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Statistical properties of the maximal entropy measure for partially hyperbolic attractors

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Abstract. We show the existence and uniqueness of the maximal entropy probability measure for partially hyperbolic diffeomorphisms which are semiconjugate to non-uniformly expanding maps. Using the theory of projective metrics on cones, we then prove exponential decay of correlations for Hölder continuous observables and the central limit theorem for the maximal entropy probability measure. Moreover, for systems derived from a solenoid, we also prove the statistical stability for the maximal entropy probability measure. Finally, we use such techniques to obtain similar results in a context containing partially hyperbolic systems derived from Anosov.

1. *Introduction*

The thermodynamical formalism from statistical mechanics was introduced into the study of dynamical systems by the former works of Sinai, Ruelle and Bowen [**Sin72**, **Bow75**, **BR75**] for uniformly hyperbolic maps and Hölder potentials in the beginning of the 70s. Beyond the uniformly hyperbolic context, the theory is still quite incomplete. Several contributions do exist, for example, [**BK98**, **BF09**, **BKL01**, **BMD02**, **BS03**, **BT08**, **Cas02**, **Cas04**, **DL08**, **GL06**, **LM13**, **MT14**, **OV08**, **Pin08**, **Sar99**, **SV09**, **VV10**, **Yur03**].

In recent years, the thermodynamical formalism of a class of partial hyperbolic diffeomorphisms introduced by Alves *et al* [**ABV00**] and Castro [**Cas98**] has been developed under some conditions that resemble or may lead to some mostly expanding or mostly contracting assumption in the central direction.

In the non-invertible setting, this has been studied by Castro, Oliveira, Varandas and Viana [**OV08**, **VV10**, **CV13**]. Given a compact metric space M and a local homeomorphism $f : M \to M$ with Lipschitz inverse branches that admit some expanding

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and some possibly contracting domains of invertibility, it was proved in [**VV10**] that for every Hölder continuous potential ϕ satisfying a small variation condition there are finitely many ergodic equilibrium states for f with respect to ϕ. Furthermore, the equilibrium states are absolutely continuous with respect to some conformal measure and there exists a unique equilibrium state provided that the dynamical system is topologically exact. Later on, using a functional analytic approach by means of projective metrics techniques to the study of the spectral properties of Ruelle–Perron–Frobenius operators on the space of $C^{r+\alpha}$ observables ($r \in \mathbb{N}$, $\alpha > 0$), Castro and Varandas [**CV13**] presented a more general proof for the uniqueness of equilibrium states for this class of maps and deduced many statistical properties, such as exponential decay of correlations, the central limit theorem and both statistical and spectral stabilities.

In this paper, our motivation is to contribute to the study of the thermodynamical formalism of a large class of partially hyperbolic diffeomorphisms with strong stable foliations. We are interested in two different settings. The first setting consists in partially hyperbolic diffeomorphisms which are semiconjugate to the class of local diffeomorphisms in [**CV13**]. The second setting consists in a class of partially hyperbolic attractors exhibiting a Markov partition (whose iterates need not have diameters going to zero). These settings include many examples of partially hyperbolic diffeomorphisms that arise as local bifurcations of Axiom A diffeomorphisms and will be mostly expanding with respect to some conformal measure, including a robust (open) class of systems derived from Anosov, introduced by Mañé in [**Mane78**].

Sinai–Ruelle–Bowen (SRB) measures for large classes of partially hyperbolic diffeomorphisms have been constructed by [**Car93**, **ABV00**, **BV00**, **Cas98**], and the existence and uniqueness of maximal entropy measures have been proved by Buzzi *et al* [**BFSV12**] for derived-from-Anosov diffeomorphisms, by Buzzi and Fisher [**BF13**] for a wide class of deformations of Anosov diffeomorphisms that include some examples by Bonatti and Viana of robustly transitive non-partially hyperbolic diffeomorphisms, and by Ures [**Ur12**] for partially hyperbolic diffeomorphisms of $\mathbb{T}^3$ homotopic to a hyperbolic automorphism. In most of these cases, the approach is to establish a semiconjugacy between the dynamical system and some uniformly hyperbolic one and prove that the points that remain in a non-hyperbolic region do not contribute much to the topological entropy. More recently, Climenhaga *et al* [**CFT15**] proved uniqueness for equilibrium states for some robust classes of examples of [**Mane78**, **BV00**]. The drawback is that these methods are not enough to deduce some good statistical properties for the original dynamical system, especially the exponential decay of correlations. To illustrate this fact, let us mention that in the case of non-uniformly expanding maps, the Ruelle–Perron–Frobenius transfer operator acts in the space of Hölder continuous functions and the dominant eigenvector of its adjoint operator leads to the measure of maximal entropy, while in the invertible context, any invariant measure is an eigenvector for the adjoint operator. For that reason, the method of invariant cones used in [**CV13**] could not be applied here. In fact, the results and their proofs in this paper are independent from those in the above-mentioned paper, except that we use the existence and uniqueness of the entropy maximizing measure to guarantee the uniqueness in this new context.

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So, to deduce exponential decay of correlations for the original dynamical systems, we introduce a suitable Banach space and prove that the transfer operator does preserve some cone of functions. Such a cone of functions is constructed by constructing a family of probability measures on stable leaves that is equidistributed and holonomy invariant. Very laborious work is done in order to prove the invariance of such a suitable cone of functions by the transfer operator and that the image of this by the transfer operator has finite diameter in the projective metrics, which implies that the transfer operator is a contraction with respect to the projective metrics. From that and the duality properties of transfer and Koopman operators, we derive the exponential decay of correlations and the central limit theorem as consequences.

It is worth mentioning other recent work (for example, [**BL12**, **Mel14**, **LT15**]) concerning fast mixing of SRB and Gibbs measures in some non-uniformly hyperbolic settings. The techniques used in such papers are either compactness arguments provided by Lasota–Yorke estimates, or Young towers [**You98**] and operator renewal theory. Even though such techniques have the advantage of reaching a kind of spectral gap for transfer operator rather directly, they need asymptotic assumptions, and stronger transitivity assumptions than the cone approach. For instance, the non-transitive situation that we obtain by bifurcating the Manneville–Pomeau map so that we create a sink cannot be properly worked out by a Lasota–Yorke approach. However, the method of invariant cones for transfer operators, used, for example, in [**CV13**], easily contemplates such an example without any additional hypotheses. The approach here also gathers the same advantage of a kind of mild transitivity assumption, such as in [**CV13**]. So, our paper deals with different and robust classes of examples that are not under the hypotheses of the previously cited works.

This paper is organized as follows. In the next section, we give precise definitions of the family of partially hyperbolic diffeomorphisms that we consider and state the main results. A robust class of examples is also discussed. In §§3 and 4, we establish the existence and uniqueness of equilibrium states, and, restricting to the skew products and deriving from the solenoid case, in §5 we also prove statistical stability of the equilibrium states, meaning that the measure varies continuously in the weak-* topology with the dynamics and the potential. In the remaining sections up to §10, we prove that the maximal entropy measure satisfies good statistical properties, namely, exponential decay of correlations and the central limit theorem in the space of Hölder continuous observables. In the last section, §11, we apply the methods that we developed for the case of partially hyperbolic attractors with Markov partition, including some robust classes of attractors derived from Anosov, introduced by Mañé [**Mane78**].

2. *Context and statement of the main results*

In this paper, we will work with two contexts of partially hyperbolic diffeomorphisms with strong stable direction. We deal with partially hyperbolic systems that are semiconjugate to non-uniformly expanding endormorphisms (see [**CV13**]) and with diffeomorphisms that include systems derived from Anosov. Although both classes of dynamical systems present a partially hyperbolic behavior, the study of their thermodynamical properties requires different approaches due to crucial geometrical differences.

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First setting. Let N be a connected compact Riemannian manifold, and let $g : N \to N$ be a *local homeomorphism* with Lipschitz inverse branches. By that, we mean there exists $L(x) \geq 0$ such that, for all $x \in N$, there is a neighborhood $U_x \ni x$ such that $g_x := g|_{U_x} : U_x \to g(U_x)$ is invertible and

$$d(g_x^{-1}(y), g_x^{-1}(z)) \leq L(x)d(y, z) \quad \text{for all } y, z \in g(U_x). \tag{2.1}$$

Let us denote by $\deg(g)$ the degree of g, which coincides with the number of pre-images of any $x \in N$ by g. We also assume that there exist $0 < \lambda_u < 1$ and an open region $\Omega \subset N$ such that:

(H1) $L(x) \leq L$ for $x \in \Omega$ and $L(x) < \lambda_u$ for $x \notin \Omega$, for some L close to 1; and

(H2) there exists a covering $\mathcal{U}$ of N by injective domain of g, such that Ω can be covered by $q < \deg(g)$ elements of $\mathcal{U}$.

Let M be a compact invariant manifold and $f : M \to M$ a diffeomorphism onto its image. Suppose there exists a continuous and surjective $\Pi : M \to N$ such that

$$\Pi \circ f = g \circ \Pi. \tag{2.2}$$

Given $y \in N$, set $M_y = \Pi^{-1}(y)$. Therefore, $M = \bigcup_{y \in N} M_y$. Note that $f(M_y) \subset M_{g(y)}$, and also suppose that there exists $0 < \lambda_s < 1$ such that

$$d(f(z), f(w)) \leq \lambda_s d(z, w) \tag{2.3}$$

for all $z, w \in M_y$.

As the maximizing entropy measure is f-invariant, by Poincaré's recurrence theorem such a measure is supported in the attractor

$$\Lambda := \bigcap_{n=0}^{\infty} f^n(M).$$

Note that Λ is compact and invariant by f. So, it is sufficient to study the dynamics of f restricted to Λ.

Given $x, y \in M$, write $\hat{x} := \Pi(x)$, $\hat{y} := \Pi(y)$. We assume that there exist holonomies $\pi_{\hat{x},\hat{y}} : M_{\hat{x}} \cap \Lambda \to M_{\hat{y}} \cap \Lambda$ satisfying

$$\frac{1}{C}[d_N(\hat{x}, \hat{y}) + d_M(\pi_{\hat{x},\hat{y}}(x), y)] \leq d_M(x, y) \leq C[d_N(\hat{x}, \hat{y}) + d_M(\pi_{\hat{x},\hat{y}}(x), y)] \tag{2.4}$$

for some constant $C > 0$, where d_M, d_N are the metrics of M, N, respectively. For simplicity, we shall write d for any such metrics.

We suppose such holonomies are invariant by f, that is,

$$f(\pi_{\hat{x},\hat{y}}(z)) = \pi_{g(\hat{x}),g(\hat{y})}(f(z)) \tag{2.5}$$

for all $z \in M_{\hat{x}} \cap \Lambda$.

Second setting. Let M be a compact Riemannian manifold and $f : M \to M$ be a C^{1+} diffeomorphism. Assume that there exists a compact subset Λ of M with the following properties.

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(1) There exists an open f-invariant neighborhood Q of Λ such that $f(\overline{Q}) \subset Q$ and

$$\Lambda = \bigcap_{n=0}^{\infty} f^n(Q).$$

(2) The subset Λ is partially hyperbolic, in the sense that there exists a Df-invariant dominated splitting

$$T_\Lambda M = E^{\mathrm{ss}} \oplus E^{uc}, \quad \dim(E^{\mathrm{ss}}) > 0$$

of the tangent bundle restricted to Λ such that, for fixed Riemannian metrics in M, we have:

(a) E^{ss} contracts uniformly: $\|Df^n|E^{\mathrm{ss}}_x\| \le C\lambda_s^n$; and

(b) E^{uc} is dominated by E^{ss}: $\|Df^n|E^{\mathrm{ss}}_x\|\|Df^{-n}|E^{uc}_{f^n(x)}\| \le C\lambda_s^n$ for all $n \ge 1$ and $x \in \Lambda$, with $0 < \lambda_s < 1$.

(3) There exists an f-invariant center-unstable foliation $\mathcal{F}^{uc}_{\mathrm{loc}}$ of a neighborhood Λ, which is tangent to the center-unstable subbundle E^{uc} in Λ. There is also an f-invariant stable foliation $\mathcal{F}^{s}_{\mathrm{loc}}$ tangent to the stable subbundle E^{ss} in Λ.

In order to proceed with our considerations on the dynamics f, we recall the concept of Markov partition in this partially hyperbolic context.

Definition 2.1. We call $R \subset \Lambda$ a Markov *proper rectangle* if for all x and y in R there exists a unique point $z := [x, y] \in R$ which is the intersection of the local (strong) stable manifold passing by x and the local center-unstable manifold passing by y. Moreover, R is the closure of its interior (in the relative topology of Λ) and, in particular, is closed.

We observe that the boundaries of Markov proper rectangles are unions of local (strong) stable manifolds and local center-unstable manifolds.

Definition 2.2. A collection $\mathcal{R} = \{R_1, \ldots, R_p\}$ of proper rectangles is a *Markov partition* for f restricted to Λ if:

(a) $\Lambda = \bigcup_{i=1}^{p} R_i$;

(b) $\mathrm{int}(R_i) \cap \mathrm{int}(R_j) = \emptyset$ for $i \ne j$; and

(c) if γ is the intersection of a local (strong) stable manifold with R_i and $f(\gamma) \cap R_j \ne \emptyset$ then $f(\gamma) \subset R_j$. Analogously, if Γ is the intersection of a local center-unstable manifold with R_i and $f^{-1}(\Gamma) \cap R_k \ne \emptyset$ then $f^{-1}(\Gamma) \subset R_k$.

(4) The diffeomorphism f restricted to Λ admits a Markov partition $\mathcal{R} = \{R_1, \ldots, R_p\}$, $p \ge 2$ with the (mild) mixing property: given $i, j \in \{1, \ldots, p\}$, there exists $n_0 \ge 1$ such that

$$f^n(R_i) \cap R_j \ne \emptyset \quad \text{for all } n \ge n_0.$$

We distinguish two kinds of rectangles in $\mathcal{R}$ according to its behavior in the direction E^{uc}. Fixing $0 < \zeta < 1$, we call $R_i \in \mathcal{R}$ a *good rectangle* if

$$\|Df|_{E^{uc}_x}\|^{-1} \le \zeta$$

for all $x \in R_i$. That is, E^{uc} expands uniformly in R_i, for one iterate. The other rectangles will be called *bad rectangles*.

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(5) There exists at least one good rectangle, and for all x in a bad rectangle,

$$\|Df|_{E^{uc}_x}\|^{-1} \leq L$$

for some $L \geq 1$ close to 1 (depending on ζ and the combinatorics of the partition).

2.1. *Statement of the main results.* We recall the definition of topological entropy due to Bowen, using (n, ϵ)-separable sets. A compact set K contained in a metric space (X, d) is (n, ϵ)-separable if

$$\text{for all } x, y \in K, x \neq y \quad \max\{d(f^j(x)), d(f^j(y)); j = 0, \ldots, n-1\} > \epsilon.$$

We denote by $S(n, \epsilon, K)$ the greatest cardinality of an (n, ϵ)-separate subset of K. The *relative entropy* of f with respect to a (not necessarily invariant) compact $K \subset X$ is given by

$$h(f, K) := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log S(n, \epsilon, K).$$

For a uniformly continuous map $f : X \to X$, (X not necessarily compact), the *topological entropy* is defined by

$$h(f) := \sup\{h(f, K); K \text{ compact}\}.$$

In our context $X = \Lambda$ is a compact set, and f is automatically uniformly continuous. We also have by [**W82**] that $h(f) = h(f, X)$ does not depend on the metrics.

For an invariant measure μ, we also recall the definition by Kolmogorov of its metric entropy $h_\mu(f)$. Given a probability space $(X, \mathcal{B}, \mu)$, if μ is f-invariant, we define the entropy of a finite partition $\mathcal{P}$ of X by

$$h_\mu(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P).$$

Then the entropy of a partition with respect to f is

$$h_\mu(f, \mathcal{P}) := \lim_{n \to \infty} \frac{1}{n} h_\mu(\mathcal{P} \vee f^{-1}(\mathcal{P}) \vee \cdots \vee f^{n-1}(\mathcal{P})),$$

and the metric entropy of f with respect to μ is given by

$$h_\mu(f) := \sup_{\mathcal{P}}\{h_\mu(f, \mathcal{P})\}.$$

Denote by $\mathcal{M}^1_f(X)$ the set of all f-invariant probabilities. The *variational principle* establishes that, for a continuous map f on a compact metric space X, the equation

$$h(f) = \sup\{h_\mu(f); \mu \in \mathcal{M}^1_f(X)\}$$

holds. We say that an invariant probability μ is a *maximal entropy measure* for f if $h(f) = h_\mu(f)$. We now state the main results in this work.

THEOREM A. (Existence and uniqueness of maximal entropy measure) *Let $f : \Lambda \to \Lambda$ be a diffeomorphism in the first setting, as described in §2 (that is, the conditions given by equations (2.2)–(2.5)). Then there exists a unique maximal entropy measure μ for f.*

As a by-product of the proof we also obtain the following corollary.

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COROLLARY 2.3. (Statistical stability in the derived-from-solenoid case) *Let f_n be a sequence of derived-from-solenoid diffeomorphisms (see Example 2.7) and call μ_n the maximal entropy probability measure for f_n. If $f_n \to f$ in the C^1-topology then μ_n converges to the maximal entropy probability measure for f in the weak-$*$ topology.*

Using the theory of projective metrics over invariant cones, we prove the following theorem.

THEOREM B. (Exponential decay of correlations) *The maximal measure entropy μ for $f : \Lambda \to \Lambda$ has exponential decay of correlations for Hölder continuous observables; that is, there exists some $0 < \tau < 1$ such that for α-Hölder continuous φ, ψ, there exists $K(\varphi, \psi) > 0$ satisfying*

$$\left| \int (\varphi \circ f^n)\psi \, d\nu - \int \varphi \, d\nu \int \psi \, d\nu \right| \leq K(\varphi, \psi) \cdot \tau^n \quad \text{for all } n \geq 1.$$

For the maximal entropy measure μ, the following theorem also holds.

THEOREM C. (Central limit theorem) *Let μ be the maximal entropy measure for $f : \Lambda \to \Lambda$, as in (2.2), and let φ be a Hölder continuous function. If*

$$\sigma_\varphi^2 := \int \phi^2 \, d\mu + 2 \sum_{j=1}^{\infty} \int \phi \cdot (\phi \circ f^j) \, d\mu \quad \text{with } \phi = \varphi - \int \varphi \, d\mu$$

then $\sigma_\varphi < \infty$ and $\sigma_\varphi = 0$ if and only if $\varphi = u \circ f - u$ for some $u \in L^2(\mu)$. Moreover, if $\sigma_\varphi > 0$ then for all intervals $A \subset \mathbb{R}$,

$$\lim_{n\to\infty} \mu\left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi(f^j(x)) - \int \varphi \, d\mu \right) \in A \right) = \frac{1}{\sigma_\varphi \sqrt{2\pi}} \int_A e^{-t^2/(2\sigma_\varphi^2)} \, dt$$

holds.

In what follows, we shall describe the results for the class of partially hyperbolic diffeomorphisms considered in the second setting. In this other context, we construct a dominant eigenmeasure μ for the dual of the transfer operator acting in a suitable space of distributions. We then prove the following theorem.

THEOREM D. *The measure μ exhibits exponential decay of correlations in the space of Hölder continuous observables. Furthermore, the central limit theorem holds for the measure μ.*

We note that, even in the second setting, the Markov partition permits us to construct a quotient map from the original one (see §11.2 for the precise definition). In the cases in which the maximizing entropy measure exists and is unique for the quotient system, one can repeat the arguments in Theorem A to conclude that the measure μ is the (unique) maximal entropy measure for the system (f, Λ).

In particular, by using [**LSV98**], we obtain the following corollary.

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COROLLARY 2.4. *Let f be a system satisfying (1)–(5) of the second setting. Suppose also that the center-unstable spaces of f are one-dimensional. Then f has a unique maximizing entropy measure, which has exponential decay of correlations and satisfies the central limit theorem for Hölder continuous observables.*

2.2. *Some examples.* Let us start with examples of the first setting.

