*Ergod. Th. & Dynam. Sys.* (1997), **17**, 643–648
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# Chains that realize the Gromov invariant of hyperbolic manifolds

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(*Received* 23 *January* 1995 *and revised* 5 *March* 1996)

*Abstract.* For any closed hyperbolic manifold of dimension $n \geq 3$, suppose a sequence of $n$-cycles representing the fundamental homology class have norms converging to the Gromov invariant. We show that this sequence must converge to the uniform measure on the space of maximal-volume ideal simplices. As a corollary, we show that for a hyperbolic $n$-manifold $L$ ($n \geq 3$) with totally-geodesic boundary, the Gromov norm of $(L, \partial L)$ is strictly greater than the volume of $L$ divided by the maximal volume of an ideal $n$-simplex.

## 1. *Introduction*

Let $C_*(X)$ denote the singular chain complex of a space $X$ with real coefficients. There is a norm on $C_*(X)$, namely, for a chain $c = \sum a_i \sigma_i$ (where $\sigma_i$ are simplices), $\|c\| = \sum |a_i|$. This norm passes to a pseudo-norm on homology called the 'Gromov norm': $\|h\| = \inf \|c\|$, where the inf is taken over all chains $c$ that represent $h$. For discussions of this norm, see [**G**] and [**T**]. If $M$ is a closed manifold with fundamental class $z$, then the Gromov invariant of $M$ is $\|z\|$. We will denote this invariant simply as $\|M\|$.

When $M$ has strictly negative curvature, we can straighten the simplices of $M$ (as in [**G**]): for a simplex $\sigma$, choose any lift $\tilde{\sigma}$ to the universal cover of $M$, and suppose the lift has vertices $x_0, x_1, \ldots, x_k$. Define straight $(\tilde{\sigma})$ inductively as the geodesic cone from $x_k$ to the straight face spanned by $x_0, x_1, \ldots, x_{k-1}$, and define straight $(\sigma)$ as the projection to $M$ of straight $(\tilde{\sigma})$. This function extends linearly to a chain map straight: $C_*(M) \to C_*(M)$. This chain map is homotopy equivalent to the identity, and for any chain $c$, $\|\text{straight}(c)\| \leq \|c\|$, so we will only consider straightened chains. When it will not cause any confusion, we will sometimes use the term 'simplex' to also refer to the image of a simplex.

Assume that $M$ is a closed hyperbolic manifold of dimension $n \geq 2$. Then $\|M\|$ is equal to the volume of $M$ divided by the volume $\text{vol}_n$ of a maximal-volume ideal hyperbolic $n$-simplex (that is, a regular ideal $n$-simplex) [**T**]. There is no cycle $c$ representing the fundamental class such that $\|c\| = \|M\|$; that is, no cycle realizes

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the inf. Suppose $\{c_i\}$ is a sequence of straight cycles for which $\|c_i\| \to \|M\|$. Let $\tilde{c_i}$ be the sum of all lifts of $c_i$ to $H^n$. We ask the question, what are all possible limits for subsequences of $\{\tilde{c_i}\}$? The limit of such a subsequence will be a measure $\mu$ on the space of simplices of $\overline{H^n}$ (where $\overline{H^n}$ denotes $H^n$ union the sphere at infinity).

One such limit is the uniform norm on the space of all positively-oriented maximal-volume ideal simplices [**T**]. Furthermore, one can replace any simplices by simplices with the same images and same orientation but permuted vertices, or by the negatives of simplices with the same image and opposite orientation. We show that in dimensions 3 and higher, these are the only possible limits. As a corollary, we show that if $L^n$ is a hyperbolic manifold with totally-geodesic boundary and with $n \geq 3$, then $\|L^n, \partial L^n\|$ is strictly greater than the volume of $L^n$ divided by $\text{vol}_n$.

## 2. *Possible limits of* $\{\tilde{c_i}\}$

We do not wish to distinguish between simplices which are the same except that the vertices are permuted, so we will define an 'alternating' function on chains by

$$\text{alt}(\sigma) = \frac{1}{n!} \sum_{\pi \in S_n} \text{sign}(\pi) \sigma \circ h_\pi,$$

where $h_\pi$ is a homeomorphism of the simplex that permutes the vertices by $\pi$. This function extends linearly to a chain map alt : $C_*(M) \to C_*(M)$. This chain map is homotopy equivalent to the identity, and for any chain $c$, $\|\text{alt}(c)\| \leq \|c\|$.

