}(\gamma_1, \ldots, \hat{\gamma_i}, \ldots, \gamma_d, \alpha))^\circ = \mathbb{R}_+ w_i + \rho_i(\mathfrak{c}),$$

while if $\langle \alpha, w_i \rangle < 0$, then

$$(\mathfrak{c}(\gamma_1, \ldots, \hat{\gamma_i}, \ldots, \gamma_d, \alpha))^\circ = \mathbb{R}_+(-w_i) + \rho_i(\mathfrak{c}).$$

Thus we have proven (b). □

As we will see now, Proposition 15 consists of particular cases of the next result, Brion–Vergne decomposition. Let L be a linear subspace of V. Let $\mathfrak{c}$ be a simplicial cone in V with generators $w_1, \ldots, w_d$. We consider the subsets $\sigma \subset \{1, \ldots, d\}$ such that the vectors w_j, for $j \in \sigma$, form a supplementary basis to L in V. In other words, we have a direct sum decomposition

$$V = \bigoplus_{j \in \sigma} \mathbb{R} w_j \oplus L.$$

The corresponding projection on L is denoted by $\rho_\sigma : V \to L$. The set of these σ is denoted by $\mathcal{B}(\mathfrak{c}, L)$.

Remark 18. At most $\binom{d}{k_0} = O(d^{k_0})$ such sets σ exist. Thus, if codim L is a fixed constant k_0, this is a polynomial quantity, and the sets $\sigma \in \mathcal{B}(\mathfrak{c}, L)$ can be enumerated by a straightforward algorithm in polynomial time. (In practice, reverse search [1] would be used for this enumeration.)

A vector a in V/L is said to be *generic* with respect to the pair $(\mathfrak{c}, L)$ if a does not belong to any hyperplane generated by vectors among the projections of $w_1, \ldots, w_d$. In other words, for any $\sigma \in \mathcal{B}(\mathfrak{c}, L)$ we have

$$a = \sum_{j \in \sigma} a_{\sigma, j} (w_j \bmod L),$$

where $a_{\sigma, j} \neq 0$ for every $j \in \sigma$.

THEOREM 19 (Brion–Vergne decomposition [10, Theorem 1.2]). *Let L be a linear subspace of V. Let $\mathfrak{c}$ be a full-dimensional simplicial cone in V with generators $w_1, \ldots, w_d$. Fix a vector $a \in V/L$ that is generic with respect to the pair $(\mathfrak{c}, L)$ and belongs to the projection of $\mathfrak{c}$ on V/L. For $\sigma \in \mathcal{B}(\mathfrak{c}, L)$, let*

$$a = \sum_{j \in \sigma} a_{\sigma, j} (w_j \bmod L).$$

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Let $\epsilon_{\sigma,j}$ be the sign of $a_{\sigma,j}$ and $\epsilon(\sigma) = \prod_{j\in\sigma} \epsilon_{\sigma,j}$. Denote by $\mathfrak{c}_\sigma \subset V$ the cone with edge generators

$$\epsilon_{\sigma,j} w_j \text{ for } j \in \sigma \quad \text{and} \quad \rho_\sigma(w_k) \text{ for } k \notin \sigma.$$

Then we have the following relation between indicator functions of cones:

$$[\mathfrak{c}] \equiv \sum_{\sigma \in \mathcal{B}(\mathfrak{c},L)} \epsilon(\sigma)[\mathfrak{c}_\sigma] \mod \mathcal{L}(V). \tag{10}$$

This theorem is illustrated in Figure 2.

Remark 20. As explained in [**10**], the case where $\dim L = 1$ reduces to case (a) of Proposition 15. Let us show that the case where $\operatorname{codim} L = 1$ reduces to case (b). A vector a in V/L is generic with respect to $(\mathfrak{c}, L)$ if and only if $a \neq 0$. Up to renumbering, we can assume that $w_1, \ldots, w_r$ lie on the same side of L as a, while $w_{s+1}, \ldots, w_d$ lie on the other side. The set $\mathcal{B}(\mathfrak{c}, L)$ consists of the singletons $\sigma_i = \{i\}$ such that $w_i \notin L$, i.e. $i \in \{1, \ldots, r\} \cup \{s+1, \ldots, d\}$. The sign $\epsilon_{\sigma_i,i}$ is $+1$ if $i \in \{1, \ldots, r\}$ and -1 if $i \in \{s+1, \ldots, d\}$; therefore equation (10) becomes precisely equation (9).