Example 2.5. The most simple family of examples is a skew product obtained from a map $g : N \to N$ as in [**CV13**] (this means that g can be taken from a robust class of non-uniformly expanding maps that, in particular, includes all expanding maps) and an endomorphism $\Phi : N \times K \to K$, by the formula

$$\begin{aligned} f : N \times K &\to N \times K \\ (x, y) &\mapsto (g(x), \Phi(x, y)) \end{aligned}$$

such that f is a diffeomorphism onto its image, and for each $x \in N$, $\Phi(x, \cdot) : K \to K$ is a λ_s-contraction. In such a case, Π is the canonical projection in the first coordinate, and $N \times K = \bigcup_{x \in N} K_x$, where $K_x = \{x\} \times K$ for all $x \in N$.

Example 2.6. As a subexample, we may take the solenoid generated in the solid torus $S^1 \times D$. We define f by

$$\begin{aligned} f : S^1 \times D &\to S^1 \times D \\ (\theta, z) &\mapsto (g(\theta), \varphi(\theta) + A(z)), \end{aligned}$$

where g is the Manneville–Pomeau map given by

$$g(\theta) = \begin{cases} \theta(1 + 2^\alpha \theta^\alpha) & \text{if } 0 \le \theta \le \frac{1}{2}, \\ (\theta - 1)(1 + 2^\alpha (1 - \theta)^\alpha) + 1 & \text{if } \frac{1}{2} < \theta \le 1, \end{cases}$$

where $\alpha \in (0, 1)$, φ is a local diffeomorphism and A is a contraction.

Example 2.7. One can modify the examples above in order to obtain robust (containing an open set) classes of examples. These are examples derived from solenoid-like systems. For the sake of simplicity, we will give a construction in dimension four, which can be easily adapted to similar higher-dimensional examples.

Let us begin with a solenoid-like C^2-skew-product hyperbolic diffeomorphism $f_0 : T^2 \times D \to T^2 \times D$ similar to Examples 2.6 and 2.7 above. We suppose that

$$\begin{aligned} f_0 : T^2 \times D &\to T^2 \times D \\ (x, y) &\mapsto (g_0(x), \Phi_0(x, y)) \end{aligned}$$

is such that g_0 is an expanding map.

We suppose that the norm of Df_0 along the stable subbundle and the norm of Df_0^{-1} along the unstable subbundle are bounded by a constant $\lambda_0 < 1/3$. Let p be a fixed point of f_0 and let $\delta > 0$ be a small constant. Define $V_0 = B(p, \delta/2)$. Then, in the same manner as in [**Cas02**], we deform f_0^{-1} inside V_0 by a isotopy, obtaining a continuous family of maps f_t, $0 < t < 2$ in such a way that the following statements are true.

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(i) The continuation p_{f_t} of the fixed point p goes through some generic bifurcation such as a flip bifurcation or a Hopf bifurcation. Points of different indexes appear in a transitive attractor for values of t between 1 and 2 (staying all the time inside V_0). For $t = 1$, we have the first moment of the Hopf (or flip) bifurcation, with f_1 conjugated to f_0. We suppose that the derivative $Df_1|_{E^{cu}}$ does not contract vectors. In the case of Hopf bifurcation, we suppose that $Df_t|_{E^{cu}}(p_{f_t})$ exhibits complex eigenvalues, for all t.

(ii) In the process, there always exist a strong stable cone field C^{ss} (cf. [**Vi97**] for definitions) and a center-unstable cone field C^{cu}, defined everywhere, such that C^{cu} contains the unstable direction of the initial map f_0. We also suppose that there exists a continuation of the torus $T^2 \times \{0\}$ that is f_0-invariant and normally hyperbolic. So, for each $t \in [1, 2]$, there exists an f_t-invariant manifold T_t that is the normally hyperbolic continuation of $T^2 \times \{0\}$.

(iii) The width of the cone fields C^{ss} and C^{cu} are bounded by a small constant $\alpha > 0$.

(iv) There exist a constant $\sigma > 1$ and a neighborhood $V_1 \subset V_0 \cap W^s(p)$ such that $J^c = \|\det Df_t^{-1}|_{E^{cu}}\| > \sigma$ outside V_1.

(v) The maps f_t^{-1} are δ-C^0 close to f_0^{-1} outside V_0 so that $\|(Df_1^{-1}|_{E^{cu}})\| < \lambda_0 < 1/3$ outside V_0.

Note that the properties stated in conditions (i)–(v), which are valid for f_t, $0 \le t \le 2$, are also valid for a whole C^1-neighborhood $\mathcal{U}$ of the set of diffeomorphisms $\{f_t, 0 \le t \le 2\}$. In particular, by [**HPS77**], conditions (i)–(iii) imply that any $f \in \mathcal{U}$ has an invariant central foliation, since the central cone field enables us to define a graph transform associated with it with domain in the space of foliations tangent to C^{cu}, which is not empty, since the unstable foliation of f_0 is tangent to it. On the other hand, all $f \in \mathcal{U}$ also exhibit a strong stable foliation varying continuously with the diffeomorphism.

As a consequence of [**BV00**, Lemma 6.1], there is a C^1-neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of the set $\{f_t, 1 < t \le 2\}$ such that for all $f \in \mathcal{U}_1$, $\Lambda = T^n$ is a partially hyperbolic attractor, which is not hyperbolic, because it is transitive and contains points with different indexes.

One can embed $T^2 \times D$ as a subset of T^4. So, it is easy to extend f_t above to T^4 in a manner that each f_t is hyperbolic (and structurally stable) outside $T^2 \times D$. So, we will assume each f_t is defined in T^4 in such way.

Now, take f in some small ball $B = B(f_1, \delta')$, $\delta' < \delta/2$. Suppose also that δ' is sufficiently small that all diffeomorphisms in $B(f_1, \delta') \subset \mathcal{U}$ are partially hyperbolic. So, if $\delta' > 0$ is small, $B(f_1, \delta')$ is an open set of diffeomorphisms of T^4 satisfying the conditions in §2.

COROLLARY 2.8. *There exists an open set of non-hyperbolic diffeomorphisms $f : T^4 \to T^4$ satisfying conditions expressed by equations (2.2)–(2.5).*

Proof. Just take the open set of diffeomorphisms $\mathcal{U}_2 = \mathcal{U}_1 \cap B(f_1, \delta')$, with δ' as given above. The conditions in equations (2.2)–(2.5) fit for every diffeomorphism in a ball $B(f_1, \delta')$. □

Example 2.9. (Derived from Anosov) Mañé [**Mane78**] has introduced a robust class of partially hyperbolic attractors by a pitchfork or Hopf bifurcation of some periodic orbit.

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In [**Cas02**], the author proved that a robust class of maps satisfying conditions (1)–(5) can be obtained. In fact, let $g : M \to M$ as in [**Cas02**, Example 2] of a system derived from Anosov. Taking $f = g^{-1}$, the strong unstable (E^{uu}) and center-stable (E^{cs}) subbundles of g become, respectively, the strong stable subbundle (E^{ss}) and center-unstable subbundle (E^{uc}) for f, and such an f satisfies (1)–(5).

3. *Construction of the maximal entropy measure*

Due to the contraction in the stable foliation, the dynamics of distinct orbits of $f : M \to M$ will be determined by the dynamical behavior of the map $g : N \to N$. As seen in [**CV13**], such a map g has only a unique maximal entropy measure, which we will denote by ν.

We start the construction of the maximal entropy measure for f by defining it on measurable sets of the form $\Pi^{-1}(A)$, where A is a Borelian set of N.

Since Π is a semiconjugation, by [**W82**] one obtains that

$$h(f) \geq h(g).$$

Moreover, due to Bowen [**Bow71**], it follows that

$$h(f) \leq h(g) + \sup\{h(f, \Pi^{-1}(y));\ y \in N\}.$$

We now prove that $h(f, \Pi^{-1}(y)) = 0$ for all $y \in N$. Indeed, since $f : M_y \to M_{g(y)}$ is a λ_s-contraction, given $\epsilon > 0$, the only (n, ϵ)-separate subsets restricted to M_y are unitary subsets. As $\Pi^{-1}(y)$ can be written as a union of $m(\epsilon) \in \mathbb{N}$ balls of ϵ-diameter, we conclude that the cardinality of any (n, ϵ)-separate subset of $\Pi^{-1}(y)$ is at most $m(\epsilon)$. By the definition of entropy due to Bowen, this implies $h(f, \Pi^{-1}(y)) = 0$ for all $y \in N$. Therefore, $h(f) \leq h(g)$, and so $h(f) = h(g)$.

This allows us to construct the maximal entropy measure for f from the one for g. Denote by ν the unique maximal entropy measure built in [**CV13**]. By the variational principle and the fact that $h(f) = h(g)$, it follows that $h_\nu(g)$ is greater than or equal to the metric entropy of any f-invariant probability. So, for the proof of existence part of the statement, it is sufficient to obtain an f-invariant probability μ whose metric entropy with respect to f is greater or equal than $h_\nu(g) = h(g)$.

For this purpose, let $\Pi_\Lambda = \Pi|_\Lambda$. Let $\mathcal{A}_N$ be the Borel σ-algebra on N. Clearly, $\mathcal{A}_0 := \Pi_\Lambda^{-1}(\mathcal{A}_N)$ is a σ-algebra on Λ. Since f is a bijection in Λ and $\Pi_\Lambda \circ f = g \circ \Pi_\Lambda$, we have

$$A = \Pi_\Lambda^{-1}(B) = f \circ \Pi_\Lambda^{-1} \circ g^{-1}(B).$$

As $g^{-1}(B)$ belongs to $\mathcal{A}_N$, it follows that $\mathcal{A}_0 \subset f(\mathcal{A}_0)$, and therefore $\mathcal{A}_n := f^n(\mathcal{A}_0)$ is a sequence of σ-algebras such that $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \cdots$. Define $\mu_n : \mathcal{A}_n \to [0, 1]$ by $\mu_n(f^n(A_0)) = \nu(\Pi_\Lambda(A_0))$, for all $A_0 \in \mathcal{A}_0$. Note that μ_n is an f-invariant probability for all $n \in \mathbb{N}$. In fact, given $A = f^n(A_0)$, where $A_0 = \Pi_\Lambda^{-1}(B)$ and $B \in \mathcal{A}_N$, due to the g-invariance of ν and the surjection of maps g and Π_Λ, we have

$$\begin{aligned}
\mu_n(f^{-1}(A)) &= \mu_n(f^{-1}(f^n(A_0))) = \mu_n(f^n(f^{-1}(A_0))) \\
&= \nu(\Pi_\Lambda(f^{-1}(A_0))) = \nu(\Pi_\Lambda(f^{-1} \circ \Pi_\Lambda^{-1}(B))) \\
&= \nu(\Pi_\Lambda(\Pi_\Lambda^{-1} \circ g^{-1}(B))) = \nu(g^{-1}(B)) \\
&= \nu(B) = \nu(\Pi_\Lambda(A_0)) = \mu_n(f^n(A_0)) = \mu_n(A).
\end{aligned}$$

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Now, as $\mathcal{A}_n \subset \mathcal{A}_{n+1}$, $\mathcal{A} := \bigcup_{n=0}^{\infty} \mathcal{A}_n$ is an algebra in Λ.

Then we define the probability $\mu : \mathcal{A} \to [0, 1]$ such that $\mu(A) = \mu_n(A)$ if $A \in \mathcal{A}_n$. By the standard measure theory arguments (see [**Mane**]), μ is σ-additive. Moreover, μ is an f-invariant probability, as μ_n are f-invariant probabilities. It remains to prove that the smallest σ-algebra that contains $\mathcal{A}$ is the Borel σ-algebra.

For that, it is sufficient to see that $\mathcal{A}$ contains a sequence of partitions whose diameters go to zero.

This is because $f : M_y \to M_{g(y)}$ is a λ_s-contraction.

In fact, for each $n \in \mathbb{N}$, by the continuity of g^n, there exists $\delta(n) > 0$ such that $d(z, w) < \delta(n)$ implies $d(g^n(z), g^n(w)) < \lambda_s^n$, for all $z, w \in N$. Taking $\mathcal{P}_n^0$, a partition of N whose diameter is less than $\delta(n)$, we define a sequence of partitions of Λ by

$$\mathcal{P}_n := f^n(\Pi_\Lambda^{-1}(\mathcal{P}_n^0)). \tag{3.1}$$

Clearly, $\operatorname{diam}(\mathcal{P}_n) \to 0$ as $n \to +\infty$. Indeed, given $\bar{x}$, $\bar{y}$ in the same element of $\mathcal{P}_n$, writing $\bar{x} = f^n(x)$ and $\bar{y} = f^n(y)$ we have $\hat{x} = \Pi(x)$, $\hat{y} = \Pi(y) \in P_n^0$. Therefore, noting that $g^n(\hat{x}) = g^n(\Pi(x)) = \Pi(f^n(x)) = \tilde{x}$ and $g^n(\hat{y}) = g^n(\Pi(y)) = \Pi(f^n(y)) = \tilde{y}$, we obtain

$$\begin{aligned} d(f^n(x), f^n(y)) &\le C[d(\tilde{x}, \tilde{y}) + d(\pi_{\tilde{x},\tilde{y}} \circ f^n(x), f^n(y))] \\ &= C[d(g^n \circ \Pi(x), g^n \circ \Pi(y)) + d(f^n(\pi_{\hat{x},\hat{y}}(x)), f^n(y))] \\ &\le C[\lambda_s^n + \lambda_s^n d(\pi_{\hat{x},\hat{y}}(x), y)] \le C[1 + \operatorname{diam}(M)]\lambda_s^n. \end{aligned}$$

By a slight abuse of notation, we also write μ for its natural extension to the Borel σ-algebra of M.

Now we prove that μ is a maximizing entropy measure for f, by proving that $h_\mu(f) \ge h_\nu(g)$. Denote by $B_\epsilon^n(g, y_0)$ an (n, ϵ)-dynamical ball of g around $y_0 \in N$, that is, the set of points $y \in N$ such that $d(g^j(y), g^j(y_0)) < \epsilon$, for all $j \in \{0, \dots, n-1\}$. By the Brin–Katok theorem, for ν-almost every $y \in N$,

$$h_\nu(g) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\nu(B_\epsilon^n(g, y))}$$

holds.

Now take $B_\epsilon^n(f, x)$, the (n, ϵ)-dynamical ball of f restricted to Λ at $x \in \Lambda$. By the uniform continuity of Π, given $\epsilon > 0$, there exists $0 < \delta < \epsilon$ such that $\Pi(B_\delta(w)) \subset B_\epsilon(\Pi(w))$ for all $w \in M$. Note that $B_\delta^n(f, x) \subset \Pi_\Lambda^{-1}(B_\epsilon^n(g, y))$ for all $x \in \Pi_\Lambda^{-1}(y)$.

In fact, given $z \in B_\delta^n(f, x)$, we shall prove that $\Pi(z) \in B_\epsilon^n(g, y)$. As $\Pi(x) = y$, we have for all $j \in \{0, \dots, n-1\}$,

$$d(g^j \circ \Pi(z), g^j(y)) = d(g^j \circ \Pi(z), g^j \circ \Pi(x)) = d(\Pi \circ f^j(z), \Pi \circ f^j(x)) < \epsilon.$$

Therefore,

$$\mu(B_\delta^n(f, x)) \le \mu(\Pi_\Lambda^{-1}(B_\epsilon^n(g, y))) = \nu(B_\epsilon^n(g, y))$$

and since $\delta \to 0$ as $\epsilon \to 0$, we obtain

$$h_\nu(g) \le \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\mu(B_\delta^n(f, x))}$$

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for μ-almost every $x \in \Lambda$. So,

$$h_\mu(f) = \int_\Lambda \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{\mu(B^n_\delta(f,x))}\, d\mu \geq \int_\Lambda h_\nu(g)\, d\mu = h_\nu(g)$$

and we conclude that $h_\mu(f) \geq h_\nu(g) = h(g) = h(f)$, which is equivalent to saying that μ is a maximal entropy measure for f.

4. *Uniqueness of maximal entropy measure*

Now we prove the uniqueness of the maximal entropy measure for f built in the last section. For this purpose, we use the uniqueness of the maximal entropy measure for g, provided by [**CV13**]. Suppose that μ_1 is another invariant maximal entropy measure for f, different to μ. Define $\nu_1 := (\Pi_\Lambda)_*\mu_1$, the push-forward of μ_1.

We claim that since μ_1 is different to μ, it follows that ν_1 is different to ν. Indeed, since $\mu_1 \neq \mu$, $\mu_1(A) \neq \mu(A)$ for some $A \in \mathcal{A} = \mathcal{A}_0 \cup f(\mathcal{A}_0) \cup \cdots \cup f^n(\mathcal{A}_0) \cup \cdots$. The fact that such algebras on $\mathcal{P}(\Lambda)$ are nested implies that there exist $A_0 \in \mathcal{A}_0$ and $n \in \mathbb{N}$ such that $f^n(A_0) = A$. By the definition of $\mathcal{A}_0$, there exists $B_0 \in \mathcal{A}_N$ such that $\Pi^{-1}_\Lambda(B_0) = A_0$. We now observe that, on the one hand,

$$\nu_1(B_0) = (\Pi_\Lambda)_*\mu_1(B_0) = \mu_1(\Pi^{-1}_\Lambda(B_0)) = \mu_1(A_0) = \mu_1(f^n(A_0)) = \mu_1(A),$$

and, on the other hand,

$$\nu(B_0) = \nu(\Pi_\Lambda(A_0)) = \mu(A_0) = \mu(f^n(A_0)) = \mu(A).$$

So, $\nu_1 \neq \nu$. By the f-invariance of μ_1, it follows that ν_1 is g-invariant.

Let us prove that ν_1 is a maximal entropy measure for g, which is a contradiction, since by [**CV13**], such a probability is unique. It is sufficient to prove that $h_{\nu_1}(g) \geq h_{\mu_1}(f)$, since $h_{\mu_1}(f) = h(f) = h(g)$.

In fact, we may suppose that the sequence $\mathcal{P}_n = f^n(\Pi^{-1}_\Lambda(\mathcal{P}^0_n))$, in (3.1), is such that $\mathcal{P}_0 \leq \mathcal{P}_1 \leq \cdots \leq \mathcal{P}_n \leq \cdots$, and as $\bigcup_{n=0}^\infty \mathcal{P}_n$ generates the Borel σ-algebra of Λ, we obtain

$$h_{\mu_1}(f) = \sup_n\{h_{\mu_1}(f,\, P_n)\}.$$

Therefore, for all $\epsilon > 0$, there exists $n \in N$ such that

$$h_{\mu_1}(f,\, P_n) \geq h_{\mu_1}(f) - \epsilon.$$

However, if follows from the definition of ν_1 that for all $n \in \mathbb{N}$,

$$h_{\nu_1}(g,\, P^0_n) = h_{\mu_1}(f,\, \Pi^{-1}_\Lambda(P^0_n)).$$

Indeed, for a partition $\mathcal{P}$ we have

$$h_{\nu_1}(g,\, \mathcal{P}) = \lim_{m\to\infty} \frac{1}{m} h_{\nu_1}(\mathcal{P} \vee g^{-1}(\mathcal{P}) \vee \cdots \vee g^{-(m-1)}(\mathcal{P})).$$

From the definition of ν_1 and the semiconjugation between f and g, we obtain

$$\begin{aligned}\nu_1\Big(\bigvee_{j=0}^{m-1} g^{-j}(P_{i_j})\Big) &= \mu_1\Big(\Pi^{-1}_\Lambda\Big(\bigvee_{j=0}^{m-1} g^{-j}(P_{i_j})\Big)\Big)\\ &= \mu_1\Big(\bigvee_{j=0}^{m-1} f^{-j}(\Pi^{-1}_\Lambda(P_{i_j}))\Big),\end{aligned}$$

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which guarantees $h_{\nu_1}(\bigvee_{j=0}^{m-1} g^{-j}(\mathcal{P})) = h_{\mu_1}(\bigvee_{j=0}^{m-1} f^{-j}(\Pi_\Lambda^{-1}(\mathcal{P})))$, and so we have

$$h_{\nu_1}(g, \mathcal{P}) = h_{\mu_1}(f, \Pi_\Lambda^{-1}(\mathcal{P})).$$

From the f-invariance of μ_1, it follows that

$$h_{\mu_1}(f, \Pi_\Lambda^{-1}(\mathcal{P}_n^0)) = h_{\mu_1}(f, \mathcal{P}_n),$$

because $P_{n_j} \in \mathcal{P}_n$ if and only if there exist $P_{n_j}^0 \in \mathcal{P}_n^0$ such that $P_{n_j} = f^n(\Pi_\Lambda^{-1}(P_{n_j}^0))$. Therefore,

$$\mu_1\Big(\bigvee_{j=0}^{m-1} f^{-j}(P_{n_j})\Big) = \mu_1\Big(\bigvee_{j=0}^{m-1} f^{-j}(f^n(\Pi_\Lambda^{-1}(P_{n_j}^0)))\Big)$$
$$= \mu_1\Big(\bigvee_{j=0}^{m-1} f^{-j}(\Pi_\Lambda^{-1}(P_{n_j}^0))\Big).$$

We then obtain that for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$h_{\nu_1}(g) \geq h_{\nu_1}(g, P_n^0) = h_{\mu_1}(f, \Pi_\Lambda^{-1} P_n^0)$$
$$= h_{\mu_1}(f, P_n) \geq h_{\mu_1}(f) - \epsilon,$$

and this proves $h_{\nu_1}(g) \geq h_{\mu_1}(f)$ and the uniqueness of the maximal entropy measure.

