AN INFINITE INTEGRAL

By G. N. WATSON

Received 22 April 1942

In a recent note in these *Proceedings** Prof. Copson gave two methods of obtaining an ascending series for the integral

$$I = \int_0^\infty \frac{\sin xu}{1+u^2} du,$$

one method being his own and the other being due to Prof. Hardy. It seems worth while to point out that this integral supplies an excellent illustration of a straightforward method of evaluating integrals which contain a Fourier kernel. The method consists in replacing the kernel by an integral of Mellin's type, changing the order of the integrations, effecting the new inner integration, and finally evaluating the resulting integral by means of the calculus of residues.

In the example under consideration, we write $\sin \xi$ in the form

$$\sum_{n=1}^{\infty} \frac{(-)^{n-1} \xi^{2n-1}}{(2n-1)!},$$

and observe that the general term is the residue of

$$-\frac{\pi}{\sin s\pi} \frac{\xi^{2s-1}}{\Gamma(2s)}$$

at $s = n$. We thus obtain the integral of Mellin's type

$$\sin \xi = \frac{1}{2i} \int_{c-\infty i}^{c+\infty i} \frac{\xi^{2s-1} ds}{\sin s\pi \, \Gamma(2s)},$$

in which, to secure absolute convergence, we suppose that ξ is positive and that $\frac{3}{4} < c < 1$. These conditions are supplied by Stirling's formula, which shows that there is a constant K (depending only on c) such that, for $0 < c < 1$,

$$\frac{1}{|\sin s\pi \, \Gamma(2s)|} < \frac{K}{|t|^{2c-\frac{1}{2}} + 1} \quad (s = c + it),$$

while a slightly different application of Stirling's formula shows that the integral, when taken round a semicircle on the right of the contour, tends to zero as the radius of the semicircle tends to infinity through such values that the semicircle does not pass through any of the zeros of $\sin s\pi$; this shows that the integral along the straight line is equal to $-2\pi i$ times the sum of the residues of the integrand at its poles on the right of the line.

* E. T. Copson, *Proc. Cambridge Phil. Soc.* 37 (1941), 102–4.

https://doi.org/10.1017/S030500410002199X Published online by Cambridge University Press

We thus get (with $u^2 = v$)

$$I = \frac{1}{2i}\int_0^\infty \int_{c-\infty i}^{c+\infty i} \frac{(xu)^{2s-1}}{\sin s\pi\,\Gamma(2s)} \frac{ds\,du}{1+u^2}$$
$$= \frac{1}{4i}\int_{c-\infty i}^{c+\infty i} \frac{x^{2s-1}}{\sin s\pi\,\Gamma(2s)} \int_0^\infty \frac{v^{s-1}dv}{1+v}\,ds$$
$$= \frac{\pi}{4i}\int_{c-\infty i}^{c+\infty i} \frac{x^{2s-1}\,ds}{\sin^2 s\pi\,\Gamma(2s)},$$

when we change the order of the integrations and remember that

$$\int_0^\infty \frac{v^{s-1}\,dv}{1+v} = \frac{\pi}{\sin s\pi} \quad (0<c<1).$$

Now the residue of $$\frac{\pi^2}{\sin^2 s\pi} \frac{x^{2s-1}}{\Gamma(2s)}$$

at $s = m+1$ $(m = 0, 1, 2, \ldots)$ is equal to

$$\frac{x^{2m+1}}{(2m+1)!}\{2\log x - 2\psi(2m+2)\},$$

and so, by Cauchy's theorem, we finally get

$$I = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}\{\psi(2m+2) - \log x\} = \sum_{m=0}^{\infty} \frac{x^{2m+1}\psi(2m+2)}{(2m+1)!} - \log x \sinh x.$$

This is the result obtained by Copson and Hardy.

The only step which still needs justification is the inversion of the order of the integrations. To do this, we observe that

$$\left|\frac{(xu)^{2s-1}}{\sin s\pi\,\Gamma(2s)} \frac{1}{1+u^2}\right| < Kx^{2c-1} \frac{u^{2c-1}}{1+u^2} \frac{1}{|t|^{2c-\frac{1}{2}}+1} \quad (s = c+it),$$

and that both of the integrals

$$\int_0^\infty \frac{u^{2c-1}\,du}{1+u^2}, \quad \int_{-\infty}^\infty \frac{dt}{|t|^{2c-\frac{1}{2}}+1} \quad (\tfrac{3}{4}<c<1)$$

are convergent; this justifies the inversion.

The only snag in this procedure is the necessity of using Stirling's formula with the variable complex in order to make the work rigorous; apart from this, the analysis seems decidedly simpler than in either method in the original note. In point of fact, I came upon the note when starting on a half-hour's railway journey, and had no difficulty in working out the contents of the present note in the course of the journey. Probably there are few mathematicians (I am certainly not one of them myself) who would care to attack the problem by either of the other methods in such circumstances.

THE UNIVERSITY
BIRMINGHAM

https://doi.org/10.1017/S030500410002199X Published online by Cambridge University Press