THEOREM 21 (Efficient construction of a Brion–Vergne decomposition). *Fix a number k_0. There exists a polynomial-time algorithm for the following problem. Given a rational simplicial cone $\mathfrak{c}$ and a rational subspace L of V of codimension at most k_0, compute a decomposition (10).*

Proof. By Remark 18, we can enumerate the bases $\sigma \in \mathcal{B}(\mathfrak{c}, L)$ in polynomial time.

A rational generic vector a can be constructed in polynomial time as follows; this technique was already used in [**5**]. Fix one of the bases $\bar\sigma \in \mathcal{B}(\mathfrak{c}, L)$ as a basis of V/L. Then the projections of $w_1, \ldots, w_d$ have rational coordinate vectors $\bar w_1, \ldots, \bar w_d \in \mathbb{Q}^{k_0}$ with respect to this basis.

Consider the *moment curve* $\bar a(t) = (1, t, t^2, \ldots, t^{k_0-1}) \in \mathbb{R}^{k_0}_+$ for $t \geqslant 0$. For any $\sigma \in \mathcal{B}(\mathfrak{c}, L)$ and $j \in \sigma$, we have by Cramer's rule that the coefficient $a_{\sigma,j}(t)$ for $a = a(t)$ is non-zero if and only if $f_{\sigma,j}(t) = \det(a(t); \bar w_i : i \in \sigma, i \neq j) \neq 0$. The functions $f_{\sigma,j}(t)$ are polynomial functions in t of degree at most $k_0(k_0-1)$ which are not identically zero. Hence each has at most $k_0(k_0-1)$ zeros. Thus, by computing the coefficients $a_{\sigma,j}(t)$ for $M = k_0(k_0-1)|\mathcal{B}(\mathfrak{c}, L)| + 1$ different values for t, we will find one $t = \bar t$ that is not a zero of any $f_{\sigma,j}$. Clearly, this search can be implemented in polynomial time.

Then the vector $a = \sum_{i\in\sigma} \bar a_i(t)(w_i \mod L) \in V/L$ is generic with respect to the pair $(\mathfrak{c}, L)$, and since $\bar a_i(t) \geqslant 0$, it also lies in the projection of $\mathfrak{c}$ on V/L. (We could also use explicit deterministic constructions of generic vectors of polynomial encoding size, without search, similar to [**12, 13**].) □

Remark 22 (Relation to variants of Barvinok's decomposition). Barvinok's method for efficiently computing the discrete generating function $S(s+\mathfrak{c})(\xi)$ of a cone in fixed dimension decomposes a simplicial cone into cones with smaller indices (and ultimately into unimodular cones). The original method of

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Barvinok [5] used a primal decomposition and inclusion–exclusion to take care of lower-dimensional cones. The dual of the stellar decomposition is used in the variant of Barvinok's method that was presented in [7] and later implemented in *LattE* [11]; this avoids having to take care of lower-dimensional cones. Instead of dualizing, one can directly use formula (8), the case where $\dim L = 1$. We use this primal variant of Barvinok's decomposition in our implementation. Other primal variants were studied in [12, 13] and implemented in *LattE macchiato*.

3.3. *Complexity of the intermediate generating function.* We will use the following notation.

Definition 23. For a and a positive integer q, we write

$$\lfloor n \rfloor_q := q \left\lfloor \frac{1}{q} n \right\rfloor \in q\mathbb{Z}, \qquad \{n\}_q := (n \bmod q) \in [0, q),$$

which give the unique decomposition

$$n = \lfloor n \rfloor_q + \{n\}_q.$$

Thus $\lfloor a \rfloor_1 = \lfloor a \rfloor$ and $\{a\}_1 = \{a\}$ will be the ordinary "floor" and "fractional part" of a.