5. *Statistical stability*

Now we prove the statistical stability for the maximizing probability measure μ. That is, given $f_n \to f$ in the C^1-topology, then $\mu_n \to \mu$ in weak-$*$ topology, where μ_n (respectively, μ) is the maximizing entropy measure for f_n (respectively, f).

Let us fix such f, and consider the collection $\mathcal{C}$ whose elements are open subsets $A \subset M$ whose frontiers are μ-zero sets with the form $A = \bigcup_{x \in B} M_x$, for some ball $B \subset N$ with ν-zero frontier. Denote by $\hat{\mathcal{C}} \supset \mathcal{C}$ the collection whose elements are non-negative iterates of some element of $\mathcal{C}$. Observe that, if we fix $k \in \mathbb{N}$, $f^k(\bigcup_{x \in N} M_x)$ is a neighborhood for the attractors Λ_n where μ_n are supported for all sufficiently big n. Note that $\hat{\mathcal{C}}$ is a neighborhood basis for Λ.

The key ingredient for the proof is the following lemma.

LEMMA 5.1. *Let $\hat{A} \in \hat{\mathcal{C}}$. Then $\mu_n(\hat{A}) \to \mu(\hat{A})$ as $n \to +\infty$.*

Proof. Let $\hat{A} = f^k(A)$, with $A = \bigcup_{x \in B} M_x$. We start with the case $k = 0$; that is, first we prove that $\mu_n(A) \to \mu(A)$ as $n \to +\infty$.

Set $A_n := \Pi_n^{-1}(B)$. Therefore, $\mu_n(A_n) = \nu_n(B)$, where ν_n is the maximizing measure g_n as in [**CV13**]. We also have $\mu(A) = \nu(B)$, where ν is the entropy maximizing probability associated with g, as in [**CV13**].

Given $\epsilon > 0$, take $B^+ \supset B \supset B^-$, with ν-zero frontier such that

$$\nu(B^+) - \epsilon/3 < \nu(B) < \nu(B^-) + \epsilon/3,$$

and let us also assume that $A_n^\pm := \Pi_n^{-1}(B^\pm)$, with μ-zero frontier such that there exists n_2 such that for all $n \geq n_2$, $A_n^+ \supset A \supset A_n^-$ and

$$\mu(A_n^+) - \epsilon/3 < \mu(A) < \mu(A_n^-) + \epsilon/3,$$

hold.

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Such sets exist by the C^0-convergence of (strong stable/center-unstable) foliations for f_n to the respective foliations for f.

On the one hand, there exists $n_1 \geq n_2$ such that

$$\mu(A) - \mu_n(A) \leq \mu(A) - \mu_n(A) \leq \mu(A) - \mu_n(A_n^-) = \nu(B) - \nu_n(B^-) \leq \frac{2\epsilon}{3},$$

for all $n \geq n_1$, as $\nu_n(B^-) \to \nu(B^-)$ by the statistical stability for g proved in [**CV13**].

In the same manner, we prove the other inequality, implying there exists $n_0 \geq n_1$ such that

$$|\mu(A) - \mu_n(A)| < \epsilon \quad \text{for all } n \geq n_0.$$

The same arguments are also valid for the case $k > 0$.

This finishes the proof of the lemma. □

THEOREM 5.2. *Let $\varphi : M \to \mathbb{R}$ be a continuous function. Then $\int_M \varphi \, d\mu_n \to \int_M \varphi \, d\mu$.*

Proof. Let $\epsilon > 0$ be given, and the $\delta > 0$ obtained by the uniform continuity of φ associated with $\epsilon/9$. Take a covering $\bigcup_{j=1}^k C_j$, $C_j \in \mathcal{C}$ of Λ, with diameter less than $\delta/3$. There is also n_0 such that $\bigcup_{j=1}^k C_j \supset \Lambda_n$, for all $n \geq n_0$. In particular, $\mu_n(M \setminus \bigcup_{j=1}^k C_j) = 0$, for all $n \geq n_0$.

Consider a partition of unity $\{\psi_j,\ j = 1, \ldots, k\}$ associated to $\bigcup_{j=1}^k C_j$.

For each C_j, take $x_j \in C_j$ and set $\hat{\varphi} := \sum_{j=1}^k \varphi(x_j)\psi_j$. Therefore, $\|\varphi - \hat{\varphi}\|_\infty < \epsilon/3$. Now, take $n_1 \geq n_0$ such that

$$|(\mu_n - \mu)(C_j)| < \frac{\epsilon}{3k\|\varphi\|_\infty} \quad \text{for all } n \geq n_1.$$

So, we conclude that

$$\begin{aligned}\left|\int_M \varphi \, d\mu_n - \int_M \varphi \, d\mu\right| &\leq \left|\int_M \varphi \, d\mu_n - \int_M \hat{\varphi} \, d\mu_n\right| + \left|\int_M \hat{\varphi} \, d\mu_n - \int_M \hat{\varphi} \, d\mu\right| \\ &\quad + \left|\int_M \varphi \, d\mu - \int_M \hat{\varphi} \, d\mu\right| \\ &\leq \|\varphi - \hat{\varphi}\|_\infty + \sum_{j=1}^k \|\varphi\|_\infty |\mu_n(C_j) - \mu(C_j)| \\ &\quad + \|\varphi - \hat{\varphi}\|_\infty < \epsilon \quad \text{for all } n \geq n_0.\end{aligned}$$

□

6. *Cones and projective metrics*

We recall here some necessary results of projective metrics defined in cones whose proofs can be found in [**Li95**, **Bal00**, **Vi97**].

Given a linear space E we call $C \subset E \setminus \{0\}$ a convex cone if

$$t > 0 \quad \text{and} \quad v \in C \Rightarrow t \cdot v \in C$$

and

$$t_1, t_2 > 0 \quad \text{and} \quad v_1, v_2 \in C \Rightarrow t_1 \cdot v_1 + t_2 \cdot v_2 \in C.$$

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We define $\overline{C}$ to be the set of points $w \in E$ such that there exists $v \in C$ and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$, going to zero, such that $w + t_n \cdot v \in C$ for all $n \in \mathbb{N}$. We will only consider the so-called *projective cones*, such that

$$\overline{C} \cap (-\overline{C}) = \{0\}.$$

We then define

$$\alpha_C(v, w) = \sup\{t > 0;\ w - t \cdot v \in C\}$$

and

$$\beta_C(v, w) = \inf\{s > 0;\ s \cdot v - w \in C\}.$$

By convention, $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. The projective metrics associated with C are given by

$$\theta(v, w) = \log \frac{\beta_C(v, w)}{\alpha_C(v, w)}.$$

PROPOSITION 6.1. *Given a projective cone C, $\theta(\cdot, \cdot) : \overline{C} \times \overline{C} \to [0, +\infty]$ is a metric in the projective space of C, that is:*

- $\theta(v, w) = \theta(w, v)$;
- $\theta(u, w) \le \theta(u, v) + \theta(v, w)$; *and*
- $\theta(v, w) = 0$ *if and only if there exists* $t > 0$ *such that* $v = t \cdot w$.

The proof of the following essential result can be found in [**Vi97**, Proposition 2.3].

THEOREM 6.2. *Let E_1 and E_2 be linear spaces and let $C_1 \subset E_1$ and $C_2 \subset E_2$ be projective cones. If $L : E_1 \to E_2$ is a linear operator such that $L(C_1) \subset C_2$ and*

$$D = \sup\{\theta_2(L(v), L(w));\ v, w \in C_1\} < \infty$$

then

$$\theta_2(L(v), L(w)) \le (1 - e^{-D})\theta_1(v, w),$$

for all $v, w \in C_1$.

7. *Ruelle–Perron–Frobenius operator and invariant cones*

We recall that the main goal of this work is to deduce good statistical properties of the maximal entropy probability measure associated with the dynamics of f. The technique presented uses the Ruelle–Perron–Frobenius operator (for simplicity, called the transfer operator) and its duality with the Koopman operator, $U(\varphi) = \varphi \circ f$, to obtain the exponential decay of correlations and consequently the central limit theorem.

However, this technique may also be useful to prove exponential decay of correlations and consequently the central limit theorem for more general equilibrium states, not just particularly for measures of maximum entropy. We recall that given a map $f : \Lambda \to \Lambda$ and a fixed potential $\phi : \Lambda \to \mathbb{R}$, we say that a measure η is an equilibrium state for f with respect to ϕ if

$$h_\eta(f) + \int \phi\, d\eta = \sup\left\{h_\mu(f) + \int \phi\, d\mu \mid \mu \text{ is an } f\text{-invariant probability}\right\}.$$

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That is, the variational principle tells us that η carries out the topological pressure $P(f, \phi)$. The reader can easily see that in the case where the potential ϕ is a constant, obtaining an equilibrium state is equivalent to obtaining a maximum entropy measure. What we do in this section is to obtain some preliminary results, for more general potentials than constant potentials, namely, low variation potentials. That is, we assume that $\sup \phi - \inf \phi < \varepsilon$ for some small enough ε. Moreover, such potential must belong to the following cone:

$$|e^{\phi}|_{\alpha} \leq \varepsilon \inf e^{\phi} \tag{7.1}$$

where $|e^{\phi}|_{\alpha} = \inf\{C > 0 \mid |e^{\phi}(x) - e^{\phi}(y)| \leq C d(x, y)^{\alpha}, \forall x, y \in \Lambda\}$. Let E be the space of continuous functions $\varphi : \Lambda \to \mathbb{R}$. Define the Ruelle–Perron–Frobenius operator $\mathcal{L} : E \to E$ given

$$\mathcal{L}(\varphi)(y) = \varphi(f^{-1}(y)) e^{\phi(f^{-1}(y))}$$

where ϕ satisfies the above conditions.

Our inspiration is the work developed in [**CV13**], where the exponential decay of correlations and other good statistical and regularity properties are proven for the unique equilibrium state in a non-uniformly expanding context. Castro–Varandas defined suitable cones for the Ruelle–Perron–Frobenius (or transfer) operator $\mathcal{L}$, proving the invariance and the finite diameter for the image of such cones by $\mathcal{L}$.

More precisely, the basic cone used by [**CV13**] is the cone of Hölder continuous, positive functions φ such that $|\varphi|_{\alpha} \leq \kappa \inf \varphi$. The invariance of such a cone by f is due to some increase in the regularity given by the contraction of some inverse branch of f. In our context, however, we always have backward expansion in stable directions for the points into each strong stable manifold $\Pi^{-1}(y)$, instead of contraction. Since for the case of entropy (potential $\phi \equiv 0$), the transfer operator $\mathcal{L}$ is just the composition of each observable φ with f^{-1}, it is obvious that the Hölder constants of $\mathcal{L}(\varphi)$ cannot be improved if one takes a cone as in [**CV13**].

In order to avoid this undesirable effect in stable directions, we will analyze the action of $\mathcal{L}$ in some kind of average taken in each stable leaf restricted to the attractor Λ. We will write the lowercase letter γ to denote a stable leaf (intersected with Λ) and $\mathcal{F}^s$ will denote the stable foliation.

Fixing $y \in N$, let y_j be such that $g(y_j) = y$, where $j \in \{1, \ldots, \deg(g)\}$. Writing $\gamma = \Pi_{\Lambda}^{-1}(y)$ and $\gamma_j = \Pi_{\Lambda}^{-1}(y_j)$, it follows that $f(\gamma_j) \subset \gamma$, since $\Pi \circ f(x) = g \circ \Pi(x) = g(y_j) = y$, for all $x \in \gamma_j$.

Let p be the degree of g. Let us construct a family of measures $\{\mu_{\gamma}\}_{\gamma \in \mathcal{F}^s}$ supported in Λ, such that for all $\hat{\gamma}$, where $f^n(\hat{\gamma}) \subset \gamma$, we have $\mu_{\gamma}(f^n(\hat{\gamma})) = 1/p^n$. In particular, $\mu_{\gamma}(\gamma) = 1$. Furthermore, for all γ_j, with $f(\gamma_j) \subset \gamma$ we will obtain

$$\int_{f(\gamma_j)} \psi \, d\mu_{\gamma} = \frac{1}{p} \int_{\gamma_j} \psi \circ f \, d\mu_{\gamma_j}.$$

The construction of such family of measures is rather natural. Fix $\gamma = \Pi_{\Lambda}^{-1}(y)$ and $n \in \mathbb{N}$, $n > 0$. By setting $\gamma_j := \Pi_{\Lambda}^{-1}(y_j)$, where $y_j \in g^{-n}(y)$, one can write $\gamma = \dot{\bigcup}_{j=1}^{p^n} f^n(\gamma_j)$, since f^n is a bijection in Λ and $\Pi \circ f^n = g^n \circ \Pi$. Therefore, $\{f^n(\gamma_j)\}_{j=1}^{p^n}$ is a sequence of partitions in γ. As $\gamma_j = \Pi_{\Lambda}^{-1}(y_j)$ and $f^n : M_{y_j} \to M_{g^n(y_j)}$ is a λ_s^n-contraction, it follows

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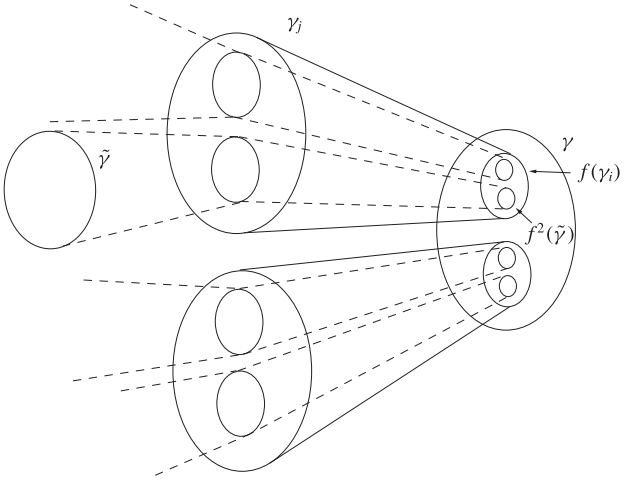


FIGURE 1. Mass distribution.

that the diameter of $\{f^n(\gamma_j)\}_{j=1}^{p^n}$ goes to zero. So, we just define μ_γ in the elements of such partition by mass distribution (see Figure 1):

$$\mu_\gamma(f^n(\gamma_j)) = \frac{1}{p^n}$$

and extend μ_γ by approximation to any Borelian $A \subset \Lambda$.

If $\gamma_j = \Pi_\Lambda^{-1}(x_j)$, $x_j \in g^{-1}(x)$, then

$$\mu_\gamma(A) = \mu_\gamma(A \cap \gamma) = \mu_\gamma\left(A \cap \bigcup_{j=1}^{p} f(\gamma_j)\right)$$

$$= \mu_\gamma\left(\bigcup_{j=1}^{p}(A \cap f(\gamma_j))\right) = \sum_{j=1}^{p} \mu_\gamma(A \cap f(\gamma_j)).$$

Setting $\mu_{\gamma_j}(A) := p \cdot \mu_\gamma(f(A \cap \gamma_j))$, we obtain $\mu_\gamma(A \cap f(\gamma_j)) = (1/p)\mu_{\gamma_j}(f^{-1}(A))$ and so

$$\mu_\gamma(A) = \frac{1}{p}\sum_{j=1}^{p} \mu_{\gamma_j}(f^{-1}(A)).$$

We conclude that for any measurable set A, its indicator function χ_A satisfies

$$\int_{f(\gamma_j)} \chi_A \, d\mu_\gamma = \frac{1}{p}\int_{\gamma_j} \chi_A \circ f \, d\mu_{\gamma_j}.$$

By Lebesgue's dominated convergence theorem, for any $g : \Lambda \to \mathbb{R}$ continuous we have

$$\int_{f(\gamma_j)} g \, d\mu_\gamma = \frac{1}{p}\int_{\gamma_j} g \circ f \, d\mu_{\gamma_j}. \tag{7.2}$$

Note also that for all $\hat{\gamma}$, $f^n(\hat{\gamma}) \subset \gamma$, we have $\mu_\gamma(f^n(\hat{\gamma})) = 1/p^n$. So it follows that for all $\widetilde{\gamma}$ such that $f^n(\widetilde{\gamma}) \subset \gamma_j$ and $f(\gamma_j) \subset \gamma$,

$$\mu_{\gamma_j}(f^n(\widetilde{\gamma})) = p\mu_\gamma(f(f^n(\widetilde{\gamma}) \cap \gamma_j)) = p\mu_\gamma(f^{n+1}(\widetilde{\gamma})) = \frac{p}{p^{n+1}} = \frac{1}{p^n}$$

holds.

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That is, μ_{γ_j} is the mass distribution measure constructed for γ_j.

Moreover, for $y \in N$ and y_j such that $g(y_j) = y$, $j \in \{1, \dots, p\}$, if we consider $\gamma = \Pi_\Lambda^{-1}(y)$ and $\gamma_j = \Pi_\Lambda^{-1}(y_j)$, $f(\gamma_j) \subset \gamma$, then $\gamma = \dot{\bigcup}_{j=1}^{p} f(\gamma_j)$. Therefore, for all measurable bounded functions $\psi : \gamma \to \mathbb{R}$ it follows that

$$\int_\gamma \psi \, d\mu_\gamma = \sum_{j=1}^{p} \int_{f(\gamma_j)} \psi \, d\mu_\gamma.$$

For $\rho : \gamma \to \mathbb{R}$, we conclude that

$$\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma = \sum_{j=1}^{p} \int_{f(\gamma_j)} \mathcal{L}(\varphi)\rho \, d\mu_\gamma = \sum_{j=1}^{p} \frac{1}{p} \int_{\gamma_j} \varphi \cdot e^{\phi} \cdot \rho \circ f \, d\mu_{\gamma_j}.$$

Defining $\rho_j := (1/p)\rho \circ f e^{\phi}$, we have

$$\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma = \sum_{j=1}^{p} \int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}.$$

We will study the action of the transfer operator in the strong stable leaves via its action on the integrals of densities in suitable cones of functions which are defined in each strong stable leaf. More precisely, for each $\gamma \in \mathcal{F}^s$ we define the auxiliary cone of Hölder continuous functions

$$\mathcal{D}(\gamma, \kappa) := \{\rho : \gamma \to \rho > 0 \text{ and } |\rho|_\alpha < \kappa \inf \rho\}$$

with $|\rho|_\alpha = \inf\{C > 0;\ |\rho(x) - \rho(y)| \le C d(x, y)^\alpha, \forall x, y \in \gamma\}$.