Throughout this section, $M$ will be a closed hyperbolic manifold of dimension $n \geq 3$, and $\{c_i\}$ will be a sequence of alternating, straight cycles representing the fundamental homology class of $M$ such that $\|c_i\|$ converges to $\|M\|$. Let $\tilde{c_i}$ denote the sum of all lifts of $c_i$ to the universal cover $H^n$ of $M$. By taking a subsequence, we may assume that $\{\tilde{c_i}\}$ converges in the weak* topology to a measure $\mu$ on the space of singular simplices of $\overline{H^n}$.

Using notation similar to that in [**T**], let $h$ denote Haar measure on the space of isometries of $H^n$, normalized so that the set of isometries that move a point $x \in H^n$ to a region $R \in H^n$ is equal to the volume of $R$. Now for any simplex $\rho$, the measure $h$ induces a measure $\text{smear}(\rho)$ supported on the set of isometric copies of $\rho$ in $H^n$. The main purpose of this paper is to prove the following.

THEOREM 2.1.

$$\mu = \frac{1}{2\,\text{vol}_n}(\text{smear}(\rho_+) - \text{smear}(\rho_-)),$$

*where $\rho_+$ (respectively $\rho_-$) is a positively-oriented (respectively negatively-oriented) maximal-volume ideal simplex.*

It is clear that $\mu$ has support on the maximal-volume simplices or else $\|c_i\|$ would be bounded away from $\|M\|$. For any maximal-volume simplex $\sigma$ with ordered vertices $v_0, \ldots, v_n$, there is a unique maximal-volume simplex $\sigma_j'$ with ordered vertices $v_0, \ldots, v_{j-1}, w_j, v_{j+1}, \ldots, v_n$ (for some $w_j$) which has the opposite orientation (namely, $\sigma_j'$ is the reflection of $\sigma$ through the face $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$). Define the map $r_j$ by $r_j(\sigma) = -\sigma_j'$.

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LEMMA 2.2. *The measure $\mu$ is $r_j$-invariant for each $j$.*

*Proof.* All cases being equivalent, we will prove the lemma only for $j = n$, and we will write $r_n$ simply as $r$. Let $X$ denote the space of straight simplices in $\overline{H^n}$, and define the distance between two elements of $X$ to be the volume of their symmetric difference. For any set $S \subset X$, define $B(S, \epsilon)$ to be the $\epsilon$-neighborhood of $S$; that is, the set of simplices with distance $< \epsilon$ from any element of $S$. Let $K \subset X$ be an arbitrary compact set. Let $K_M$ be the subset of $K$ consisting of maximal-volume ideal simplices. We will show that $\mu(K_M) = \mu(r(K_M))$. Since $K$ was arbitrary, it will follow that $\mu$ is $r$-invariant. We will assume also that $K$ consists entirely of positively-oriented simplices, the proof being equivalent for negatively-oriented simplices.

Any chain $c = \sum a_j \sigma_j$ can be thought of as a measure on the space of simplices given by $c(S) = \sum_{\sigma_j \in S} a_j$. Since $\tilde{c_i} \to \mu$, for any $\epsilon_1 > 0$ and $\delta > 0$, for sufficiently large $i$,

$$\tilde{c_i}(B(K_m, \delta)) \geq \mu(K_M) - \epsilon_1.$$

The set of cycles with rational coefficients is dense in the space of cycles with real coefficients, so we can assume that each $c_i$ has rational coefficients. If we let $m$ be the least common denominator of the coefficients of $c_i$, then we can write $c_i = \frac{1}{m}(\sum \sigma_p - \sum \sigma_q)$ (where $\sigma_p$ and $\sigma_q$ are not-necessarily-distinct simplices, but no $\sigma_p$ equals any $\sigma_q$). Since $c_i$ is a cycle and is alternating, there is a (not-necessarily unique) pairing of $\sigma_p$'s with $\sigma_q$'s such that each simplex is paired with a simplex with the same $n$th face. Now suppose that $\sigma_1$ and $\sigma_2$ are two simplices that are paired. For any $\epsilon_2 > 0$, there is a $\delta > 0$ such that if $\sigma_1 \in B(\rho, \delta)$ for some maximal simplex $\rho$, and $\sigma_2$ is within $\delta$ of any maximal-volume simplex with the opposite orientation of $\rho$, then $\sigma_2 \in B(r(\rho), \epsilon_2)$. Since $\|c_i\| \to \|M\|$, for sufficiently large $i$, all but a measure-$\epsilon_1$ set of simplices of $c_i$ will be within $\delta$ of maximal-volume simplices. Thus for any $\epsilon_1, \epsilon_2 > 0$, for sufficiently large $i$ and small $\delta$,

$$\tilde{c_i}(B(r(K_M), \epsilon_2)) \geq \tilde{c_i}(B(K_M, \delta)) - \epsilon_1.$$