THEOREM 24 (Short formula for $S^L(s + \mathfrak{c})(\xi)$). *Fix a non-negative integer k_0. There exists a polynomial-time algorithm for the following problem.*

Suppose the following input is given:

(I_1) *a number d in unary encoding;*

(I_2) *a simplicial cone $\mathfrak{c} = \mathfrak{c}(v_1, \dots, v_d) \subset \mathbb{R}^d$, represented by the primitive vectors $v_1, \dots, v_d \in \mathbb{Z}^d$ in binary encoding;*

(I_3) *a rational subspace $L \subseteq \mathbb{R}^d$ of codimension k_0, represented by $d - k_0$ linearly independent vectors $b_1, \dots, b_{d-k_0} \in \mathbb{Z}^d$ in binary encoding.*

Compute the following output in binary encoding:

(O_1) *a finite set Γ;*

(O_2) *for every γ in Γ, integers $\alpha^{(\gamma)}$ and lattice vectors $w_i^{(\gamma)} \in \mathbb{Z}^d$ for $i = 1, \dots, d$.*

The output should have the following properties.

(i) *For every γ, the family $(w_i^{(\gamma)})$ for $i = 1, \dots, d$ is a basis of $\mathbb{R}^d$, and if we denote the dual basis by $(\eta_i^{(\gamma)})$ for $i = 1, \dots, d$, then $\eta_i^{(\gamma)} \in \mathbb{Z}^d$ for $i = 1, \dots, k_0$.*

(ii) *For every $s \in \mathbb{R}^d$, the following equality of meromorphic functions of ξ holds:*

$$S^L(s + \mathfrak{c}, \mathbb{Z}^d)(\xi) = \mathrm{e}^{\langle \xi, s \rangle} \sum_{\gamma \in \Gamma} \alpha^{(\gamma)} \prod_{i=1}^{k_0} T(\{-\langle \eta_i^{(\gamma)}, s \rangle\}, \langle \xi, w_i^{(\gamma)} \rangle) \cdot \frac{1}{\prod_{i=1}^{d} \langle \xi, w_i^{(\gamma)} \rangle}. \quad (11)$$

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Remark 25. Consider the term corresponding to $\gamma \in \Gamma$ in (11). Define

$$s^{(\gamma)} = \sum_{i=1}^{k_0} \{-\langle \eta_i^{(\gamma)}, s\rangle\} w_i^{(\gamma)}.$$

The point $s + s^{(\gamma)}$ is in

$$\bigoplus_{i=1}^{k_0} \mathbb{Z} w_i^{(\gamma)} \oplus \bigoplus_{i=k_0+1}^{d} \mathbb{R} w_i^{(\gamma)},$$

and formula (11) can also be expressed as

$$S^L(s + \mathfrak{c}, \mathbb{Z}^d)(\xi) = \sum_{\gamma\in\Gamma} \alpha^{(\gamma)} \mathrm{e}^{\langle \xi, s+s^{(\gamma)}\rangle} \frac{1}{\prod_{i=1}^{k_0}(1 - \mathrm{e}^{\langle \xi, w_i^{(\gamma)}\rangle})} \frac{1}{\prod_{i=k_0+1}^{d} \langle \xi, w_i^{(\gamma)}\rangle}. \tag{12}$$

Formula (12) is easier to grasp, while (11) is used in the program, where we avoid all the singular hyperplanes $\langle \xi, w_i^{(\gamma)}\rangle = 0$, for $\gamma \in \Gamma$ and $i = 1, \ldots, d$, by deforming the linear form ξ into $\xi + \epsilon$.

Proof. Using Theorem 19, we construct a signed decomposition of $\mathfrak{c}$ into cones $\mathfrak{c}^\sigma$ which have a face parallel to L, so that for any $s \in \mathbb{R}^d$ we have

$$[s + \mathfrak{c}] \equiv \sum_{\sigma} \epsilon^\sigma [s + \mathfrak{c}^\sigma] \quad \text{(modulo cones containing lines)}, \tag{13}$$

where $\epsilon^\sigma \in \{\pm 1\}$. Here σ runs through the set $\mathcal{B}(\mathfrak{c}, L)$ of Theorem 19, which can be enumerated in polynomial time by Remark 18. This makes the construction of the signed decomposition a polynomial-time algorithm. As the valuation S^L vanishes on cones which contain lines, we get

$$S^L(s + \mathfrak{c})(\xi) = \sum_{\sigma\in\mathcal{B}(\mathfrak{c},L)} \epsilon^\sigma S^L(s + \mathfrak{c}^\sigma)(\xi).$$