Note that for ρ in a cone $\mathcal{D}(\gamma, \kappa)$ we have $\sup \rho \le \inf \rho(1 + \kappa \cdot \operatorname{diam} M^\alpha)$.

The next lemma is about the invariance of the auxiliary cones under the action of the transfer operator.

LEMMA 7.1. *There exist sufficiently small $0 < \lambda < 1$ and $\kappa > 0$ such that the following items hold:*

(1) *if $\rho \in \mathcal{D}(\gamma, \kappa)$ then $\rho_j \in \mathcal{D}(\gamma_j, \lambda\kappa)$ for all $j \in \{1, \dots, p\}$;*
(2) *for all $\gamma \in \mathcal{F}^s_{\text{loc}}$, if $\rho, \hat{\rho} \in \mathcal{D}(\gamma, \lambda\kappa)$ then $\theta(\rho, \hat{\rho}) \le 2 \log((1 + \lambda)/(1 - \lambda))$; and*
(3) *if $\rho', \rho'' \in \mathcal{D}(\gamma, \kappa)$ then there exists $\Lambda_1 = 1 - ((1 - \lambda)/(1 + \lambda))^2$ such that $\theta_j(\rho'_j, \rho''_j) \le \Lambda_1 \theta(\rho', \rho'')$ for all $j \in \{1, \dots, p\}$;*

where θ_j and θ are the projective metrics associated with $\mathcal{D}(\gamma_j, \kappa)$ and $\mathcal{D}(\gamma, \kappa)$, respectively.

Proof. (1) In our context, we suppose $\sup \phi - \inf \phi < \varepsilon$ and $|e^{\phi}|_\alpha < \varepsilon \inf e^{\phi}$. Therefore,

$$\begin{aligned}
\frac{|\rho_j|_\alpha}{\inf\{\rho_j\}} &= \frac{|(1/p)\rho \circ f \cdot e^{\phi}|_\alpha}{\inf\{(1/p)\rho \circ f \cdot e^{\phi}\}} = \frac{|\rho \circ f \cdot e^{\phi}|_\alpha}{\inf\{\rho \circ f \cdot e^{\phi}\}} \\
&\le \frac{|\rho \circ f|_\alpha \cdot e^{\sup \phi} + \sup\{\rho \circ f\} \cdot |e^{\phi}|_\alpha}{\inf \rho \cdot e^{\inf \phi}} \\
&\le \frac{\lambda_s^\alpha \kappa \inf \rho \cdot e^{\sup \phi}}{\inf \rho \cdot e^{\inf \phi}} + \frac{(1 + \kappa \cdot \operatorname{diam} M^\alpha) \inf \rho \cdot |e^{\phi}|_\alpha}{\inf \rho \cdot e^{\inf \phi}} \\
&\le \lambda_s^\alpha \kappa e^{\varepsilon} + (1 + \kappa \cdot \operatorname{diam} M^\alpha)\varepsilon = (\lambda_s^\alpha e^{\varepsilon} + \operatorname{diam} M^\alpha \epsilon)\kappa + \varepsilon.
\end{aligned}$$

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In order to guarantee a $0 < \lambda < 1$ such that

$$(\lambda_s^\alpha e^\varepsilon + \operatorname{diam} M^\alpha \varepsilon)\kappa + \varepsilon < \lambda\kappa,$$

it is sufficient to obtain

$$\frac{(\lambda_s^\alpha e^\varepsilon + \operatorname{diam} M^\alpha \varepsilon)\kappa + \varepsilon}{\kappa} < \lambda < 1.$$

For that we just need

$$\frac{(\lambda_s^\alpha e^\varepsilon + \operatorname{diam} M^\alpha \varepsilon)\kappa + \varepsilon}{\kappa} < 1$$

or, equivalently,

$$\kappa > \frac{\varepsilon}{1 - (\lambda_s^\alpha e^\varepsilon + \operatorname{diam} M^\alpha \varepsilon)}. \tag{7.3}$$

Note that λ and κ can be chosen in order to satisfy the above equation since in our hypothesis we choose $\varepsilon > 0$ and $0 < \lambda_s < 1$ suitably small.

(2) By a triangular argument, it is sufficient to bound $\theta(1, \rho)$ for $\rho \in \mathcal{D}(\gamma, \lambda\kappa)$. There is no loss of generality in assuming that $\inf \rho = 1$. So, for $t = 1 - \lambda$, we have

$$\frac{|\rho - t|_\alpha}{\inf(\rho - t)} = \frac{|\rho|_\alpha}{\inf \rho - t} < \frac{\lambda\kappa}{1 - t} = \frac{\lambda\kappa}{\lambda} = \kappa.$$

Since $\inf \rho = 1$, it follows that $\rho - t \geq \inf \rho - (1 - \lambda) = \lambda > 0$, which guarantees $\alpha(1, \rho) \geq 1 - \lambda$. On the other hand, by setting $s = 1 + \lambda$, we obtain

$$\frac{|s - \rho|_\alpha}{\inf(s - \rho)} = \frac{|\rho|_\alpha}{s - \inf \rho} < \frac{\lambda\kappa}{s - 1} = \frac{\lambda\kappa}{\lambda} = \kappa.$$

As $\sup \rho \leq \inf \rho(1 + \kappa \operatorname{diam} M^\alpha) = 1 + \kappa \operatorname{diam} M^\alpha$ taking κ such that $\lambda > \kappa \operatorname{diam} M^\alpha$, it follows that $s - \rho = 1 + \lambda - \rho \geq 1 + \lambda - \sup \rho \geq 1 + \lambda - (1 + \kappa \operatorname{diam} M^\alpha) > 0$. Therefore, $\beta(1, \rho) \leq 1 + \lambda$. So we conclude that $\theta(\rho, \hat{\rho}) \leq 2 \log((1 + \lambda)/(1 - \lambda))$.

Finally, in order to prove (3) it is sufficient to note that by item (1) we have $\rho_j \in \mathcal{D}(\gamma_j, \lambda\kappa)$ for all $j \in \{1, \ldots, p\}$ and by item (2) the diameter $\mathcal{D}(\gamma_j, \lambda\kappa)$ in $\mathcal{D}(\gamma_j, \kappa)$ is, at most, $2 \log((1 + \lambda)/(1 - \lambda))$. Therefore, the result follows by Theorem 6.2. Considering $\Delta = 2 \log((1 + \lambda)/(1 - \lambda))$ and the linear map

$$\rho \mapsto \frac{1}{p} \rho \circ f e^\phi,$$

we have $\theta_j(\rho_j', \rho_j'') \leq \Lambda_1 \theta(\rho', \rho'')$, where

$$\Lambda_1 = 1 - e^{-\Delta} = 1 - \left(\frac{1 - \lambda}{1 + \lambda}\right)^2. \qquad \square$$

For the definition of the main cone on which we will apply the transfer operator, we need to define a notion of distance between two strong stable leaves γ and $\widetilde{\gamma}$ in $\mathcal{F}^s$. Given $x, y \in N$, let $\gamma = \Pi_\Lambda^{-1}(x)$ and $\widetilde{\gamma} = \Pi_\Lambda^{-1}(y)$. Suppose $\pi = \pi_{x,y} : \widetilde{\gamma} \to \gamma$ satisfies

$$\int_\gamma \varphi \, d\mu_\gamma = \int_{\widetilde{\gamma}} \varphi \circ \pi \, d\mu_{\widetilde{\gamma}}$$

for all continuous functions φ and define the distance $d(\gamma, \widetilde{\gamma}) = \sup\{d(\pi(p), p);\ p \in \widetilde{\gamma}\}$.

Now, let us define our main cone. Denote by $\mathcal{D}_1(\gamma)$ the set of densities $\rho \in \mathcal{D}(\gamma, \kappa)$ such that $\int_\gamma \rho \, d\mu_\gamma = 1$. Given $b > 0$, $c > 0$ and κ as in Lemma 7.1, let $C(b, c, \alpha)$ be the cone of functions $\varphi \in E$ satisfying for all $\gamma \in \mathcal{F}^s$ the following.

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(A) For all $\rho \in \mathcal{D}(\gamma, \kappa)$:

$$\int_\gamma \varphi\rho \, d\mu_\gamma > 0.$$

(B) For all $\rho', \rho'' \in \mathcal{D}_1(\gamma)$:

$$\left|\int_\gamma \varphi\rho' \, d\mu_\gamma - \int_\gamma \varphi\rho'' \, d\mu_\gamma\right| < b\theta(\rho', \rho'') \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_\gamma \varphi\rho \, d\mu_\gamma\right\}.$$

(C) Given any $\widetilde{\gamma}$ sufficiently close to γ:

$$\left|\int_\gamma \varphi \, d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi \, d\mu_{\widetilde{\gamma}}\right| < cd(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma \left\{\int_\gamma \varphi \, d\mu_\gamma\right\}.$$

The proof of the next lemma follows from standard arguments.

LEMMA 7.2. *$C(b, c, \alpha)$ is a projective cone.*

PROPOSITION 7.3. *Let ϕ be constant. There exists $0 < \sigma < 1$ such that $\mathcal{L}(C(b, c, \alpha)) \subset C(\sigma b, \sigma c, \alpha)$ for sufficiently large $b,c > 0$.*

Proof. Invariance of condition (A): Let $\varphi \in C(b, c, \alpha)$. We know that $\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma = \sum_{j=1}^p \int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}$ and, by Lemma 7.1, $\rho_j \in \mathcal{D}(\gamma_j, \kappa)$. Therefore, $\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma > 0$.

Invariance of condition (B): Denoting $\rho_j/(\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j})$ by $\hat{\rho}_j$ we can write

$$\begin{aligned}
\inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma\right\} &\geq \sum_{j=1}^p \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}\right\} \\
&= \sum_{j=1}^p \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \varphi\hat{\rho}_j \, d\mu_{\gamma_j} \int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\right\} \\
&\geq \sum_{j=1}^p \inf_{\rho\in\mathcal{D}_1(\gamma_j)} \left\{\int_{\gamma_j} \varphi\rho \, d\mu_{\gamma_j}\right\} \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\right\}.
\end{aligned}$$

Given $\rho', \rho'' \in \mathcal{D}_1(\gamma)$, writing $\rho'_j/\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j}$ and $\rho''_j/\int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}$ for $\overline{\rho}_j$ and $\overline{\overline{\rho}}_j$, respectively, it follows that

$$\begin{aligned}
\left|\int_\gamma \mathcal{L}(\varphi)\rho' \, d\mu_\gamma - \int_\gamma \mathcal{L}(\varphi)\rho'' \, d\mu_\gamma\right| &\leq \sum_{j=1}^p \left|\int_{\gamma_j} \varphi\overline{\rho}_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \varphi\overline{\overline{\rho}}_j \, d\mu_{\gamma_j}\right| \int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} \\
&\quad + \sum_{j=1}^p \int_{\gamma_j} \varphi\overline{\overline{\rho}}_j \, d\mu_{\gamma_j} \left|\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}\right|. \qquad (7.4)
\end{aligned}$$

By hypothesis, φ is in the cone, and by Lemma 7.1, we have

$$\begin{aligned}
\left|\int_{\gamma_j} \varphi\overline{\rho}_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \varphi\overline{\overline{\rho}}_j \, d\mu_{\gamma_j}\right| &\leq b\theta_j(\overline{\rho}_j, \overline{\overline{\rho}}_j) \inf_{\rho\in\mathcal{D}_1(\gamma_j)} \left\{\int_{\gamma_j} \varphi\rho \, d\mu_{\gamma_j}\right\} \\
&\leq b\Lambda_1\theta(\rho', \rho'') \inf_{\rho\in\mathcal{D}_1(\gamma_j)} \left\{\int_{\gamma_j} \varphi\rho \, d\mu_{\gamma_j}\right\}. \qquad (7.5)
\end{aligned}$$

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For all $\hat{\rho} \in \mathcal{D}_1(\gamma)$ we obtain the following estimate:

$$\frac{\int_{\gamma_j} (\hat{\rho})_j \, d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}} \leq (1 + \kappa \operatorname{diam} M^\alpha)^2. \tag{7.6}$$

In fact, given $\delta > 0$, there exists $\widetilde{\rho} \in \mathcal{D}_1(\gamma)$ such that $\int_{\gamma_j} (\widetilde{\rho})_j \, d\mu_{\gamma_j} \leq (1 + \delta) \inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}$. Moreover, as $\hat{\rho}$ and $\widetilde{\rho}$ are normalized, we necessarily have $\inf \hat{\rho} \leq 1$ and $\sup \widetilde{\rho} \geq 1$. Therefore,

$$\frac{(\hat{\rho})_j}{(\widetilde{\rho})_j} = \frac{(1/p)\hat{\rho} \circ f e^{\phi}}{(1/p)\widetilde{\rho} \circ f e^{\phi}} \leq \frac{\sup \hat{\rho}}{\inf \widetilde{\rho}} \leq \frac{(1 + \kappa \operatorname{diam} M^\alpha) \inf \hat{\rho}}{(1 + \kappa \operatorname{diam} M^\alpha)^{-1} \sup \widetilde{\rho}} = (1 + \kappa \operatorname{diam} M^\alpha)^2.$$

And so, $\int_{\gamma_j} (\hat{\rho})_j \, d\mu_{\gamma_j} \leq (1 + \kappa \operatorname{diam} M^\alpha)^2 \int_{\gamma_j} (\widetilde{\rho})_j \, d\mu_{\gamma_j}$, and we obtain, for all $\delta > 0$,

$$\begin{aligned} \frac{\int_{\gamma_j} (\hat{\rho})_j \, d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}} &\leq \frac{(1 + \delta)(1 + \kappa \operatorname{diam} M^\alpha)^2 \int_{\gamma_j} (\widetilde{\rho})_j \, d\mu_{\gamma_j}}{\int_{\gamma_j} (\widetilde{\rho})_j \, d\mu_{\gamma_j}} \\ &\leq (1 + \delta)(1 + \kappa \operatorname{diam} M^\alpha)^2, \end{aligned}$$

giving the estimate we wish.

Now, for fixed j, we obtain

$$\begin{aligned} &\frac{|\int_{\gamma_j} \varphi \overline{\rho}_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \varphi \overline{\overline{\rho}}_j \, d\mu_{\gamma_j}| \int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma_j)} \{\int_{\gamma_j} \varphi \rho \, d\mu_{\gamma_j}\} \inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\} \theta(\rho', \rho'')} \\ &\quad \leq \frac{b \Lambda_1 \theta(\rho', \rho'') \int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\} \theta(\rho', \rho'')} \\ &\quad \leq (1 + \kappa \operatorname{diam} M^\alpha)^2 \Lambda_1 b. \end{aligned} \tag{7.7}$$

Let us analyze the second parcel of (7.4). First, note that for all $\hat{\rho} \in \mathcal{D}_1(\gamma)$, denoting $(\hat{\rho})_j / \int_{\gamma_j} (\hat{\rho})_j \, d\mu_{\gamma_j}$ by $\bar{\hat{\rho}}_j$, we claim that

$$\frac{\int_{\gamma_j} \varphi \bar{\hat{\rho}}_j \, d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \varphi \overline{\rho}_j \, d\mu_{\gamma_j}\}} < b \log\left(\frac{1 + \lambda}{1 - \lambda}\right)^2 + 1.$$

In fact, analogously to what was done in (7.6), it is sufficient to note that, since φ is in the cone, we have

$$\frac{\int_{\gamma_j} \varphi \bar{\hat{\rho}}_j \, d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi \overline{\rho}_j \, d\mu_{\gamma_j}} < b\theta(\bar{\hat{\rho}}_j, \overline{\rho}_j) + 1 = b\theta((\hat{\rho})_j, \rho_j) + 1.$$

By (7.1), we conclude the proof of our claim.

Now, we establish the other necessary estimate:

$$\frac{|\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}|}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)} \{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}} \leq 2(1 + \kappa \operatorname{diam} M^\alpha)^2.$$

In order to prove this last estimate, we observe that

$$\frac{\rho'_j}{\rho''_j} \leq \frac{\sup \rho'}{\inf \rho''} \leq \frac{\sup \rho'/\inf \rho'}{\inf \rho''/\sup \rho''} = e^{\theta_+(\rho', \rho'')} \leq e^{\theta(\rho', \rho'')}.$$

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Therefore, by assuming without loss of generality that $\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} \geq \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}$, we obtain

$$\frac{|\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}|}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}} \leq \frac{(e^{\theta(\rho', \rho'')} - 1) \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}}.$$

For $\theta(\rho', \rho'') \leq 1$ it follows that $(e^{\theta(\rho', \rho'')} - 1)/\theta(\rho', \rho'') < 2$ and so we obtain our estimate.

If $\theta(\rho', \rho'') \geq 1$ we also have that

$$\begin{aligned}\frac{|\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}|}{\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}} &\leq \frac{|\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}|}{\inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}} \\ &\leq 2(1 + \kappa \operatorname{diam} M^\alpha)^2,\end{aligned}$$

and again for fixed j and by writing $M(\kappa, \alpha)$ for $(1 + \kappa \operatorname{diam} M^\alpha)^2$,

$$\begin{aligned}&\frac{\int_{\gamma_j} \varphi \overline{\overline{\rho}}_j \, d\mu_{\gamma_j} |\int_{\gamma_j} \rho'_j \, d\mu_{\gamma_j} - \int_{\gamma_j} \rho''_j \, d\mu_{\gamma_j}|}{\inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}\} \inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}\}\theta(\rho', \rho'')} \\ &\quad \leq \frac{\int_{\gamma_j} \varphi \overline{\overline{\rho}}_j \, d\mu_{\gamma_j}}{\inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}\}} 2M(\kappa, \alpha) \\ &\quad \leq \left(b \log\left(\frac{1+\lambda}{1-\lambda}\right)^2 + 1\right) 2M(\kappa, \alpha) \leq 2M(\kappa, \alpha) \log\left(\frac{1+\lambda}{1-\lambda}\right)^2 b + 2M(\kappa, \alpha). \quad (7.8)\end{aligned}$$

The inequalities (7.7) and (7.8) do not depend on j, so

$$\begin{aligned}&\frac{|\int_\gamma \mathcal{L}(\varphi)\rho' \, d\mu_\gamma - \int_\gamma \mathcal{L}(\varphi)\rho'' \, d\mu_\gamma|}{\inf_{\rho \in \mathcal{D}_1(\gamma)}\{\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma\}\theta(\rho', \rho'')} \\ &\quad \leq M(\kappa, \alpha)\Lambda_1 b + 2M(\kappa, \alpha) \log\left(\frac{1+\lambda}{1-\lambda}\right)^2 b + 2M(\kappa, \alpha) \\ &\quad = \left(\Lambda_1 + 2 \log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right) M(\kappa, \alpha) b + 2M(\kappa, \alpha).\end{aligned}$$

We need the term which multiplies b above to be less than 1. Recall that by Lemma 7.1, $\Lambda_1 = 1 - ((1-\lambda)/(1+\lambda))^2$. So, we need to guarantee that

$$\left(1 - \left(\frac{1-\lambda}{1+\lambda}\right)^2 + 2 \log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right)(1 + \kappa \operatorname{diam}(M)^\alpha)^2 < 1.$$

Also by Lemma 7.1, we can choose κ such that $\kappa \operatorname{diam} M^\alpha < \lambda$, for fixed λ. So let us find a bound $0 < \lambda < 1$ such that

$$\left(1 - \left(\frac{1-\lambda}{1+\lambda}\right)^2 + 2 \log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right)(1 + \lambda)^2 < 1.$$

It is possible because, by Lemma 7.1, $0 < \lambda < 1$ can be taken sufficiently small depending on the contraction rate in the strong stable directions. So, there exists $0 < \tilde{\sigma}_1 < 1$ such that

$$\left(\Lambda_1 + 2 \log\left(\frac{1+\lambda}{1-\lambda}\right)^2\right) M(\kappa, \alpha) < \tilde{\sigma}_1.$$

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Since $M(\kappa, \alpha)$ does not depend on b, for sufficiently large b we can obtain $\sigma_1 < 1$ such that

$$\frac{|\int_\gamma \mathcal{L}(\varphi)\rho'\, d\mu_\gamma - \int_\gamma \mathcal{L}(\varphi)\rho''\, d\mu_\gamma|}{\inf_{\rho\in\mathcal{D}_1(\gamma)}\{\int_\gamma \mathcal{L}(\varphi)\rho\, d\mu_\gamma\}\theta(\rho', \rho'')} \le \sigma_1 b.$$

This proves the strict invariance of condition (B).