Finally, since $\tilde{c_i} \to \mu$, for any $\epsilon_1 > 0$ we can select $\epsilon_2 > 0$ such that for sufficiently large $i$,

$$\mu(r(K_M)) + \epsilon_1 \geq \tilde{c_i}(B(r(K_M), \epsilon_2)).$$

We can combine these three inequalities to get $\mu(r(K_M)) \geq \mu(K_M) - 3\epsilon_1$, for any $\epsilon_1$. Letting $\epsilon_1 \to 0$, we get $\mu(r(K_M)) \geq \mu(K_M)$. We can now apply the same reasoning to $r^{-1}$ to conclude that $\mu(K_M) = \mu(r(K_M))$. □

*Proof of Theorem 2.1.* Maximal-volume ideal simplices in $\overline{H^n}$ can be represented by orthonormal frames in $H^n$ in the following inductive way. The simplex $v_0, v_1, v_2$ is represented by a frame based at the centroid of the simplex, with a first vector directed towards $v_0$ and a second vector directed towards edge $v_0v_1$. To represent the simplex $v_0, v_1, \ldots, v_n$, first represent $v_1, \ldots, v_n$ by an orthonormal frame based at the centroid $c$ of $v_1, \ldots, v_n$, then translate this frame along the geodesic $v_0c$ to the centroid of $v_0, \ldots, v_n$, then insert the vector pointing towards $v_0$ as the first vector of the frame. This produces a homeomorphism between the space of maximal-volume ideal simplices in

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$\overline{H^n}$ and the space of orthonormal frames in $H^n$, which is in turn naturally homeomorphic to Isom($H^n$).

Now $\mu$ corresponds to a measure $\mu'$ on the space of frames, and the functions $r_j$ correspond to functions $r_j'$. As before, $\mu'$ is $r_j'$-invariant, and is thus invariant under the closure of the group $\langle r_0', r_1', \ldots, r_n' \rangle$. The transformations $r_j'$ act on a frame $\sigma'$ by reflection in the faces of a regular ideal $n$-simplex. Observe that $r_1', \ldots, r_n'$ preserve the horosphere centred at the zeroth vertex of a simplex. Restricting to such a horosphere, $r_1', \ldots, r_n'$ act on a frame $\sigma'$ by reflection in the faces of a regular Euclidean $(n-1)$-simplex.

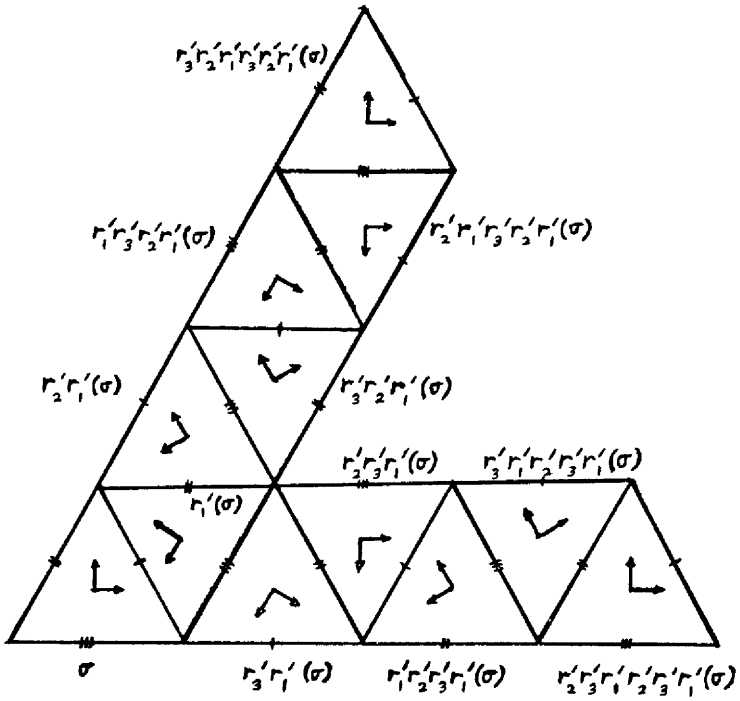



FIGURE 1.