Each cone $\mathfrak{c}^\sigma$ is given by its edge generators w_i^σ, $i = 1, \ldots, d$, which are such that (w_i^σ), for $i = k_0 + 1, \ldots, d$, is a basis of L. Then we apply [**2**, Theorem 28] (short formula for $S^{L_I}(s + \mathfrak{c}, \mathbb{Z}^d)(\xi)$) with input $\mathfrak{c}^\sigma$ and $I = \{k_0 + 1, \ldots, d\}$. See Examples 7 and 8. □

Example 26 (Continuation of Example 17). Again, $\mathfrak{c}$ is the cone

$$\mathfrak{c}((0, -2, 0), (1, -2, 0), (0, -2, 3)).$$

Here is the output of our Maple program for computing the generating functions $S^L(s + \mathfrak{c})(\xi)$ (formula (12)). We set $s = (a, b, c)$ and $\xi = (x, y, z)$.

(i) Let L be the subspace with basis $(0, 1, 1)$. In this case, the three cones which occur in the decomposition of Example 17 have non-unimodular projections in V/L, so that the projections are further decomposed into

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unimodular cones. This is why we have six terms in the expression for $S^L(s+\mathfrak{c})$:

$$\begin{aligned} &S^L(s+\mathfrak{c})(x,\,y,\,z)\\ &\quad= \mathrm{e}^{ax+by+cz}\Bigg(\frac{\mathrm{e}^{\{3a-b+c\}(\frac{2}{5}y-\frac{3}{5}z)+\{a\}(-x+3z)}}{(1-\mathrm{e}^{\frac{2}{5}y-\frac{3}{5}z})(1-\mathrm{e}^{-x+3z})(y+z)}\\ &\qquad\qquad+\frac{\mathrm{e}^{\{a\}(-x+2y)+\{2a+b-c\}(-\frac{2}{5}y+\frac{3}{5}z)}}{(1-\mathrm{e}^{-x+2y})(1-\mathrm{e}^{-\frac{2}{5}y+\frac{3}{5}z})(y+z)}\\ &\qquad-\frac{\mathrm{e}^{-\{3a-b+c\}z+\{-a\}(x-3z)}}{(1-\mathrm{e}^{-z})(1-\mathrm{e}^{x-3z})(-y-z)}+\frac{\mathrm{e}^{-\{-b+c\}z+\{-a\}x}}{(1-\mathrm{e}^{-z})(1-\mathrm{e}^{x})(-y-z)}\\ &\qquad+\frac{\mathrm{e}^{\{-b+c\}y-\{a\}x}}{(1-\mathrm{e}^{y})(1-\mathrm{e}^{-x})(-y-z)}+\frac{\mathrm{e}^{\{a\}(-x+2y)-\{2a+b-c\}y}}{(1-\mathrm{e}^{-x+2y})(1-\mathrm{e}^{-y})(-y-z)}\Bigg). \end{aligned}$$

(ii) Let L be the subspace with basis $(1,\,0,\,0)$, $(0,\,1,\,1)$. Then

$$\begin{aligned} &S^L(s+\mathfrak{c})(x,\,y,\,z)\\ &\quad= \mathrm{e}^{ax+by+cz}\Bigg(-6\,\frac{\mathrm{e}^{\{-b+c\}(-\frac{1}{2}x+y)}}{(1-\mathrm{e}^{-\frac{1}{2}x+y})x(-5x+6y+6z)}\\ &\qquad\qquad+6\frac{\mathrm{e}^{\{-b+c\}(\frac{1}{3}x-z)}}{(1-\mathrm{e}^{\frac{1}{3}x-z})x(-5x+6y+6z)}\Bigg). \end{aligned}$$

Example 27. We consider the four triangles T_i in Figure 3, which subdivide the square $\mathfrak{q}=[0,\,4]\times[0,\,4]$. Let $L=\mathbb{R}(0,\,1)$. Let us apply the inclusion–exclusion formula together with the valuation property of S^L. As S^L vanishes on the edges and vertices which occur in the intersections of the triangles, it follows that the sum $\sum_{i=1}^4 S^L(T_i)$ must be equal to $S^L(\mathfrak{q})$.