Invariance of condition (C): This is where we need that ϕ is constant. We have that

$$\inf_\gamma\left\{\int_\gamma \mathcal{L}\varphi\, d\mu_\gamma\right\} \ge e^\phi \inf_\gamma\left\{\int_\gamma \varphi\, d\mu_\gamma\right\}.$$

For g as in (2.1), every $y \in N$ has at least one pre-image out of the region Ω. So, for $\gamma = \Pi^{-1}(y)$ and $\widetilde{\gamma} = \Pi^{-1}(\tilde{y})$ sufficiently close to γ such that $\tilde{y}_i$ is a pre-image of $\tilde{y}$ close to y_i, lying in U_{y_i} we obtain that $d(\gamma_i, \widetilde{\gamma}_i) \le \lambda_u d(\gamma, \widetilde{\gamma})$.

In fact, let $x \in \widetilde{\gamma}_i$ realizing the distance $d(\gamma_i, \widetilde{\gamma}_i)$. By a slight abuse of notation, we write d for the product distance equivalent to the original metrics. So,

$$\begin{aligned} d(\gamma_i, \widetilde{\gamma}_i) &= d(x, \pi_{\tilde{y}_i, y_i}(x)) = d(\tilde{y}_i, y_i) \le \lambda_u d(g(\tilde{y}_i), g(y_i)) \\ &= \lambda_u d(\tilde{y}, y) = \lambda_u [d(\tilde{y}, y) + d(\pi_{\tilde{y},y}(f(x)), \pi_{\tilde{y},y}(f(x)))] \\ &= \lambda_u d(f(x), \pi_{\tilde{y},y}(f(x))) \le \lambda_u d(\widetilde{\gamma}, \gamma). \end{aligned}$$

Analogously, in the other cases we have $d(\gamma_j, \widetilde{\gamma}_j) \le L d(\gamma, \widetilde{\gamma})$. Furthermore, we can assume, with no loss of generality, that $d(\gamma_1, \widetilde{\gamma}_1) \le \tilde{\lambda}_u d(\gamma, \widetilde{\gamma})$, and for other pre-images we have $d(\gamma_j, \widetilde{\gamma}_j) \le \tilde{L} d(\gamma, \widetilde{\gamma})$. Note that condition (C) could not hold for such other pre-images with the same c, since they are more distant than the initial leaves $\gamma, \widetilde{\gamma}$. However, just as in [**CV13**, Lemma 3.5], it holds with constant $c(1 + (\tilde{L} - 1)^\alpha)$ instead of c. It follows that

$$\begin{aligned} &\left|\int_\gamma \mathcal{L}\varphi\, d\mu_\gamma - \int_{\widetilde{\gamma}} \mathcal{L}\varphi\, d\mu_{\widetilde{\gamma}}\right| \\ &\le \frac{e^\phi}{p}\sum_{j=1}^{p}\left|\int_{\gamma_j} \varphi\, d\mu_{\gamma_j} - \int_{\widetilde{\gamma}_j} \varphi\, d\mu_{\widetilde{\gamma}_j}\right| \\ &\le \frac{e^\phi c}{p}\inf_\gamma\left\{\int_\gamma \varphi\, d\mu_\gamma\right\}\sum_{j=1}^{p} d(\gamma_j, \widetilde{\gamma}_j)^\alpha \\ &\le \frac{\tilde{\lambda}_u^\alpha + (p-1)(1 + (\tilde{L} - 1)^\alpha)\tilde{L}^\alpha}{p} c d(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma\left\{\int_\gamma \mathcal{L}\varphi d\mu_\gamma\right\}. \end{aligned}$$

We should obtain $(\tilde{\lambda}_u^\alpha + (p-1)(1 + (\tilde{L} - 1)^\alpha)\tilde{L}^\alpha)/p < 1$. This is equivalent to $(1 + (\tilde{L} - 1)^\alpha)\tilde{L}^\alpha < (p - \tilde{\lambda}_u^\alpha)/(p-1)$. By the fact that $\tilde{L} \ge 1$, we have $(p - \tilde{\lambda}_u^\alpha)/(p-1) \ge 1$, because $\tilde{\lambda}_u < 1$. Therefore, there exists $0 < \sigma_2 < 1$ such that

$$\left|\int_\gamma \mathcal{L}\varphi\, d\mu_\gamma - \int_{\widetilde{\gamma}} \mathcal{L}\varphi\, d\mu_{\widetilde{\gamma}}\right| < \sigma_2 c d(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma\left\{\int_\gamma \mathcal{L}\varphi\, d\mu_\gamma\right\}, \tag{7.9}$$

which proves (C).

By setting $\sigma = \max\{\sigma_1, \sigma_2\}$, we finish the proof of the proposition. □

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8. *Finite diameter of the main cone*

From now on, up to the end of our text, ϕ will be always constant. In this section, we prove the strict invariance of the main cone $C(b, c, \alpha)$ by the Ruelle–Perron–Frobenius operator $\mathcal{L}$. First, let us calculate the projective metrics Θ. Recall that $\alpha_C(\varphi, \psi) = \sup\{t > 0;\ \psi - t\varphi \in C(b, c, \alpha)\}$. By (A), for all $\gamma \in \mathcal{F}^s_{\mathrm{loc}}$ and $\rho \in \mathcal{D}(\gamma)$ we have $\int_\gamma (\psi - t\varphi)\rho\, d\mu_\gamma > 0$, that is, $t < (\int_\gamma \psi\rho\, d\mu_\gamma)/(\int_\gamma \varphi\rho\, d\mu_\gamma)$. By condition (B), one obtains

$$\left|\int_\gamma (\psi - t\varphi)\rho'\, d\mu_\gamma - \int_\gamma (\psi - t\varphi)\rho''\, d\mu_\gamma\right| < b\theta(\rho', \rho'') \inf_{\rho\in\mathcal{D}_1(\gamma)} \left\{\int_\gamma (\psi - t\varphi)\rho\, d\mu_\gamma\right\}$$

and so, for all ρ', ρ'', and $\hat\rho$ in $\mathcal{D}_1(\gamma)$ we have

$$t < \frac{\int_\gamma \psi\rho'\, d\mu_\gamma - \int_\gamma \psi\rho''\, d\mu_\gamma + b\theta(\rho', \rho'') \int_\gamma \psi\hat\rho\, d\mu_\gamma}{\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma + b\theta(\rho', \rho'') \int_\gamma \varphi\hat\rho\, d\mu_\gamma}$$

and

$$t < \frac{\int_\gamma \psi\rho''\, d\mu_\gamma - \int_\gamma \psi\rho'\, d\mu_\gamma + b\theta(\rho', \rho'') \int_\gamma \psi\hat\rho\, d\mu_\gamma}{\int_\gamma \varphi\rho''\, d\mu_\gamma - \int_\gamma \varphi\rho'\, d\mu_\gamma + b\theta(\rho', \rho'') \int_\gamma \varphi\hat\rho\, d\mu_\gamma}.$$

By condition (C),

$$\left|\int_\gamma (\psi - t\varphi)\, d\mu_\gamma - \int_{\widetilde\gamma} (\psi - t\varphi)\, d\mu_{\widetilde\gamma}\right| < cd(\gamma, \widetilde\gamma)^\alpha \inf_\gamma \left\{\int_\gamma (\psi - t\varphi)\, d\mu_\gamma\right\}.$$

Therefore, for all γ, $\hat\gamma \in \mathcal{F}^s_{\mathrm{loc}}$ and $\widetilde\gamma$ sufficiently close to γ, we have

$$t < \frac{\int_{\widetilde\gamma} \psi\, d\mu_{\widetilde\gamma} - \int_\gamma \psi\, d\mu_\gamma + cd(\gamma, \widetilde\gamma) \int_{\hat\gamma} \psi\, d\mu_{\hat\gamma}}{\int_{\widetilde\gamma} \varphi\, d\mu_{\widetilde\gamma} - \int_\gamma \varphi\, d\mu_\gamma + cd(\gamma, \widetilde\gamma) \int_{\hat\gamma} \varphi\, d\mu_{\hat\gamma}}$$

and

$$t < \frac{\int_\gamma \psi\, d\mu_\gamma - \int_{\widetilde\gamma} \psi\, d\mu_{\widetilde\gamma} + cd(\gamma, \widetilde\gamma) \int_{\hat\gamma} \psi\, d\mu_{\hat\gamma}}{\int_\gamma \varphi\, d\mu_\gamma - \int_{\widetilde\gamma} \varphi\, d\mu_{\widetilde\gamma} + cd(\gamma, \widetilde\gamma) \int_{\hat\gamma} \varphi\, d\mu_{\hat\gamma}}.$$

By defining

$$\xi(\gamma, \rho', \rho'', \hat\rho, \varphi, \psi) = \frac{(\int_\gamma \psi\rho''\, d\mu_\gamma - \int_\gamma \psi\rho'\, d\mu_\gamma)/\int_\gamma \psi\hat\rho\, d\mu_\gamma + b\theta(\rho', \rho'')}{(\int_\gamma \varphi\rho''\, d\mu_\gamma - \int_\gamma \varphi\rho'\, d\mu_\gamma)/\int_\gamma \varphi\hat\rho\, d\mu_\gamma + b\theta(\rho', \rho'')}$$

and

$$\eta(\gamma, \widetilde\gamma, \hat\gamma, \hat\rho, \varphi, \psi) = \frac{(\int_\gamma \psi\, d\mu_\gamma - \int_{\widetilde\gamma} \psi\, d\mu_{\widetilde\gamma})/\int_{\hat\gamma} \psi\, d\mu_{\hat\gamma} + cd(\gamma, \widetilde\gamma)}{(\int_\gamma \varphi\, d\mu_\gamma - \int_{\widetilde\gamma} \varphi\, d\mu_{\widetilde\gamma})/\int_{\hat\gamma} \varphi\, d\mu_{\hat\gamma} + cd(\gamma, \widetilde\gamma)}$$

we can write

$$\alpha_C(\varphi, \psi) = \inf\left\{\frac{\int_\gamma \psi\rho\, d\mu_\gamma}{\int_\gamma \varphi\rho\, d\mu_\gamma}, \frac{\int_\gamma \psi\hat\rho\, d\mu_\gamma}{\int_\gamma \varphi\hat\rho\, d\mu_\gamma}\xi(\gamma, \rho', \rho'', \hat\rho, \varphi, \psi), \frac{\int_{\hat\gamma} \psi\, d\mu_{\hat\gamma}}{\int_{\hat\gamma} \varphi\, d\mu_{\hat\gamma}}\eta(\gamma, \widetilde\gamma, \hat\gamma, \hat\rho, \varphi, \psi)\right\}$$

as $\beta_C(\varphi, \psi) = \alpha_C(\psi, \varphi)^{-1}$ to obtain

$$\beta_C(\varphi, \psi) = \sup\left\{\frac{\int_\gamma \varphi\rho\, d\mu_\gamma}{\int_\gamma \psi\rho\, d\mu_\gamma}, \frac{\int_\gamma \varphi\hat\rho\, d\mu_\gamma}{\int_\gamma \psi\hat\rho\, d\mu_\gamma}\xi(\gamma, \rho', \rho'', \hat\rho, \psi, \varphi), \frac{\int_{\hat\gamma} \varphi\, d\mu_{\hat\gamma}}{\int_{\hat\gamma} \psi\, d\mu_{\hat\gamma}}\eta(\gamma, \widetilde\gamma, \hat\gamma, \hat\rho, \psi, \varphi)\right\}.$$

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Now, we prove that the Θ-diameter of $\mathcal{L}(C(b, c, \alpha))$ is finite.

PROPOSITION 8.1. *For all sufficiently large $b > 0$, $c > 0$ and for $\alpha \in (0, 1]$, we have*

$$\Delta := \sup\{\Theta(\mathcal{L}\varphi, \mathcal{L}\psi);\ \varphi, \psi \in C(b, c, \alpha)\} < \infty.$$

Proof. Given $\varphi, \psi \in C(\sigma b, \sigma c, \alpha)$, note that

$$\frac{1-\sigma}{1+\sigma} < \xi(\gamma, \rho', \rho'', \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}$$

and

$$\frac{1-\sigma}{1+\sigma} < \eta(\gamma, \widetilde{\gamma}, \hat{\gamma}, \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}.$$

Indeed, given $\rho', \rho'', \hat{\rho} \in \mathcal{D}_1(\gamma)$,

$$\frac{\int_\gamma \varphi\rho''\, d\mu_\gamma - \int_\gamma \varphi\rho'\, d\mu_\gamma}{\int_\gamma \varphi\hat{\rho}\, d\mu_\gamma} \le \frac{\sigma b\theta(\rho', \rho'') \inf_{\rho\in\mathcal{D}_1(\gamma)}\{\int_\gamma \varphi\rho\, d\mu_\gamma\}}{\int_\gamma \varphi\hat{\rho}\, d\mu_\gamma} \le \sigma b\theta(\rho', \rho'')$$

and

$$\frac{\int_\gamma \varphi\rho''\, d\mu_\gamma - \int_\gamma \varphi\rho'\, d\mu_\gamma}{\int_\gamma \varphi\hat{\rho}\, d\mu_\gamma} \ge \frac{-\sigma b\theta(\rho', \rho'') \inf_{\rho\in\mathcal{D}_1(\gamma)}\{\int_\gamma \varphi\rho\, d\mu_\gamma\}}{\int_\gamma \varphi\hat{\rho}\, d\mu_\gamma} \ge -\sigma b\theta(\rho', \rho'')$$

hold.

The same is valid for ψ, and, as $\sigma < 1$, we conclude that

$$\frac{1-\sigma}{1+\sigma} < \frac{(\int_\gamma \psi\rho''\, d\mu_\gamma - \int_\gamma \psi\rho'\, d\mu_\gamma)/\int_\gamma \psi\hat{\rho}\, d\mu_\gamma + b\theta(\rho', \rho'')}{(\int_\gamma \varphi\rho''\, d\mu_\gamma - \int_\gamma \varphi\rho'\, d\mu_\gamma)/\int_\gamma \varphi\hat{\rho}\, d\mu_\gamma + b\theta(\rho', \rho'')} < \frac{1+\sigma}{1-\sigma}.$$

That is,

$$\frac{1-\sigma}{1+\sigma} < \xi(\gamma, \rho', \rho'', \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}.$$

In a similar way, we prove that

$$\frac{1-\sigma}{1+\sigma} < \eta(\gamma, \widetilde{\gamma}, \hat{\gamma}, \hat{\rho}, \psi, \varphi) < \frac{1+\sigma}{1-\sigma}.$$

Denoting by Θ_+ the projective metrics associated with the cone defined just by condition (A),

$$\Theta_+(\varphi, \psi) = \log \sup_{\gamma,\rho\in\mathcal{D}(\gamma),\hat{\gamma},\hat{\rho}\in\mathcal{D}(\hat{\gamma})} \left\{ \frac{\int_\gamma \varphi\rho\, d\mu_\gamma \int_{\hat{\gamma}} \psi\hat{\rho}\, d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} \varphi\hat{\rho}\, d\mu_{\hat{\gamma}} \int_\gamma \psi\rho\, d\mu_\gamma} \right\},$$

and it follows by the expression of Θ that

$$\Theta(\varphi, \psi) < \Theta_+(\varphi, \psi) + \log\left(\frac{1+\sigma}{1-\sigma}\right)^2.$$

So, we just need to prove that the Θ_+-diameter of $\mathcal{L}(C(b, c, \alpha))$ is finite. By a triangular argument, it is sufficient to show that $\{\Theta_+(\mathcal{L}\varphi, 1);\ \varphi \in C(b, c, \alpha)\}$ is finite. For that, we just need to find an upper bound for $(\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho}\, d\mu_{\hat{\gamma}})/(\int_\gamma \mathcal{L}\varphi\rho\, d\mu_\gamma)$ for all $\varphi \in C(b, c, \alpha)$, $\rho \in \mathcal{D}_1(\gamma)$ and $\hat{\rho} \in \mathcal{D}_1(\hat{\gamma})$. First, we note that

$$\frac{\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho}\, d\mu_{\hat{\gamma}}}{\int_\gamma \mathcal{L}\varphi\rho\, d\mu_\gamma} = \frac{\sum_{j=1}^p \int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j\, d\mu_{\hat{\gamma}_j}}{\sum_{j=1}^p \int_{\gamma_j} \varphi\rho_j\, d\mu_{\gamma_j}}$$

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and we reduce our problem to bounding

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}} = \frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi \, d\mu_{\hat{\gamma}_j}} \frac{\int_{\hat{\gamma}_j} \varphi \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi \, d\mu_{\gamma_j}} \frac{\int_{\gamma_j} \varphi \, d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}}.$$

Denoting $\rho_j/(\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j})$ and $(\hat{\rho})_j/(\int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j})$ by $\overline{\rho}_j$ and $\overline{\overline{\rho}}_j$, respectively, then applying (B) and Lemma 7.1, we obtain

$$\begin{aligned}\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi \, d\mu_{\hat{\gamma}_j}} &= \frac{\int_{\hat{\gamma}_j} \varphi\overline{\overline{\rho}}_j \, d\mu_{\hat{\gamma}_j} \int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi \, d\mu_{\hat{\gamma}_j}} \le (1 + b\theta_j(\overline{\overline{\rho}}_j, 1)) \int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j} \\ &\le \left(1 + b \log\left(\frac{1+\lambda}{1-\lambda}\right)\right) \int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j} \end{aligned} \tag{8.1}$$

and

$$\begin{aligned}\frac{\int_{\gamma_j} \varphi \, d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}} &= \frac{\int_{\gamma_j} \varphi \, d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\overline{\rho}_j \, d\mu_{\gamma_j} \int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}} \\ &\le \frac{(1 + b\theta_j(1, \overline{\rho}_j))}{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}} \le \frac{(1 + b\log((1+\lambda)/(1-\lambda)))}{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}}.\end{aligned}$$

We know that $(\hat{\rho})_j = (1/p)\hat{\rho} \circ f \cdot e^{\phi}$ and $\rho_j = (1/p)\rho \circ f \cdot e^{\phi}$. Since ρ and $\hat{\rho}$ are normalized, it follows that $(\hat{\rho})_j \le (1/p)(1 + \kappa \operatorname{diam}(M)^{\alpha})e^{\phi}$ and $\rho_j \ge (1/p)(1 + \kappa \operatorname{diam}(M)^{\alpha})^{-1}e^{\phi}$. Therefore,

$$\frac{\int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}} < (1 + \kappa \operatorname{diam}(M)^{\alpha})^2 \frac{\int_{\hat{\gamma}_j} e^{\phi} \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} e^{\phi} \, d\mu_{\gamma_j}}.$$

On the other hand, $|e^{\phi}|_{\alpha} < \varepsilon \inf e^{\phi}$, and so $\sup e^{\phi} < (1 + \varepsilon \operatorname{diam}(M)^{\alpha}) \inf e^{\phi}$. We then obtain $(\int_{\hat{\gamma}_j} e^{\phi} \, d\mu_{\hat{\gamma}_j})/(\int_{\gamma_j} e^{\phi} \, d\mu_{\gamma_j}) < 1 + \varepsilon \operatorname{diam}(M)^{\alpha}$, and, by consequence,

$$\frac{\int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}} < (1 + \kappa \operatorname{diam}(M)^{\alpha})^2 (1 + \varepsilon \operatorname{diam}(M)^{\alpha}).$$

Moreover, for γ e $\hat{\gamma}$ such that we can apply (C), we have

$$\frac{\int_{\hat{\gamma}_j} \varphi \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi \, d\mu_{\gamma_j}} \le 1 + cd(\hat{\gamma}_j, \gamma_j)^{\alpha} \le 1 + c \cdot \operatorname{diam}(M)^{\alpha},$$

implying that

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}} < \left(1 + b \log\left(\frac{1+\lambda}{1-\lambda}\right)\right)^2 (1 + \max\{\kappa, c, \varepsilon\} \operatorname{diam}(M)^{\alpha})^4,$$

finishing the proof of the proposition. □

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9. *Exponential decay of correlations*

In this section, we prove the exponential decay of correlation for Hölder continuous observables.