Consider first the case $n > 3$. It is well known that the group generated by reflections in the faces of a regular Euclidean $(n-1)$-simplex is dense in Isom($\mathbf{R}^{n-1}$) (for $n > 3$). Therefore the closure of the group $\langle r_1', \ldots, r_n' \rangle$ contains the horospherical flow in Isom($H^n$). Veech [**V**] showed that this horospherical flow is uniquely ergodic, and so the action of $\langle r_0', r_1', \ldots, r_n' \rangle$ is uniquely ergodic.

Now consider the case $n = 3$. In this case, the elements $r_2'r_3'r_1'r_2'r_3'r_1'$ and $r_3'r_2'r_1'r_3'r_2'r_1'$ are translations that generate a cocompact lattice in the horospherical flow. (Figure 1 illustrates this, showing the view from the zeroth vertex.) Ratner [**R**] has shown that the action of such a cocompact lattice is uniquely ergodic, and so again the action of $\langle r_0', r_1', \ldots, r_n' \rangle$ is uniquely ergodic.

In each case, it follows from the unique ergodicity that the only invariant measure on the space of positively-oriented (respectively negatively-oriented) maximal-volume

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simplices are proportional to smear($\rho_+$) (respectively smear($\rho_-$)). Thus, $\mu = \alpha_1 \operatorname{smear}(\rho_+) + \alpha_2 \operatorname{smear}(\rho_-)$, for some constants $\alpha_1$, $\alpha_2$. Since all $\tilde{c_i}$ are alternating chains, we have $\alpha_1 = -\alpha_2$. Finally, each $c_i$ represents the fundamental class of $M$ and $\tilde{c_i} \to \mu$, so we must have $\alpha_1 = 1/2 \operatorname{vol}_n$. □



FIGURE 2.

*Remark.* Theorem 2.1 is not true in dimension 2. The proof fails in this case because all ideal simplices have maximal volume, and so any maximal-volume simplex shares a boundary-face with infinitely many other maximal-volume simplices. As an example, suppose that $S$ is a hyperbolic surface. Decompose $S$ into pairs of pants with geodesic boundaries, and decompose each pair of pants into two ideal triangles as in Figure 2. (The figure shows a pair of pants decomposed into a white triangle and a black triangle.) Let $\mu$ be the sum of all of the lifts of all of these triangles to $\overline{H^2}$. Then there exist sequences $\{c_i\}$ of cycles representing the fundamental class of $S$ such that $\|c_i\| \to \|S\|$, and $\{c_i\} \to \mu$.

*Question.* Suppose that $S$ is a closed hyperbolic surface, and $\{c_i\}$ is a convergent sequence of cycles representing the fundamental homology class of $S$ with $\|c_i\| \to \|S\|$. Is there a nice way to categorize the set of possible limits of $\{c_i\}$?

We conclude with an application of Theorem 2.1. The Gromov invariant can be defined for manifolds with boundary as follows. For a manifold $M$ with boundary $\partial M$, let $C_*(M, \partial M)$ denote the relative chain group. Then we define the norm of a chain as before, and we define the Gromov invariant of $(M, \partial M)$ to be the infimum of the norms of relative cycles representing the fundamental homology class of $(M, \partial M)$.

COROLLARY 2.3. *Let $L$ be a compact hyperbolic manifold of dimension $n > 2$, and suppose that $L$ has non-empty, totally-geodesic boundary. Then $\|(L, \partial L)\|$ is strictly greater than the volume of $L$ divided by* $\operatorname{vol}_n$.

*Proof.* Suppose to the contrary that $c_i$ is a sequence of chains representing the fundamental homology class of $(L, \partial L)$, and suppose that $\|c_i\| \to \operatorname{volume}(L)/\operatorname{vol}_n$. Let $2L$ be the double of $L$ obtained by reflecting $L$ through its boundary and glueing $L$ to its reflection. Let $\overline{c_i}$ be the chain obtained by reflecting $c_i$ through the boundary of $L$. Then $c_i - \overline{c_i}$

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represents the fundamental class of $2L$, and $\|c_i - \overline{c_i}\|$ converges to $\|2L\|$. But $\partial L$ does not intersect the interior of any simplex of $c_i - \overline{c_i}$, so $c_i - \overline{c_i}$ cannot converge to the uniform measure, contradicting the theorem. □

*Acknowledgements.* I would like to thank Andrew Casson for introducing me to this subject, Dave Witte for helpful suggestions, and Geoff Mess for numerous helpful conversations and for pointing out the corollary to me.

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