The example has been chosen so that the intermediate valuation $S^L(\mathfrak{q})$ can also be computed directly by elementary means. This allows us to verify our methods on the example. A direct computation gives

$$S^L(\mathfrak{q})=\frac{\mathrm{e}^{4y}-1}{y}(1+\mathrm{e}^{x}+\mathrm{e}^{2x}+\mathrm{e}^{3x}+\mathrm{e}^{4x}).$$

Table 1 shows the output of our Maple program that implements the algorithm of the theorem for $S^L(T_i)$, $i=1,\ldots,4$. The equality $\sum_{i=1}^4 S^L(T_i)=S^L(\mathfrak{q})$ is readily checked on these expressions.

§4. *Ehrhart quasi-polynomials for intermediate valuations.* Let us fix a rational polytope $\mathfrak{p}$ and a weight $h(x)$. It is well known (as "Ehrhart's theorem") that when we dilate $\mathfrak{p}$ by an *integer* $t\in\mathbb{N}$, the function $t\mapsto S(t\mathfrak{p},\,h)$ is a quasi-polynomial function of t of degree $d+M$ where M is the degree of h; that is,

$$S(t\mathfrak{p},\,h)=\sum_{m=0}^{d+M}E_m(\mathfrak{p},\,h;\,t)t^m. \tag{14}$$

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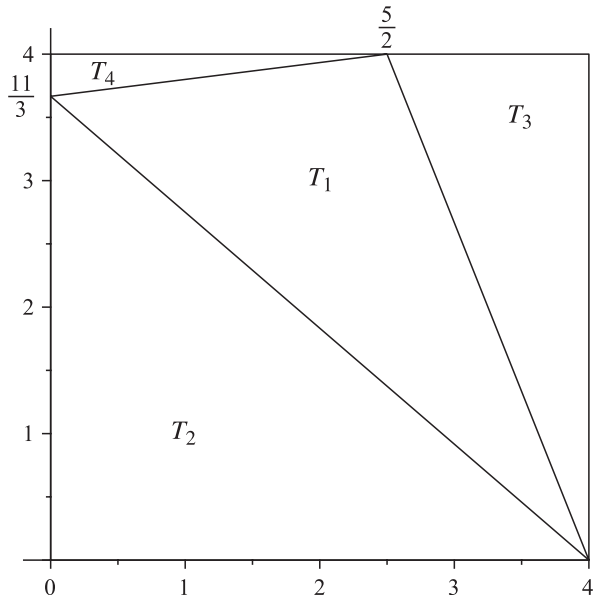


Figure 3: The square $\mathfrak{q}$ subdivided into four triangles T_i. Lattice subspaces parallel to the subspace $L = \mathbb{R}(0, 1)$ intersect the walls between the triangles at most in a point, and thus S^L vanishes on the intersections of the triangles.

Table 1: Intermediate valuations of the square and the four triangles.

$\mathfrak{p}$	Vertices $\mathcal{V}(\mathfrak{p})$	Intermediate valuation $S^L(\mathfrak{p})$
T_1	$\binom{4}{0}, \binom{\frac{5}{2}}{4}, \binom{0}{\frac{11}{3}}$	$\frac{e^{4x}}{(1-e^{-x+\frac{8}{3}y})y} - \frac{e^{4x}}{(1-e^{-x+\frac{11}{12}y})y} + \frac{e^{3x+\frac{8}{3}y}}{(1-e^{x-\frac{8}{3}y})y}$ $- \frac{e^{3x+\frac{61}{15}y}}{(1-e^{x+\frac{2}{15}y})y} - \frac{e^{\frac{11}{3}y}}{(1-e^{x-\frac{11}{12}y})y} + \frac{e^{\frac{11}{3}y}}{(1-e^{x+\frac{2}{15}y})y}$
T_2	$\binom{0}{0}, \binom{4}{0}, \binom{0}{\frac{11}{3}}$	$-\frac{1}{(1-e^{x})y} - \frac{e^{4x}}{(1-e^{-x})y} + \frac{e^{4x}}{(1-e^{-x+\frac{11}{12}y})y} + \frac{e^{\frac{11}{3}y}}{(1-e^{x-\frac{11}{12}y})y}$
T_3	$\binom{\frac{5}{2}}{4}, \binom{4}{0}, \binom{4}{4}$	$-\frac{e^{3x+\frac{8}{3}y}}{(1-e^{x-\frac{8}{3}y})y} + \frac{e^{3x+4y}}{(1-e^{x})y} - \frac{e^{4x}}{(1-e^{-x+\frac{8}{3}y})y} + \frac{e^{4x+4y}}{(1-e^{-x})y}$
T_4	$\binom{\frac{5}{2}}{4}, \binom{0}{4}, \binom{0}{\frac{11}{3}}$	$\frac{e^{2x+4y}}{(1-e^{-x})y} - \frac{e^{2x+\frac{59}{15}y}}{(1-e^{-x-\frac{2}{15}y})y} + \frac{e^{4y}}{(1-e^{x})y} - \frac{e^{\frac{11}{3}y}}{(1-e^{x+2/15y})y}$
$\mathfrak{q}$	$\binom{0}{0}, \binom{0}{4}, \binom{4}{4}, \binom{4}{0}$	$\frac{e^{4y}-1}{y}(1 + e^{x} + e^{2x} + e^{3x} + e^{4x})$