In our context, the transfer operator is just $\mathcal{L}(\varphi) = \varphi \circ f^{-1}$ acting in the space of continuous observables.

The adjoint operator of $\mathcal{L}$ is

$$\int \tilde{\mathcal{L}}\varphi \, d\mu = \int \varphi \, d\mathcal{L}^*\mu.$$

for all continuous φ and all probability measures μ. Instead of the non-uniformly expanding case, any invariant probability is an eigenmeasure of the transfer operator's adjoint.

PROPOSITION 9.1. *If f is invertible, then $\mathcal{L}^*(\mu) = \mu$ if and only if μ is f-invariant.*

Proof. Let φ be a continuous function. If $\mathcal{L}^*(\mu) = \mu$ then

$$\int \varphi \circ f^{-1} \, d\mu = \int \mathcal{L}(\varphi) \, d\mu = \int \varphi \, d\mathcal{L}^*(\mu) = \int \varphi \, d\mu.$$

Now, given an f-invariant measure μ, we have

$$\int \varphi \, d\mathcal{L}^*(\mu) = \int \tilde{\mathcal{L}}(\varphi) \, d\mu = \int \varphi \circ f^{-1} \, d\mu = \int \varphi \, d\mu. \qquad \square$$

Other important relation obtained from the f-invariance of a measure μ is that

$$\int (\varphi \circ f^n)\psi \, d\mu = \int \varphi \mathcal{L}^n(\psi) \, d\mu. \tag{9.1}$$

Indeed, as $\tilde{\mathcal{L}}(\varphi) = \varphi \circ f^{-1}$ we have

$$\int (\varphi \circ f)\psi \, d\mu = \int \varphi \circ f \circ f^{-1}\psi \circ f^{-1} \, d\mu = \int \varphi \mathcal{L}(\psi) \, d\mu,$$

and by induction,

$$\int (\varphi \circ f^n)\psi \, d\mu = \int \varphi \mathcal{L}^n(\psi) \, d\mu.$$

The exponential decay of correlations of the maximizing entropy measure will be a consequence of the strict invariance of the main cone, that we proved in the last section, and the following lemma.

LEMMA 9.2. *For all $\varphi \in C^\alpha(M)$, there exists $K(\varphi) > 0$ such that $\varphi + K(\varphi) \in C(b, c, \alpha)$.*

Proof. First we prove that there exists $K_3 = K_3(\varphi) > 0$ such that $\varphi + K_3$ satisfies condition (C) in the definition of cone $C(b, c, \alpha)$. The projections between stable leaves guarantee that

$$\int_\gamma \varphi \, d\mu_\gamma = \int_{\tilde{\gamma}} \varphi \circ \pi \, d\mu_{\tilde{\gamma}}.$$

Given $\varphi \in C^\alpha(M)$, we have

$$\frac{\varphi(x) - \varphi(\pi(x))}{d(\gamma, \tilde{\gamma})} \leq \frac{\varphi(x) - \varphi(\pi(x))}{d(\pi(x), x)} \leq |\varphi|_\alpha.$$

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So,

$$\sup_{\gamma,\widetilde{\gamma}}\left\{\frac{|\int_\gamma \varphi\, d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi\, d\mu_{\widetilde{\gamma}}|}{d(\gamma,\,\widetilde{\gamma})}\right\} \le |\varphi|_\alpha < \infty.$$

On the other hand, for $K > 0$, we have $\inf_\gamma\{\int_\gamma(\varphi + K)\, d\mu_\gamma\} = \inf_\gamma\{\int_\gamma \varphi\, d\mu_\gamma\} + K$. It is sufficient to choose $K_3 = K_3(\varphi) > 0$ such that

$$c \inf_\gamma\left\{\int_\gamma (\varphi + K_3)\, d\mu_\gamma\right\} > |\varphi|_\alpha.$$

In order to see that there exists $K_2 = K_2(\varphi)$ such that $\varphi + K_2$ satisfies condition (B), just note that

$$\sup_{\rho',\rho''\in\mathcal{D}_1(\gamma)}\left\{\frac{|\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma|}{\theta(\rho',\,\rho'')}\right\} < \infty.$$

Indeed, as $\rho',\, \rho'' \in \mathcal{D}_1(\gamma)$, we have $\rho'/\rho'' \le e^{\theta(\rho',\rho'')}$ and so, for all bounded φ,

$$\begin{aligned}\left|\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma\right| &= \left|\int_\gamma\left(\frac{\rho'}{\rho''} - 1\right)\varphi\rho''\, d\mu_\gamma\right| \le \int_\gamma\left|\frac{\rho'}{\rho''} - 1\right||\varphi|\rho''\, d\mu_\gamma\\ &\le \sup\left|\frac{\rho'}{\rho''} - 1\right| \sup\varphi \sup\rho'' = \left|\sup\frac{\rho'}{\rho''} - 1\right| \sup\varphi \sup\rho''\\ &\le |e^{\theta(\rho',\rho'')} - 1| \sup\varphi \sup\rho''.\end{aligned}$$

Let B such that $\sup(\varphi + B) = 1$. It follows that

$$\begin{aligned}\frac{|\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma|}{\theta(\rho',\,\rho'')} &= \frac{|\int_\gamma(\varphi + B)\rho'\, d\mu_\gamma - \int_\gamma(\varphi + B)\rho''\, d\mu_\gamma|}{\theta(\rho',\,\rho'')}\\ &\le \frac{(e^{\theta(\rho',\rho'')} - 1)\sup\rho''}{\theta(\rho',\,\rho'')}.\end{aligned}$$

If $\theta(\rho',\,\rho'') < 1$ then $(e^{\theta(\rho',\rho'')} - 1)/\theta(\rho',\,\rho'') < 2$, and as $\rho'' \in \mathcal{D}_1(\gamma)$, we have

$$\frac{|\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma|}{\theta(\rho',\,\rho'')} \le 2(1 + \kappa\ \mathrm{diam}(M)^\alpha).$$

Now, if $\theta(\rho',\,\rho'') \ge 1$, we obtain

$$\begin{aligned}\frac{|\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma|}{\theta(\rho',\,\rho'')} &\le \left|\int_\gamma(\varphi + B)\rho'\, d\mu_\gamma - \int_\gamma(\varphi + B)\rho''\, d\mu_\gamma\right|\\ &\le \int_\gamma\left|(\varphi + B)(\rho' - \rho'')\right| d\mu_\gamma\\ &\le \sup(\varphi + B)(\sup\rho' + \sup\rho'') \le 2(1 + \kappa\ \mathrm{diam}(M)^\alpha)\end{aligned}$$

and this implies

$$\sup_{\rho',\rho''\in\mathcal{D}_1(\gamma)}\left\{\frac{|\int_\gamma \varphi\rho'\, d\mu_\gamma - \int_\gamma \varphi\rho''\, d\mu_\gamma|}{\theta(\rho',\,\rho'')}\right\} < \infty.$$

The choice of $K_2 = K_2(\varphi)$ is similar to what we have done for (C). On condition (A) , since φ is continuous with compact domain, there exists $K_1 = K_1(\varphi)$ such that $\varphi + K_1 > 0$ and so $\int_\gamma(\varphi + K_1)\rho\, d\mu_\gamma > 0$, for all $\gamma \in F^s_{\mathrm{loc}}$ and $\rho \in \mathcal{D}(\gamma)$. We complete the proof by taking $K(\varphi) = \max\{K_1,\, K_2,\, K_3\}$. □

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Now, denote by $\mu_\gamma \times \nu$ the measure given by

$$\mu_\gamma \times \nu(\varphi) := \int \int_\gamma \varphi \, d\mu_\gamma \, d\nu(\gamma).$$

By unicity of the maximal entropy probability measure, we notice that $\mu = \mu_\gamma \times \nu$, where ν is the maximal entropy probability measure for g. Indeed, let us first show that $\mu_\gamma \times \nu$ is an f-invariant probability. In fact, for all $x \in M$, given $\gamma = \Pi_\Lambda^{-1}(x)$ and $\gamma_j = \Pi_\Lambda^{-1}(x_j)$, with $f(\gamma_j) \subset \gamma$ and $g(x_j) = x$ we have $\mu_\gamma(A) = (1/p) \sum_{j=1}^p \mu_{\gamma_j}(f^{-1}(A))$. By Castro and Varandas [**CV13**], ν is an eigenmeasure of the adjoint $\mathcal{L}_{g,\phi}^*$ given by

$$\mathcal{L}_{g,\phi}(\varphi)(x) := \sum_{g(x_j)=x} e^{\phi(x_j)} \varphi(x_j),$$

for constant potential ϕ. More precisely, if r is the spectral radius of $\mathcal{L}_{g,\phi}^*$, which is equal to the degree of g, then $\mathcal{L}_{g,\phi}^*(\nu) = r\nu$. By normalizing $\mathcal{L}_{g,\phi}^*$ by $r = p$, we obtain, for any continuous ϕ,

$$\int \varphi(x) \, d\nu = \frac{1}{r} \int \varphi(x) \, d\mathcal{L}_{g,\phi}^* \nu = \frac{1}{r} \int \mathcal{L}_{g,\phi}(\varphi)(x) \, d\nu = \int \frac{1}{p} \sum_{j=1}^p \varphi(x_j) \, d\nu.$$

Therefore, for $A \in \mathcal{A}_0$, we deduce

$$\begin{aligned}(\mu_\gamma \times \nu)(f^{-1}(A)) = \mu_\gamma \times \nu(\chi_{f^{-1}(A)}) &= \int \int_\gamma \chi_{f^{-1}(A)} \, d\mu_\gamma \, d\nu \\ &= \int \mu_\gamma(f^{-1}(A)) \, d\nu = \int \frac{1}{p} \sum_{j=1}^p \mu_{\gamma_j}(f^{-1}(A)) \, d\nu \\ &= \int \mu_\gamma(A) \, d\nu = \int \int_\gamma \chi_A \, d\mu_\gamma \, d\nu = \mu_\gamma \times \nu(A).\end{aligned}$$

As we have shown in previous sections, this implies the same equality for any Borelian A.

Furthermore, $\mu_\gamma \times \nu(A) = \mu(A)$. Indeed, let $A = \Pi_\Lambda^{-1}(A_N)$, with $A_N \in \mathcal{A}_N$.

On the one hand, we have that

$$\mu(\Pi_\Lambda^{-1}(A_N)) = \nu(\Pi_\Lambda(\Pi_\Lambda^{-1}(A_N))) = \nu(A_N) = \int_N \chi_{A_N} \, d\nu,$$

and on the other hand,

$$\mu_\gamma \times \nu(\Pi_\Lambda^{-1}(A_N)) = \int \int_\gamma \chi_{\Pi_\Lambda^{-1}(A_N)} \, d\mu_\gamma \, d\nu.$$

As $\chi_{\Pi_\Lambda^{-1}(A_N)}(x) = \chi_{A_N}(\Pi_\Lambda(x))$, and for all γ there exists $x_0 \in N$ such that $\gamma = \Pi_\Lambda^{-1}(x_0)$,

$$\int_\gamma \chi_{\Pi_\Lambda^{-1}(A_N)}(x) \, d\mu_\gamma = \int_\gamma \chi_{A_N}(\Pi_\Lambda(x)) \, d\mu_\gamma = \int_\gamma \chi_{A_N}(x_0) \, d\mu_\gamma = \chi_{A_N}(x_0)$$

and then, $\mu_\gamma \times \nu(A) = \mu(A)$ for all $A \in \mathcal{A}_0$. Now, given $A \in \mathcal{A} = \bigcup_{n=0}^\infty \mathcal{A}_n$, as $\mathcal{A}_n = f^n(\mathcal{A}_0)$, we have that there exist $n \in \mathbb{N}$ and $A_0 \in \mathcal{A}_0$ such that $A = f^n(A_0)$. Therefore,

$$\mu_\gamma \times \nu(A) = \mu_\gamma \times \nu(f^n(A_0)) = \mu_\gamma \times \nu(A_0).$$

Since μ is f-invariant, $\mu(A) = \mu(f^n(A_0)) = \mu(A_0)$, and we conclude that $\mu = \mu_\gamma \times \nu$.

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THEOREM B. *The measure μ has exponential decay of correlations for Hölder continuous observables.*

Proof. We should prove that for α-Hölder observables φ, ψ, there exist $0 < \tau < 1$ and $K(\varphi, \psi) > 0$ such that

$$\left| \int (\varphi \circ f^n)\psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n \quad \text{for all } n \geq 1.$$

By (9.1), this is equivalent to proving

$$\left| \int \varphi \tilde{\mathcal{L}}^n(\psi) \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq K(\varphi, \psi) \cdot \tau^n \quad \text{for all } n \geq 1.$$

We start with the case $\varphi_{|\gamma} \in \mathcal{D}(\gamma)$, for all $\gamma \in \mathcal{F}^s_{\text{loc}}$ and $\psi \in C(b, c, \alpha)$. We also assume $\int \varphi \, d\mu \neq 0$ and $\int \psi \, d\mu = 1$.

Recall that $\mathcal{L}(1) = 1 \circ f = 1$. Since $\varphi_{|\gamma} \in \mathcal{D}(\gamma)$ for all $\gamma \in \mathcal{F}^s_{\text{loc}}$, by (A) we have

$$\frac{\int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) \, d\mu_\gamma}{\int_\gamma \varphi \, d\mu_\gamma} \leq \beta_+(\tilde{\mathcal{L}}^n(\psi), 1).$$

Since ψ is normalized, we have $\int \tilde{\mathcal{L}}^n(\psi) \, d\mu = \int \psi \, d\mu = 1$. As $\mu = \mu_\gamma \times \nu$,

$$\int \left(\int_\gamma \tilde{\mathcal{L}}^n(\psi) \, d\mu_\gamma \right) d\nu = \int \tilde{\mathcal{L}}^n(\psi) \, d\mu = 1,$$

and so there exists $\hat{\gamma}$ such that $\int_{\hat{\gamma}} \tilde{\mathcal{L}}^n(\psi) \, d\mu_{\hat{\gamma}} \leq 1$. We conclude that

$$\alpha_+(\tilde{\mathcal{L}}^n(\psi), 1) \leq \frac{\int_{\hat{\gamma}} \tilde{\mathcal{L}}^n(\psi) \, d\mu_{\hat{\gamma}}}{\int_{\hat{\gamma}} d\mu_{\hat{\gamma}}} = \int_{\hat{\gamma}} \tilde{\mathcal{L}}^n(\psi) \, d\mu_{\hat{\gamma}} \leq 1,$$

and for all $\gamma \in \mathcal{F}^s_{\text{loc}}$,

$$\frac{\int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) \, d\mu_\gamma}{\int_\gamma \varphi \, d\mu_\gamma} \leq \frac{\beta_+(\tilde{\mathcal{L}}^n(\psi), 1)}{\alpha_+(\tilde{\mathcal{L}}^n(\psi), 1)} \leq e^{\Theta_+(\tilde{\mathcal{L}}^n(\psi), 1)} \leq e^{\Theta(\tilde{\mathcal{L}}^n(\psi), 1)}.$$

By Proposition 7.3 and Proposition 8.1, since the cone $C(\sigma b, \sigma c, \alpha)$ has Θ-diameter less than or equal to Δ, it follows from Proposition 6.2 that there exists $0 < \tau < 1$ such that for all $\varphi, \psi \in C(b, c, \alpha)$ we have $\Theta(\tilde{\mathcal{L}}^n(\varphi), \tilde{\mathcal{L}}^n(\psi)) \leq \Delta \tau^{n-1}$. In consequence,

$$\frac{\int \varphi \tilde{\mathcal{L}}^n(\psi) \, d\mu}{\int \varphi \, d\mu} = \frac{\int \int_\gamma \varphi \tilde{\mathcal{L}}^n(\psi) \, d\mu_\gamma \, d\nu}{\int \int_\gamma \varphi \, d\mu_\gamma \, d\nu} \leq e^{\Theta(\tilde{\mathcal{L}}^n(\psi), 1)} \leq e^{\Delta \tau^{n-1}}.$$

Note now that $\lim_{n\to\infty} ((e^{\Delta\tau^{n-1}} - 1)/\tau^n) = \Delta/\tau$. So, there exists $\tilde{\Delta} > 0$ such that $e^{\Delta\tau^{n-1}} - 1 < \tilde{\Delta}\tau^n$, for all $n \in \mathbb{N}$. This implies that

$$\left| \int \varphi \, d\mu \right| \left| \frac{\int \varphi \tilde{\mathcal{L}}^n(\psi) \, d\mu}{\int \varphi \, d\mu} - 1 \right| \leq \left| \int \varphi \, d\mu \right| (e^{\Delta\tau^{n-1}} - 1) \leq \left| \int \varphi \, d\mu \right| \tilde{\Delta}\tau^n.$$

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If $\int \psi \, d\mu \neq 1$, then

$$\left|\int \varphi\tilde{\mathcal{L}}^n(\psi)\,d\mu - \int \varphi\,d\mu \int \psi\,d\mu\right| = \left|\int \psi\,d\mu\right|\left|\int \varphi\tilde{\mathcal{L}}^n\left(\frac{\psi}{\int \psi\,d\mu}\right)d\mu - \int \varphi\,d\mu\right| \leq \left|\int \psi\,d\mu\right|\left|\int \varphi\,d\mu\right|\tilde{\Delta}\tau^n$$

for all $n \geq 1$.

By Lemma 9.2, given an α-Hölder continuous function ψ, there exists $K(\psi) > 0$ such that $\psi + K(\psi) \in C(b, c, \alpha)$. Therefore, $\psi = \psi + K(\psi) - K(\psi)$ and, noting that $\int \varphi\mathcal{L}^n(K(\psi))\,d\mu = \int \varphi\,d\mu \int K(\psi)\,d\mu$, we obtain

$$\begin{aligned}&\left|\int \varphi\tilde{\mathcal{L}}^n(\psi)\,d\mu - \int \varphi\,d\mu \int \psi\,d\mu\right| \\ &= \left|\int \varphi\tilde{\mathcal{L}}^n(\psi + K(\psi))\,d\mu - \int \varphi\,d\mu \int (\psi + K(\psi))\,d\mu\right| \\ &\leq \left(\left|\int \psi\,d\mu\right| + K(\psi)\right)\left|\int \varphi\,d\mu\right|\tilde{\Delta}\tau^n.\end{aligned}$$

Now, given an α-Hölder φ, note that there exists $K(\varphi) \in \mathbb{R}$ such that $\varphi_{|\gamma} + K(\varphi) + B \in \mathcal{D}(\gamma)$ for all $\gamma \in \mathcal{F}^s_{\text{loc}}$, and $\int \varphi + K(\varphi) + B\,d\mu > 0$ for all $B > 0$. Indeed,

$$|\varphi_{|\gamma} + K(\varphi)|_\alpha < \kappa \inf\{\varphi_{|\gamma} + K(\varphi)\}$$

if and only if

$$K(\varphi) > \frac{|\varphi_{|\gamma}|_\alpha}{\kappa} - \inf\{\varphi_{|\gamma}\}.$$

Set $K(\varphi) = \sup_{\gamma\in\mathcal{F}^s_{\text{loc}}}\{||\varphi_{|\gamma}|_\alpha/\kappa|\} - \inf\varphi$. Observe that $K(\varphi) \leq |\varphi|_\alpha/\kappa - \inf\varphi < \infty$. As $\varphi_{|\gamma} + K(\varphi) \geq |\varphi_{|\gamma}|_\alpha/\kappa \geq 0$ for all $\gamma \in \mathcal{F}^s_{\text{loc}}$, it follows that $\varphi_{|\gamma} + K(\varphi) + B \in \mathcal{D}(\gamma)$, and $\int (\varphi + K(\varphi))\,d\mu + B > 0$ for all $B > 0$. Analogously to the last case,

$$\begin{aligned}&\left|\int \varphi\tilde{\mathcal{L}}^n(\psi)\,d\mu - \int \varphi\,d\mu \int \psi\,d\mu\right| \\ &\leq \left(\left|\int \psi\,d\mu\right| + K(\psi)\right)\left(\left|\int \varphi\,d\mu\right| + K(\varphi) + B\right)\tilde{\Delta}\tau^n,\end{aligned}$$

and since B is any positive number,

$$\left|\int \varphi\tilde{\mathcal{L}}^n(\psi)\,d\mu - \int \varphi\,d\mu \int \psi\,d\mu\right| \leq \left(\left|\int \psi\,d\mu\right| + K(\psi)\right)\left(\left|\int \varphi\,d\mu\right| + K(\varphi)\right)\tilde{\Delta}\tau^n.$$

Since $|\int \varphi\,d\mu| - \inf\varphi \geq 0$, we have $|\int \varphi\,d\mu| + K(\varphi) \geq 0$. By taking

$$K(\varphi, \psi) := \left(\left|\int \psi\,d\mu\right| + K(\psi)\right)\left(\left|\int \varphi\,d\mu\right| + K(\varphi)\right)\tilde{\Delta},$$

we conclude the proof of the theorem. □

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10. *Central limit theorem*

Let $\mathcal{G}$ be the Borel σ-algebra of M and let $\mathcal{G}_n := f^{-n}(\mathcal{G})$ be a non-increasing family of σ-algebras. A function $\xi : M \to \mathbb{R}$ is $\mathcal{G}_n$-measurable if and only if $\xi = \xi_n \circ f^n$ for some $\mathcal{G}$- measurable ξ_n. Let $L^2(\mathcal{G}_n) = \{\xi \in L^2(\mu) \mid \xi \text{ is } \mathcal{G}_n\text{-measurable}\}$. Note that $\mathcal{L}^2(\mathcal{G}_{n+1}) \subset \mathcal{L}^2(\mathcal{G}_n)$ for each $n \geq 0$. Given $\varphi \in L^2(\mu)$, we will denote by $\mathbb{E}(\varphi|\mathcal{G}_n)$ the L^2-orthogonal projection of φ on $L^2(\mathcal{G}_n)$.

We will apply the following adaption of a result due to Gordin, whose proof can be found in [**Vi97**].

THEOREM 10.1. (Gordin) *Let $(M, \mathcal{F}, \mu)$ be a probability space, and let $\phi \in L^2(\mu)$ be such that $\int \phi \, d\mu = 0$. Assume that $f : M \to M$ is an invertible bimeasurable map and that μ is an f-ergodic invariant probability. Let $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_n := f^{-n}(\mathcal{F}_0)$, $n \in \mathbb{Z}$, is a non-increasing family of σ-algebras. Define*

$$\sigma_\phi^2 := \int \phi^2 \, d\mu + 2 \sum_{j=1}^{\infty} \phi \cdot (\phi \circ f^j) \, d\mu.$$

Assume

$$\sum_{n=0}^{\infty} \|\mathbb{E}(\phi|\mathcal{F}_n)\|_2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|\phi - \mathbb{E}(\phi|\mathcal{F}_{-n})\|_2 < \infty.$$

Then $\sigma_\phi < \infty$ and $\sigma_\phi = 0$ if and only if $\phi = u \circ f - u$ for some $u \in L^1(\mu)$. Moreover, if $\sigma_\phi > 0$ then for any interval $A \subset \mathbb{R}$,

$$\mu\left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} (\phi(f^j(x))) \in A\right) \to \frac{1}{\sigma_\phi \sqrt{2\pi}} \int_A e^{-t^2/(2\sigma_\phi^2)} \, dt$$

as $n \to \infty$.

Let $\mathcal{F}_0$ be the σ-algebra whose elements are Borelian subsets of Λ which are union local stable leaves (intersected with Λ). Note that if φ is F_0-measurable then φ is constant along local stable leaves.

We start by proving a statement of exponential decay of correlation concerning a function in $L^1(F_0)$.

PROPOSITION 10.2. *Let $\varphi \in L^1(F_0)$ and ψ be a α-Hölder continuous function. Then there exist constants $0 < \tau < 1$ and $C(\psi) > 0$ such that*

$$\left| \int (\varphi \circ f^n) \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \right| \leq C(\psi) \int |\varphi| \, d\mu \cdot \tau^n$$

for all $n \geq 1$.

Proof. Since φ is F_0-measurable, it is constant restricted to local stable leaves, so $|\varphi|_\gamma|_\alpha = 0$, for all $\gamma \in \mathcal{F}^s_{\text{loc}}$. Suppose $\varphi \geq 0$, and let $K(\varphi)$ and $K(\psi)$ as in the proof of Theorem B. Therefore,

$$K(\varphi) = \sup_{\gamma \in \mathcal{F}^s_{\text{loc}}} \left\{ \left| \frac{|\varphi|_\gamma|_\alpha}{\kappa} \right| \right\} - \inf \varphi = -\inf \varphi.$$

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Since $|\int \varphi\, d\mu| - \inf \varphi \le \int |\varphi|\, d\mu$, just as in the proof of Theorem B, it follows that

$$\left|\int (\varphi \circ f^n)\psi\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu\right| \le \left(\left|\int \psi\, d\mu\right| + K(\psi)\right) \int |\varphi|\, d\mu \cdot \tau^n.$$

Now, we can write $\varphi = \varphi^+ - \varphi^-$ where $\varphi^\pm = \frac{1}{2}(|\varphi| \pm \varphi)$. Noting that $\int |\varphi^\pm|\, d\mu \le \int |\varphi|\, d\mu$ from linearity of the integral, we obtain

$$\left|\int (\varphi \circ f^n)\psi\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu\right| \le C(\psi) \int |\varphi|\, d\mu \cdot \tau^n$$

with $C(\psi) := 2(|\int \psi\, d\mu| + K(\psi))$. □

As a consequence of the proposition we are able to prove the following lemma.

LEMMA 10.3. *For every Hölder continuous function φ with $\int \varphi\, d\mu = 0$, there is $R = R(\varphi)$ such that $\|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 \le R\tau^n$ for all $n \ge 0$.*

Proof. By the last proposition, if $\psi \in L^1(F_0)$ and $\int \psi\, d\mu \le 1$, then

$$\left|\int (\psi \circ f^n)\varphi\, d\mu - \int \psi\, d\mu \int \varphi\, d\mu\right| \le C(\varphi) \cdot \tau^n.$$

As $\|\psi\|_1 \le \|\psi\|_2$ and $\int \varphi\, d\mu = 0$, we have

$$\begin{aligned}\|\mathbb{E}(\varphi|\mathcal{F}_n)\|_2 &= \sup\left\{\int \xi\varphi\, d\mu;\ \xi \in L^2(\mathcal{F}_n),\ \|\xi\|_2 = 1\right\} \\ &= \sup\left\{\int (\psi \circ f^n)\varphi\, d\mu;\ \psi \in L^2(F_0),\ \|\psi\|_2 = 1\right\} \le R(\varphi)\tau^n.\end{aligned}$$ □

Now we can prove the following theorem.

THEOREM C. (Central limit theorem) *Let μ be the maximal entropy probability for $f : \Lambda \to \Lambda$, as in (2.2). Given a Hölder continuous function φ and*

$$\sigma_\varphi^2 := \int \phi^2 d\mu + 2\sum_{j=1}^{\infty} \int \phi \cdot (\phi \circ f^j)\, d\mu \quad \text{with } \phi = \varphi - \int \varphi\, d\mu,$$

$\sigma_\varphi < \infty$ and $\sigma_\varphi = 0$ if and only if $\varphi = u \circ f - u$ for some $u \in L^1(\mu)$. Moreover, if $\sigma_\varphi > 0$ then for all intervals $A \subset \mathbb{R}$,

$$\lim_{n\to\infty} \mu\left(x \in M : \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1}\left(\varphi(f^j(x)) - \int \varphi\, d\mu\right) \in A\right) = \frac{1}{\sigma_\varphi\sqrt{2\pi}} \int_A e^{-t^2/(2\sigma_\varphi^2)}\, dt.$$

Proof. By the last lemma, $\sum_{n=0}^{\infty} \|\mathbb{E}(\phi|\mathcal{F}_n)\|_2 < \infty$, so the first condition for Gordin's theorem holds. The second condition follows from the Hölder continuity of φ. In fact, $\mathbb{E}(\phi, \mathcal{F}_{-n})$ is constant in each n-image $\eta = f^n(\gamma)$ of a stable leaf γ, and

$$\inf(\phi|_\gamma) \le \mathbb{E}(\phi, \mathcal{F}_{-n}) \le \sup(\phi|_\gamma).$$

Since the diameter of η is less than $C_s\lambda_s^n$ for some constant C_s which does not depend on γ, $\lambda_s \in (0, 1)$, and ϕ is (A, α)-Hölder for some constant $A > 0$, we obtain that

$$\|\phi - \mathbb{E}(\phi, \mathcal{F}_{-n})\|_2 \le \|\phi - \mathbb{E}(\phi, \mathcal{F}_{-n})\|_0 \le AC_s^\alpha \lambda_s^{\alpha n},$$

which guarantees $\sum_{n=0}^{\infty} \|\phi - \mathbb{E}(\phi, \mathcal{F}_{-n})\|_2 < \infty$. The result then follows as a consequence of Gordin's theorem. □

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Remark 10.4. We have seen that the invariance of a suitable cone of distributions with respect to the transfer operator implied in a rather economic way the exponential decay of correlations and the central limit. Once we have such a cone, one can define a (basically unique) anisotropic space E associated with it. Such a space is a Banach space in which the transfer operator exhibits a spectral gap. The procedure to define such a space is very simple. Roughly speaking, such a cone C defines a order relation $\prec$ in a Banach Lattice B of bounded functions given by

$$v \prec w \Leftrightarrow w - v \in C.$$

Let e be some function in B such that for any function φ in B, there exists a constant c_φ such that $-c_\varphi e \prec \varphi \prec c_\varphi e$. For example, in our case, take $e \equiv 1$. Setting $\|\varphi\| := \inf\{c_\phi;\ -c_\varphi e \prec \varphi \prec c_\varphi e\}$, one easily checks that this is a norm. The anisotropic space is obtained by completing B with respect to this norm. This approach, associated with Lasota–Yorke estimates instead of projective cones, was used by Baladi, Gouezel, Liverani and Tsujii, among others, in several works [**Bal05**, **BT07**, **BG10**, **BL12**], to study fine spectral properties of the transfer operator in hyperbolic contexts.

11. *Systems derived from Anosov*

In this section, we study partially hyperbolic dynamics as in the second setting described in §2. Recall that such a setting contains an open class of derived-from-Anosov systems. Using the techniques and results in the sections before, we prove again the strict invariance of a suitable cone of functions by the transfer operator. Furthermore, from the convergence of such a cone, we construct a measure exhibiting exponential decay of correlations and satisfying the central limit theorem.

11.1. *Invariant Cones.* Given $x \in \Lambda$, we will denote by γ the intersection of the local (strong) stable manifold x with the Markov rectangle in which x belongs. By the mixing property of the Markov partition, there is no great loss of generality in supposing that for any rectangles R_i and R_j of $\mathcal{R}$, we have

$$f(R_i) \cap R_j \neq \emptyset.$$

Otherwise, one can work with some positive iterate of f. Given $\gamma = W^s_{\mathrm{loc}}(x) \cap R_i$, for some fixed Markov rectangle R_i, by the Markovian property we have $\gamma = \bigcup_{j=1}^{p_\gamma} f(\gamma_j)$ for γ_j, $j = 1, \ldots, p_\gamma$ corresponding to strong stable manifolds whose images by f intersect the interior of γ. Observe also that by the Markovian property, all γ in R_i have pre-images in the same rectangles. In particular, p_γ depends only on R_i. This is important for the calculation of the finite diameter of the cone and slightly modifies the definition of the auxiliary probabilities μ_γ on the leaves $\gamma \in \mathcal{F}^s_{\mathrm{loc}}$.

Let $\gamma \in \mathcal{F}^s_{\mathrm{loc}}$, $\gamma = \bigcup_{j=1}^{p_\gamma} f(\gamma_j)$. Given $n \in \mathbb{N}$, $n \geq 1$, by induction we can write

$$\gamma = \bigcup_{i_1=1}^{p_{i_0}} \bigcup_{i_2=1}^{p_{i_1}} \cdots \bigcup_{i_n=1}^{p_{i_{n-1}}} f^n(\gamma_{i_1 \cdots i_n})$$

where $f(\gamma_{i_1 \cdots i_{n+1}}) \subset \gamma_{i_1 \cdots i_n}$, $\gamma_{i_1 \cdots i_n} = \bigcup_{i_{n+1}=1}^{p_{i_n}} f(\gamma_{i_1 \cdots i_{n+1}})$ and p_{i_k}, $k \in \mathbb{N}$. Note also that $p_{i_0} = p_\gamma$. Since the last union is disjoint, for each $n \in \mathbb{N}$, $n \geq 1$, the sets $f^n(\gamma_{i_1 \cdots i_n})$ with

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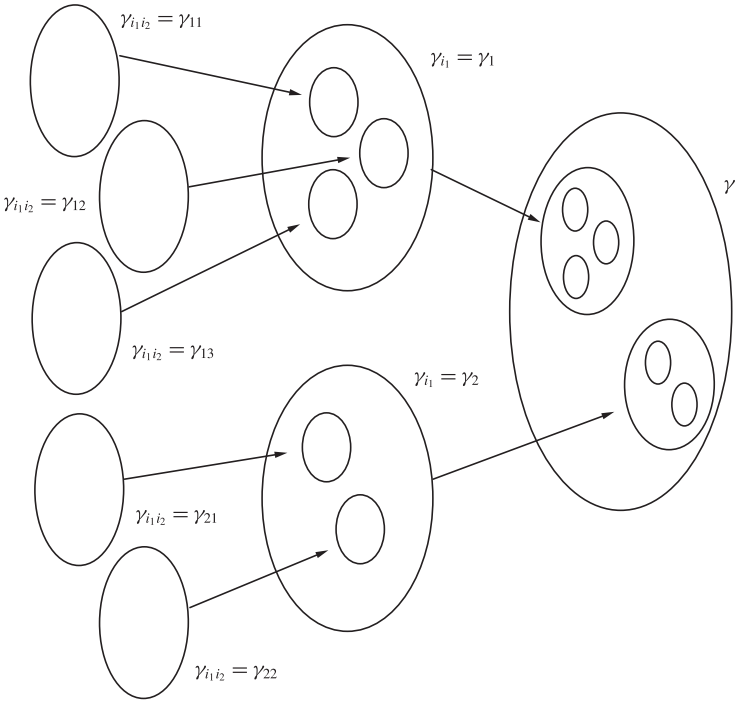


FIGURE 2. Mass distribution for derived-from-Anosov systems.

$i_k \in \{1, \ldots, p_{i_{k-1}}\}$, $k \in \mathbb{N}$, $k \geq 1$ define a partition for γ. So, it is natural to define a probability measure in γ as:

$$\mu_\gamma(f^n(\gamma_{i_1 \cdots i_n})) := \frac{1}{p_{i_0} p_{i_1} \cdots p_{i_{n-1}}}.$$

Since $f^n(\gamma_{i_1 \cdots i_n}) \subset \gamma$, we have that μ_γ is a Borelian measure of γ (see Figure 2). In fact, it is a probability measure, as

$$\mu_\gamma(\gamma) = \sum_{i_1=1}^{p_{i_0}} \sum_{i_2=1}^{p_{i_1}} \cdots \sum_{i_n=1}^{p_{i_{n-1}}} \mu_\gamma(f^n(\gamma_{i_1 \cdots i_n})) = \sum_{i_1=1}^{p_{i_0}} \sum_{i_2=1}^{p_{i_1}} \cdots \sum_{i_n=1}^{p_{i_{n-1}}} \frac{1}{p_{i_0} p_{i_1} \cdots p_{i_{n-1}}} = 1.$$

Of course, such measure can be seen as a conditional measure on Λ. Let A be a Borelian subset of Λ. So,

$$\begin{aligned} \mu_\gamma(A) = \mu_\gamma(A \cap \gamma) &= \mu_\gamma\left(A \cap \bigcup_{j=1}^{p_\gamma} f(\gamma_j)\right) \\ &= \mu_\gamma\left(\bigcup_{j=1}^{p_\gamma} (A \cap f(\gamma_j))\right) = \sum_{j=1}^{p_\gamma} \mu_\gamma(A \cap f(\gamma_j)). \end{aligned}$$

Defining $\mu_{\gamma_j}(A) := p_\gamma \mu_\gamma(f(A \cap \gamma_j))$, we obtain that $(1/p_\gamma)\mu_{\gamma_j}(f^{-1}(A)) = \mu_\gamma(A \cap$

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$f(\gamma_j))$ and then

$$\mu_\gamma(A) = \frac{1}{p_\gamma} \sum_{j=1}^{p_\gamma} \mu_{\gamma_j}(f^{-1}(A)).$$

Moreover, $\mu_{\gamma_j}(f^n(\gamma_{j_1 \cdots j_n})) = 1/(p_{j_0} p_{j_1} \cdots p_{j_{n-1}})$. In particular, it is a probability in γ_j. Note also that $\mu_\gamma(A \cap f(\gamma_j)) = (1/p_\gamma)\mu_{\gamma_j}(f^{-1}(A))$ implies that for any measurable set A, its characteristic function χ_A satisfies

$$\int_{f(\gamma_j)} \chi_A \, d\mu_\gamma = \frac{1}{p_\gamma} \int_{\gamma_j} \chi_A \circ f \, d\mu_{\gamma_j}.$$

Therefore, by the dominated convergence theorem, for any continuous function $g : \Lambda \to \mathbb{R}$, one concludes that

$$\int_{f(\gamma_j)} g \, d\mu_\gamma = \frac{1}{p_\gamma} \int_{\gamma_j} g \circ f \, d\mu_{\gamma_j}. \tag{11.1}$$

Applying the transfer operator $\mathcal{L}(\varphi)(x) = \varphi(f^{-1}(x))e^{\phi(f^{-1}(x))}$ and using (11.1), we obtain

$$\int_\gamma \mathcal{L}(\varphi)\rho \, d\mu_\gamma = \sum_{j=1}^{p_\gamma} \int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}$$

with $\rho_j := (1/p_\gamma)\rho \circ f e^\phi$. We will adapt the same kind of main cone previously defined for a diffeomorphism semiconjugated to a Castro–Varandas map.