The coefficients $E_m(\mathfrak{p}, h; t)$ depend only on $\{t\}_q$ where q is an integer such that $q\mathfrak{p}$ has lattice vertices. When $\mathfrak{p}$ is a lattice polytope, then the coefficients $E_m(\mathfrak{p}, h; t)$ do not depend on t, and $t \mapsto S(t\mathfrak{p}, h)$ becomes a polynomial.

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Linke [**14**] has proved that the expression (14) is still valid for *real dilations* $t \in \mathbb{R}$. This is the generalized setting that we are going to use in this section. Note that if we allow real dilations, then even for lattice polytopes $\mathfrak{p}$ we obtain a quasi-polynomial rather than a polynomial; the fractional-part function $\{\cdot\}_q$ with $q = 1$ appears in the expressions for the coefficients $E_m(\mathfrak{p}, h; t)$. For instance, the number of integers in the interval $[0, t] = t[0, 1]$ is $t + 1 - \{t\}_1$.

We are going to extend Ehrhart's theorem to the intermediate valuation $S^L(\mathfrak{p})$. Indeed, one still has

$$S^L(t\mathfrak{p}, h) = \sum_{m=0}^{d+M} E_m^L(\mathfrak{p}, h; t)t^m \tag{15}$$

where the coefficients $E_m^L(\mathfrak{p}, h; t)$ depend only on $\{t\}_q$. Note that when $L = V$, $S^L(t\mathfrak{p}, h)$ is a polynomial, and it coincides with the integral of h on $t\mathfrak{p}$.

More precisely, we will show that the Ehrhart coefficients are *step-polynomial* functions of t (cf. [**15**]), i.e. polynomials in modular expressions $(\zeta_i t) \bmod q_i$ with integers ζ_i and q_i, for example $(2t) \bmod 3$. In the next theorem, we prove these results and, furthermore, show that the coefficients $E_m^L(\mathfrak{p}, h; t)$ are given by short formulas, provided that in the input $(L, \mathfrak{p}, h)$, the subspace L has fixed codimension, $\mathfrak{p}$ is a simple polytope given by its vertices, and the weight is a power of a linear form $h(x) = \langle \ell, x \rangle^M$.

THEOREM 28. *For every fixed number $k_0 \in \mathbb{N}$, there exists a polynomial-time algorithm for the following problem.*
Input:

- (I_1) *a number $d \in \mathbb{N}$ in unary encoding, with $d \geqslant k$;*
- (I_2) *a rational subspace $L \subseteq \mathbb{R}^d$ of codimension k_0, represented by $d - k_0$ linearly independent vectors $b_1, \ldots, b_{d-k_0} \in \mathbb{Q}^d$ in binary encoding;*
- (I_3) *a finite index set $\mathcal{V}$;*
- (I_4) *a simple rational polytope $\mathfrak{p}$, given by the set $\mathcal{V}$ of its vertices, rational vectors $s_i \in \mathbb{Q}^d$ in binary encoding;*
- (I_5) *a rational vector $\ell \in \mathbb{Q}^d$ in binary encoding;*
- (I_6) *a number $M \in \mathbb{N}$ in unary encoding.*

Output:

- (O_1) *a finite index set Γ;*
- (O_2) *polynomials $f^{\gamma,m} \in \mathbb{Q}[r_1, \ldots, r_{k_0}]$ and numbers $\zeta_i^{\gamma,m} \in \mathbb{Z}$, $q_i^{\gamma,m} \in \mathbb{N}$ for $\gamma \in \Gamma$, $i = 1, \ldots, k_0$ and $m = 1, \ldots, d + M$;*

such that for $t \in \mathbb{R}$ we have $S^L(t\mathfrak{p}, \langle \ell, x \rangle^M) = \sum_{m=0}^{d+M} E_m(t)t^m$, where $E_m(t)$ is given by the step-polynomial

$$E_m(t) = \sum_{\gamma \in \Gamma} f^{\gamma,m}(\{\zeta_1^{\gamma,m} t\}_{q_1^{\gamma,m}}, \ldots, \{\zeta_{k_0}^{\gamma,m} t\}_{q_{k_0}^{\gamma,m}}).$$

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Remark 29. Similar results hold for a more general weight $h(x)$, as in [3] for integrals over a simplex. One can assume that the weight is given as a polynomial in a fixed number D of linear forms, $h(x) = f(\langle \ell_1, x\rangle, \ldots, \langle \ell_D, x\rangle)$, or which has a fixed degree D.

Proof of Theorem 28. As was first observed in [4], the path from generating functions to Ehrhart quasi-polynomials relies on the following important property of functions $\phi(\xi) \in \mathcal{M}_\ell(V^*)$: each such function has a unique expansion in homogeneous rational functions as follows. Consider $\phi(\tau\xi)$ as a meromorphic function of one variable $\tau \in \mathbb{C}$, which we write as

$$\phi(\tau\xi) = \sum_{m \geqslant m_0} \tau^m \phi_{[m]}(\xi),$$

where m_0 is the lowest degree. We call the function $\phi_{[m]}$ the *homogeneous component* of ϕ of degree m. For instance, ξ_1/ξ_2 is homogeneous of degree 0. From this example we observe that a function in $\mathcal{M}_\ell(V^*)$ which only has terms of non-negative degree need not be analytic.

For a fixed $t \in \mathbb{R}$, the polynomial function $\xi \mapsto S^L(t\mathfrak{p}, \langle \xi, x\rangle^M/M!)$ is the term of degree M of the holomorphic function $S^L(t\mathfrak{p})(\xi)$. Let $\mathfrak{c}_s$ be the supporting cone of $\mathfrak{p}$ at the vertex s. By Brion's theorem applied to the semi-rational polytope $t\mathfrak{p}$, we write $S^L(t\mathfrak{p})(\xi)$ as the sum of the intermediate generating functions of the cones at the vertices ts, $s \in \mathcal{V}$, of the dilated polytope $t\mathfrak{p}$. The crucial point is that the cone $\mathfrak{c}_s$ does not change when the polytope is dilated:

$$S^L(t\mathfrak{p})(\xi) = \sum_{s \in \mathcal{V}} S^L(ts + \mathfrak{c}_s)(\xi). \tag{16}$$

Let q_s be the smallest integer such that $q_s s$ is a lattice point. Then $(t - \{t\}_{q_s})s$ is a lattice point, and therefore we have

$$\begin{aligned} S^L(ts + \mathfrak{c}_s)(\xi) &= \mathrm{e}^{\langle \xi, (t - \{t\}_{q_s})s\rangle} S^L(\{t\}_{q_s} s + \mathfrak{c}_s)(\xi) \\ &= \mathrm{e}^{t\langle \xi, s\rangle} \mathrm{e}^{-\langle \xi, \{t\}_{q_s} s\rangle} S^L(\{t\}_{q_s} s + \mathfrak{c}_s)(\xi). \end{aligned}$$

Expanding $\mathrm{e}^{t\langle \xi, s\rangle}$ in powers of t and looking for the coefficient of t^i, we obtain

$$S^L\left(t\mathfrak{p}, \frac{\langle \xi, x\rangle^M}{M!}\right) = \sum_{i \geqslant 0} t^i \sum_{s \in \mathcal{V}} \frac{\langle \xi, s\rangle^i}{i!} (\mathrm{e}^{-\langle \xi, \{t\}_{q_s} s\rangle} S^L(\{t\}_{q_s} s + \mathfrak{c}_s)(\xi))_{[M-i]}.$$

Thus we must analyze the term of ξ-degree $M - i$ in

$$\mathrm{e}^{-\langle \xi, \{t\}_{q_s} s\rangle} S^L(\{t\}_{q_s} s + \mathfrak{c}_s)(\xi).$$

We use formula (11) and consider the summands indexed by $\gamma \in \Gamma$, one at a time. From then on, the proof is entirely similar to [2, proof of Theorem 37], and we omit it. □

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Table 2: Intermediate Ehrhart quasi-polynomials of the square and the four triangles.