The main change is in condition (C). The comparison in condition (C) will concern just strong stable leaves in the same rectangle. In the same manner as before, we define the cone $\mathcal{D}(\gamma, \kappa)$ of densities $\rho : \gamma \to \mathbb{R}$ such that $\rho > 0$ and $|\rho|_\alpha < \kappa \inf \rho$. Lemma 7.1 is still valid; that is, there exists $0 < \lambda < 1$ and $\kappa > 0$ such that:

(1) for all $\gamma \in \mathcal{F}^s_{\text{loc}}$, if $\rho, \hat{\rho} \in \mathcal{D}(\gamma, \lambda\kappa)$ then $\theta(\rho, \hat{\rho}) \le 2\log((1+\lambda)/(1-\lambda))$;
(2) if $\rho \in \mathcal{D}(\gamma, \kappa)$ then $\rho_j \in \mathcal{D}(\gamma_j, \lambda\kappa)$, for all $j \in \{1, \ldots, p\}$; and
(3) if $\rho', \rho'' \in \mathcal{D}(\gamma, \kappa)$ then there exists $\Lambda_1 = 1 - ((1-\lambda)/(1+\lambda))^2$ such that $\theta_j(\rho'_j, \rho''_j) \le \Lambda_1 \theta(\rho', \rho'')$, for all $j \in \{1, \ldots, p\}$;

where θ_j and θ are, respectively, the projective metrics of $\mathcal{D}(\gamma_j, \kappa)$ and $\mathcal{D}(\gamma, \kappa)$.

Once more, denote by $\mathcal{D}_1(\gamma)$ the set of densities $\rho \in \mathcal{D}(\gamma, \kappa)$ such that $\int_\gamma \rho \, d\mu_\gamma = 1$. Given $b > 0$, $c > 0$ and κ as in Lemma 7.1, let $C[b, c, \alpha]$ be the cone of functions $\varphi \in E$ satisfying the conditions below for any $\gamma \in \mathcal{F}^s$.

(A) For all $\rho \in \mathcal{D}(\gamma, \kappa)$:

$$\int_\gamma \varphi\rho \, d\mu_\gamma > 0.$$

(B) For all $\rho', \rho'' \in \mathcal{D}_1(\gamma)$:

$$\left| \int_\gamma \varphi\rho' \, d\mu_\gamma - \int_\gamma \varphi\rho'' \, d\mu_\gamma \right| < b\theta(\rho', \rho'') \inf_{\rho \in \mathcal{D}_1(\gamma)} \left\{ \int_\gamma \varphi\rho \, d\mu_\gamma \right\}.$$

(C) Given two leaves γ and $\widetilde{\gamma}$ in the same Markov rectangle R_i:

$$\left| \int_\gamma \varphi \, d\mu_\gamma - \int_{\widetilde{\gamma}} \varphi \, d\mu_{\widetilde{\gamma}} \right| < cd(\gamma, \widetilde{\gamma})^\alpha \inf_\gamma \left\{ \int_\gamma \varphi \, d\mu_\gamma \right\}.$$

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Remark 11.1. Given γ and $\widetilde{\gamma}$ in the same rectangle, that is, $\gamma = W^s_{\mathrm{loc}}(x) \cap R_i$, $\widetilde{\gamma} = W^s_{\mathrm{loc}}(y) \cap R_i$ and $x, y \in \mathrm{int}(R_i)$, then $W^s_{\mathrm{loc}}(x) \cap f(R_j) \neq \emptyset$ if and only if $W^s_{\mathrm{loc}}(y) \cap f(R_j) \neq \emptyset$. Therefore, there exists $p \in \mathbb{N}$ such that $\gamma = \bigcup_{j=1}^{p} f(\gamma_j)$ and $\widetilde{\gamma} = \bigcup_{j=1}^{p} f(\hat{\gamma}_j)$, where $\gamma_j, \widetilde{\gamma}_j \subset R_j$ and $p_\gamma = p_{\widetilde{\gamma}} = p$.

Remark 11.2. Let γ and $\widetilde{\gamma}$ be in the same Markov rectangle. Assume without loss of generality that γ_1 and $\widetilde{\gamma}_1$ are in the same good rectangle, and so there exists $0 < \lambda_{uc} < 1$ such that $d(\gamma_1, \widetilde{\gamma}_1) \leq \lambda_{uc} d(\gamma, \widetilde{\gamma})$, and in the other cases, $j \neq 1$, there exists $\tilde{L} \geq 1$ close to 1 such that $d(\gamma_j, \widetilde{\gamma}_j) \leq \tilde{L} d(\gamma, \widetilde{\gamma})$.

The proof of the next proposition is entirely analogous to Proposition 7.3.

PROPOSITION 11.3. *There exists $0 < \sigma < 1$ such that $\mathcal{L}(C[b, c, \alpha]) \subset C[\sigma b, \sigma c, \alpha]$ for sufficiently big b and c.*

Let Θ denote the projective metrics of $C[b, c, \alpha]$. Now we occupy ourselves with the following important proposition.

PROPOSITION 11.4. *For all sufficiently big $b > 0$, $c > 0$ and $\alpha \in (0, 1]$, the diameter of $\mathcal{L}(C[b, c, \alpha])$ is finite; that is, there exists $\Delta := \sup\{\Theta(\mathcal{L}\varphi, \mathcal{L}\psi); \varphi, \psi \in C[b, c, \alpha]\} < \infty$.*

Proof. Denote by Θ_+ the projective metrics associated with the cone defined by condition (A).

Exactly as in Proposition 8.1, it follows from the expression of Θ that

$$\Theta(\varphi, \psi) \leq \Theta_+(\varphi, \psi) + \log\left(\frac{1+\sigma}{1-\sigma}\right)^2.$$

In order to prove that the Θ_+-diameter of $\mathcal{L}(C[b, c, \alpha])$ is finite, we just need an upper bound for $\Theta_+(\mathcal{L}(\varphi), 1)$ in the cone $C[b, c, \alpha]$. By the definition of Θ_+, this problem reduces to obtaining an upper bound for

$$\frac{\int_{\hat{\gamma}} \mathcal{L}\varphi \hat{\rho}\, d\mu_{\hat{\gamma}}}{\int_{\gamma} \mathcal{L}\varphi\rho\, d\mu_{\gamma}} = \frac{\sum_{j=1}^{p_{\hat{\gamma}}} \int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j\, d\mu_{\hat{\gamma}_j}}{\sum_{j=1}^{p_\gamma} \int_{\gamma_j} \varphi\rho_j\, d\mu_{\gamma_j}} \tag{11.2}$$

for $\rho \in \mathcal{D}_1(\gamma)$ and $\hat{\rho} \in \mathcal{D}_1(\hat{\gamma})$. First, we will bound the expression

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j\, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j\, d\mu_{\gamma_j}} = \frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j\, d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi\, d\mu_{\hat{\gamma}_j}} \frac{\int_{\hat{\gamma}_j} \varphi\, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\, d\mu_{\gamma_j}} \frac{\int_{\gamma_j} \varphi\, d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\rho_j\, d\mu_{\gamma_j}}$$

for γ_j and $\hat{\gamma}_j$ in the same Markov rectangle. This is the same kind of calculation as that beginning with equation (8.1) in Proposition 8.1. From that, we conclude that

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j\, d\mu_{\hat{\gamma}_j}}{\int_{\hat{\gamma}_j} \varphi\, d\mu_{\hat{\gamma}_j}} \leq \left(1 + b \log\left(\frac{1+\lambda}{1-\lambda}\right)\right) \int_{\hat{\gamma}_j} (\hat{\rho})_j\, d\mu_{\hat{\gamma}_j}$$

and

$$\frac{\int_{\gamma_j} \varphi\, d\mu_{\gamma_j}}{\int_{\gamma_j} \varphi\rho_j\, d\mu_{\gamma_j}} \leq \frac{(1 + b \log((1+\lambda)/(1-\lambda)))}{\int_{\gamma_j} \rho_j\, d\mu_{\gamma_j}}.$$

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Recall that $(\hat{\rho})_j = (1/p_{\hat{\gamma}})\hat{\rho} \circ f$ and $\rho_j = (1/p_\gamma)\rho \circ f$. Since ρ and $\hat{\rho}$ are normalized densities, it follows that $(\hat{\rho})_j \le (1/p_{\hat{\gamma}})(1 + \kappa \operatorname{diam}(M)^\alpha)$ and $\rho_j \ge (1/p_\gamma)(1 + \kappa \operatorname{diam}(M)^\alpha)^{-1}$. Therefore,

$$\frac{\int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}} < \frac{p_\gamma}{p_{\hat{\gamma}}}(1 + \kappa \operatorname{diam}(M)^\alpha)^2.$$

On the other hand, $|e^\phi|_\alpha < \varepsilon \inf e^\phi$, and so $\sup e^\phi < (1 + \varepsilon \operatorname{diam}(M)^\alpha) \inf e^\phi$. So we obtain

$$\frac{\int_{\hat{\gamma}_j} (\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \rho_j \, d\mu_{\gamma_j}} < \frac{p_\gamma}{p_{\hat{\gamma}}}(1 + \kappa \operatorname{diam}(M)^\alpha)^2 \le p_{\max}(1 + \kappa \operatorname{diam}(M)^\alpha)^2.$$

By condition (C), the following inequality also holds:

$$\frac{\int_{\hat{\gamma}_j} \varphi \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi \, d\mu_{\gamma_j}} \le 1 + cd(\hat{\gamma}_j, \gamma_j)^\alpha \le 1 + c \cdot \operatorname{diam}(M)^\alpha.$$

Hence,

$$\frac{\int_{\hat{\gamma}_j} \varphi(\hat{\rho})_j \, d\mu_{\hat{\gamma}_j}}{\int_{\gamma_j} \varphi\rho_j \, d\mu_{\gamma_j}} < p_{\max}\left(1 + b \log\left(\frac{1+\lambda}{1-\lambda}\right)\right)^2 (1 + \max\{\kappa, c\} \operatorname{diam}(M)^\alpha)^2.$$

However, given a leaf γ, there could exist more than one leaf γ_j in the same Markov rectangle. So, we rewrite equation (11.2) in the following manner:

$$\frac{\int_{\hat{\gamma}} \mathcal{L}\varphi\hat{\rho} \, d\mu_{\hat{\gamma}}}{\int_\gamma \mathcal{L}\varphi\rho \, d\mu_\gamma} = \frac{\sum_{k=1}^p \sum_{l=1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}}}{\sum_{k=1}^p \sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}}$$

such that for $R_k \in \mathcal{R}$ we have $\gamma_{k_l} \subset R_k$ for each $l \in \{1, \ldots, r_k(\gamma)\}$ and $\hat{\gamma}_{k_l} \subset R_k$ for each $l \in \{1, \ldots, r_k(\hat{\gamma})\}$. Moreover, $\sum_{k=1}^p r_k(\gamma) = p_\gamma$ and $\sum_{k=1}^p r_k(\hat{\gamma}) = p_{\hat{\gamma}}$. Since $\gamma_{k_l}, \hat{\gamma}_{k_l} \subset R_k$, by defining

$$\tilde{C} := p_{\max}\left(1 + b \log\left(\frac{1+\lambda}{1-\lambda}\right)\right)^2 (1 + \max\{\kappa, c\} \operatorname{diam}(M)^\alpha)^2,$$

we have that $(\int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}})/(\int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}) \le \tilde{C}$. Now, we are interested in finding an upper bound for $(\sum_{l=1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}})/(\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}})$. Suppose $r_k(\hat{\gamma}) \le r_k(\gamma)$. Since the parcels $\int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}$ are positive, we can assume $r_k(\hat{\gamma}) = r_k(\gamma)$. Therefore, $(\sum_{l=1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}})/(\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}) \le \tilde{C}$. Now, if $r_k(\hat{\gamma}) \ge r_k(\gamma)$, there exist $q = q(\gamma, \hat{\gamma})$ and $r \in \mathbb{N}$ such that $r_k(\hat{\gamma}) = qr_k(\gamma) + r$ for $0 \le r < r_k(\gamma)$ and $1 \le q$. So, we write $l(s)$ for $(s-1)r_k(\gamma) + l$ and obtain

$$\frac{\sum_{l=1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}}}{\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}} = \sum_{s=1}^{q}\left(\frac{\sum_{l=1}^{r_k(\gamma)} \int_{\hat{\gamma}_{k_{l(s)}}} \varphi(\hat{\rho})_{k_{l(s)}} \, d\mu_{\hat{\gamma}_{k_{l(s)}}}}{\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}}\right) + \frac{\sum_{l=qr_k(\gamma)+1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}}}{\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}}.$$

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Each term in the sum in $s \in \{1, \dots, q\}$ satisfies the earlier case, and therefore the first sum can be bounded by $q\tilde{C}$. As $r_k(\hat{\gamma}) - qr_k(\gamma) = r < r_k(\gamma)$, the last parcel also satisfies the first case. In short,

$$\frac{\sum_{l=1}^{r_k(\hat{\gamma})} \int_{\hat{\gamma}_{k_l}} \varphi(\hat{\rho})_{k_l} \, d\mu_{\hat{\gamma}_{k_l}}}{\sum_{l=1}^{r_k(\gamma)} \int_{\gamma_{k_l}} \varphi\rho_{k_l} \, d\mu_{\gamma_{k_l}}} \le (q+1)\tilde{C}.$$

Note that $q = q(\gamma, \hat{\gamma}) \le p_{\max}$, for any γ and $\hat{\gamma}$. Therefore, (11.2) is bounded by $(p_{\max} + 1)\tilde{C}$ and

$$\Delta := \sup\{\Theta(\mathcal{L}\varphi, \mathcal{L}\psi);\ \varphi, \psi \in C(b, c, \alpha)\} \le 2(p_{\max} + 1)\tilde{C}. \qquad \square$$

11.2. *Statistical properties.* Now we construct a measure in Λ which is a good candidate for maximal entropy measure. For such a probability measure, we prove the exponential decay of correlations for Hölder continuous observables and the central limit theorem.

Up to now, we have obtained an $\mathcal{L}$-invariant cone $C[b, c, \alpha]$ such that $\mathcal{L}(C[b, c, \alpha])$ has finite diameter. This guarantees, in some sense, the convergence of functions in the cone $C[\sigma b, \sigma c, \alpha]$.

Let $\hat{\eta}$ be any probability measure in the quotient space of Λ given by $\mathcal{F}^s_{\mathrm{loc}}$, and define the measure

$$\eta(\varphi) := \int \left(\int_\gamma \varphi \, d\mu_\gamma \right) d\hat{\eta}(\gamma). \tag{11.3}$$

For $\varphi \in C[b, c, \alpha]$, let

$$\varphi_n = \frac{\mathcal{L}^n(\varphi)}{\int \mathcal{L}^n(1) \, d\eta} = \mathcal{L}^n(\varphi),$$

As $\Theta_+(\varphi_m, \varphi_n) \le \Theta(\varphi_m, \varphi_n)$ it follows that φ_n is a Θ_+-Cauchy sequence. Moreover, observe also that $\Theta_+(\varphi_n, 1) = \Theta_+(\varphi_n, \mathcal{L}^n(1)) \to 0$. In particular,

$$\frac{\int_\gamma \varphi_n \, d\mu_\gamma}{\int_{\hat{\gamma}} \varphi_n \, d\mu_{\hat{\gamma}}} \to 1 \quad \text{for all } \gamma, \hat{\gamma} \in \mathcal{F}^s_{\mathrm{loc}}, \text{ uniformly.} \tag{11.4}$$

Note that $\int_\gamma \varphi \, d\mu_\gamma$, $\gamma \in \mathcal{F}^s_{\mathrm{loc}}$ is bounded from below far from zero, because $\varphi \in C([b, c, \alpha])$, say, by a constant $q > 0$. Since $\mathcal{L}$ is positive, this implies that $\int_\gamma \varphi_n \, d\mu_\gamma \ge \int_\gamma \mathcal{L}^n(q) \, d\mu_\gamma = q$. We have somewhat more; suppose that μ is an f-invariant probability satisfying a product measure property just as η in equation (11.3). Then, for every n there exist γ_n, $\hat{\gamma}_n$ such that $\int \varphi_n \, d\gamma_n \ge \int_M \varphi_n \, d\mu = \int_M \varphi \, d\mu \ge \int \varphi_n \, d\hat{\gamma}_n$. By equation (11.4), this means (even if η is not invariant) that the limit

$$\lim_{n\to\infty} \int \varphi_n \, d\eta = \mu(\varphi).$$

holds.

Let us see that there exists an f-invariant probability measure $\mu = \mu_\gamma \times \hat{\mu}$, which will coincide with the last limit, for any probability η satisfying equation (11.3). Consider in M the following: $x \sim y$ if and only if x and y belongs in the same leaf $\gamma \in \mathcal{F}^s_{\mathrm{loc}}$. Let $g : \mathcal{F}^s_{\mathrm{loc}} \to \mathcal{F}^s_{\mathrm{loc}}$ be the *quotient map* of f defined by $g(\tilde{x}) = \widetilde{f(x)}$. Take the σ-algebra

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generated by the Markov rectangles and its refinements by f^{-1}. It may be that such a σ-algebra does not coincide with the Borel σ-algebra in $M/\sim$. However, just as in [**Cas02**], most of the cylinders in the construction of $\mathcal{R} \vee f^{-1}(\mathcal{R}) \dots$ have arbitrarily small $M/\sim$-diameter. We give the following construction for the measure $\hat{\mu}$. Each rectangle has the same measure, and the sum of it is one. Take a rectangle, say, $R_1 \in \mathcal{R}$. Let $\mathcal{R}_1 := \{R_{1,1}, \dots, R_{1,n_1}\}$ be the set of connected intersections of f^{-1} and the several elements of $\mathcal{R}$. Then we set $\hat{\mu}(R_{1,1}) = \cdots = \hat{\mu}(R_{1,n_1}) := 1/n_1$. We continue inductively, taking the pre-images of the elements in $\mathcal{R}_1$, doing the mass distribution. Note that by the mild mixing property of the Markov partition, there is $0 < c < 1$ such that the cylinders in each phase of the construction have their $\hat{\mu}$-measure multiplied by a fraction less than c if compared with the cylinders in the phase before. In particular, if we take a minimal element S in $M/\sim$ with non-zero diameter, its $\hat{\mu}$-measure can be approached by a cylinder with arbitrarily small $\hat{\mu}$-measure. This means that $\hat{\mu}(S) = 0$, and so $\bigvee_{n=0}^{\infty} f^{-n}(\mathcal{R})$ is the Borelian σ-algebra, modulo null-$\hat{\mu}$ sets. Note that (by arguments as in [**Cas02**]) there are leaves that can be written as enumerable intersections of cylinders. We positively saturate $\hat{\mu}$ to extend it to a measure in M. A simple calculation leads to the fact that

$$\mu = \mu_\gamma \times \hat{\mu},$$

which is invariant by construction.

If one has uniqueness of the maximal entropy measure in the quotient map g defined in the last paragraph, then one can proceed as in §4 to conclude that the measure that we just constructed is the maximal entropy measure for the system in the second setting. In the case where the central-unstable direction is one-dimensional, such existence and uniqueness of the maximal entropy measure for the quotient map is guaranteed by the work of Liverani *et al* [**LSV98**]. This implies Corollary 2.4.

Now, we proceed with the proof of the exponential decay of correlations for μ.

Again, we have the following proposition.

PROPOSITION 11.5. *If f is invertible then $\mathcal{L}^*(\mu) = \mu$ if and only if μ is f-invariant.*

Therefore, μ is an f-invariant probability.

$$\int (\varphi \circ f^n)\psi \, d\mu = \int \varphi \mathcal{L}^n(\psi) \, d\mu. \tag{11.5}$$

We also have that any Hölder function can be written as a sum of functions in the cone. That is, we have the following proposition.

PROPOSITION 11.6. *For all $\varphi \in C^\alpha(M)$, there exists $K(\varphi) > 0$ such that $\varphi + K(\varphi) \in C[b, c, \alpha]$.*

The proof of the proposition above is essentially the same of that of Lemma 9.2.

Now, we are able to prove the following theorem.

THEOREM 11.7. (Exponential decay of correlations for systems in the second setting) *The measure μ exhibits exponential decay of correlations for Hölder continuous observables.*

Proof. The proof is exactly the same as for Theorem 9, using the corresponding Propositions 11.3 and 11.4 that we have just proved for systems in the second setting. □

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The central limit theorem follows again from the exponential decay of correlations and Gordin's theorem (Theorem 10.1).

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