$\mathfrak{p}$	Ehrhart quasi-polynomial $S^L(t\mathfrak{p}, 1)$
T_1	$\frac{21}{4}t^2 - \frac{7}{8}(\{4t\}_1)^2 - \frac{7}{10}\{-5t\}_2 + \frac{7}{8}\{4t\}_1 + \frac{7}{20}(\{-5t\}_2)^2$
T_2	$\frac{22}{3}t^2 + \frac{11}{6}t - \frac{11}{24}(\{4t\}_1)^2 + \frac{11}{24}\{4t\}_1$
T_3	$3t^2 + (2 - 4\{4t\}_1)t - \frac{1}{3}(\{-5t\}_2)^2 + \frac{2}{3}\{-5t\}_2 + \frac{4}{3}(\{4t\}_1)^2 - \frac{4}{3}\{4t\}_1$
T_4	$\frac{5}{12}t^2 + \frac{1}{6}t - \frac{1}{60}(\{5t\}_2)^2 + \frac{1}{30}\{5t\}_2$
$\mathfrak{q}$	$16t^2 + (4 - 4\{4t\}_1)t$

Example 30 (Example 27, continued). Table 2 shows the output of our Maple program for the Ehrhart quasi-polynomials of the four triangles in Example 27 with respect to the weight $h = 1$. Note that the dilating parameter t is real. We obtain

$$\begin{aligned}\sum_{i=1}^{4} S^L(tT_i) &= 16t^2 + 4t - 4t\{4t\}_1 - \tfrac{1}{30}\{-5t\}_2 \\ &\quad + \tfrac{1}{30}\{5t\}_2 + \tfrac{1}{60}(\{-5t\}_2)^2 - \tfrac{1}{60}(\{5t\}_2)^2 \\ &= 16t^2 + 4t - 4t\{4t\}_1,\end{aligned}$$

using a relation between fractional parts, $\{x\}^2 - \{-x\}^2 = \{x\} - \{-x\}$, for the simplification. Indeed, a direct calculation gives the same answer.

Setting aside considerations of efficient computation, we can prove the following theorem.

THEOREM 31 (Real Ehrhart theorem). *Let $\mathfrak{p} \subseteq \mathbb{R}^d$ be a rational polytope, h a polynomial function of degree M, and $L \subseteq \mathbb{R}^d$ a rational subspace of codimension k_0. Then $t \mapsto S^L(t\mathfrak{p}, h)$ is a quasi-polynomial function of $t \in \mathbb{R}_{>0}$. More precisely,*

$$S^L(t\mathfrak{p}, h) = \sum_{m=0}^{d+M} E_m^L(\mathfrak{p}, h; t)t^m, \tag{17}$$

where each $E_m^L(\mathfrak{p}, h; t)$ is a step-polynomial of the form

$$E_m(t) = \sum_{\gamma \in \Gamma} f^{\gamma,m}(\{\zeta_1^{\gamma,m}t\}_{q_1^{\gamma,m}}, \ldots, \{\zeta_{k_0}^{\gamma,m}t\}_{q_{k_0}^{\gamma,m}}) \quad \text{for } t \in \mathbb{R}_{>0},$$

with finite index set Γ and polynomials $f^{\gamma,m} \in \mathbb{Q}[r_1, \ldots, r_{k_0}]$ and numbers $\zeta_i^{\gamma,m} \in \mathbb{Z}$, $q_i^{\gamma,m} \in \mathbb{N}$ for $\gamma \in \Gamma$, $i = 1, \ldots, k_0$ and $m = 1, \ldots, d + M$.

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Proof. The polytope $\mathfrak{p}$ is no longer assumed to be simple, so we compute a triangulation of each of the vertex cones and use inclusion–exclusion to avoid overcounting. Using a decomposition of h into powers of linear forms, the method of the previous theorem then gives the result. □

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