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EMBEDDING BOOLEAN AMPLE MONOIDS AS FULL SUBMONOIDS OF BOOLEAN INVERSE MONOIDS

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This paper is dedicated to Victoria Gould on the occasion of her birthday.

Abstract

We show that, in certain circumstances, a Boolean ample monoid may be fully embedded into a Boolean inverse monoid in a way that generalises how right reversible cancellative monoids may be embedded into groups. We use groupoids of fractions and noncommutative Stone duality to prove the result.

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1. Introduction

All terms and notation will be defined later. The problem of which cancellative monoids can be embedded in groups is a difficult one. However, if the cancellative monoid C is *right reversible*, meaning that for all $a, b \in C$, the set $Ca \cap Cb$ is nonempty, then C can be embedded in a group G in such a way that $G = C^{-1}C$ [2]. We say that G is a *group of fractions*. This result can be generalised to categories. If C is now a cancellative category, we say that it is right reversible if for all $a, b \in C$ such that $\mathbf{d}(a) = \mathbf{d}(b)$, the set $Ca \cap Cb$ is nonempty. If C is right reversible, then C can be embedded in a groupoid G in such a way that $G = C^{-1}C$ [4]. We call G a *groupoid of fractions*. Such groupoids of fractions appear naturally in the theory of C^*-algebras [8, Lemma 5.5]. Suppose now that a right reversible cancellative category C is equipped with a topology that makes it étale. We would like to extend the topology on C to the groupoid of fractions G in such a way that G becomes étale. This is carried out in a special case in [8, Proposition 5.12]; one of the goals of this paper is to show that this can always be done.

We now generalise. The correct generalisation of a group is an inverse monoid; an inverse monoid with a unique idempotent is a group. What should be the correct

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generalisation of a cancellative monoid? This should be an ample monoid which we now define. These semigroups were formerly known as 'type A adequate semigroups'. The paper [3] attributes the new nomenclature to Gould.

A semigroup S is said to be *ample* if it is equipped with two unary operations, denoted by $a \mapsto a^*$ and $a \mapsto a^+$, satisfying the following axioms:

(A1) $(s^*)^* = s^*$, $(s^+)^+ = s^+$, $(s^*)^+ = s^*$ and $(s^+)^* = s^+$;
(A2) $(s^*t^*)^* = s^*t^*$ and $(s^+t^+)^+ = s^+t^+$;
(A3) $s^*t^* = t^*s^*$ and $s^+t^+ = t^+s^+$;
(A4) $ss^* = s$ and $s^+s = s$;
(A5) $(st)^* = (s^*t)^*$ and $(st)^+ = (st^+)^+$;
(A6) $t^*s = s(ts)^*$ and $st^+ = (st^+)^+s$;
(A7) if $ac = bc$, then $ac^+ = bc^+$, and if $ca = cb$, then $c^*a = c^*b$.

Elements of the form a^* or a^+ are called *projections*. It is easy to show that in an ample semigroup, the projections are precisely the idempotents; thus, the idempotents commute. An ample monoid with a unique idempotent is a cancellative monoid. We now need to explain what we mean by 'embedding' an ample monoid in an inverse monoid. We denote the set of idempotents of a semigroup S by $\mathsf{E}(S)$. We say that S is a *full* subsemigroup of a semigroup T if S is a subsemigroup of T and $\mathsf{E}(T) \subseteq S$. An embedding $\iota\colon S \to T$ is *full* if $\iota(S)$ is a full subsemigroup of T. Given an ample monoid S, we would like to know when it is possible to fully embed S in an inverse monoid T; observe that if this is possible, then for each $a \in S$, we have $a^* = a^{-1}a$ and $a^+ = aa^{-1}$. Of course, this generalises the question of when a cancellative monoid can be embedded in a group; it is therefore not always possible.

An étale groupoid G with a Boolean space of identities is called a Boolean groupoid. (The term 'ample' is used in the C^*-algebra literature. We eschew this term here for obvious reasons.) Under a generalisation of classical Stone duality, which we call 'noncommutative Stone duality', the Boolean groupoid gives rise to a Boolean inverse monoid. We can also define what we mean by a Boolean ample monoid. It is associated with a Boolean cancellative category. We can now ask for the monoid analogue of a groupoid of fractions; this is exactly what this paper is about. The proof of Theorem 4.4 is our ultimate goal.

2. Noncommutative Stone duality

In this section, we describe the duality between Boolean restriction monoids and Boolean categories. This duality was first established in [6]. We need a proof here that is closer to the classical proof of Stone duality. The details can be found in [5], so we just sketch out the theory here; one difference from [5] is that we use prime filters and not germs. We work with posets (the order will always be the natural partial order). If $(P, \leq)$ is a poset and $X \subseteq P$, then

$$X^{\uparrow} = \{p \in P\colon x \leq p \text{ for some } x \in X\}.$$

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We begin with classical Stone duality [1]. Let B be a Boolean algebra. A *filter* in B is a subset A closed under finite meets such that $A = A^{\uparrow}$. A filter A is *proper* if $0 \notin A$. A maximal proper filter is called an *ultrafilter*. A proper filter A is said to be *prime* if $a \vee b \in A$ implies that $a \in A$ or $b \in A$. In a Boolean algebra, prime filters are the same as ultrafilters. If $a \in B$, denote by X_a the set of all prime filters that contain a. Put β equal to the set of all X_a. Denote by $\mathsf{X}(B)$ the set of prime filters of B. Then, β is the basis for a topology on $\mathsf{X}(B)$, which makes it a Boolean space, that is, a compact, Hausdorff 0-dimensional space. We call $\mathsf{X}(B)$ the *Stone space* of B. However, given a Boolean space X, the set of clopen subsets $\mathsf{B}(X)$ forms a Boolean algebra. The essence of classical Stone duality is that $B \cong \mathsf{B}(\mathsf{X}(B))$ and $X \cong \mathsf{X}(\mathsf{B}(X))$.

We now turn to the generalisation of classical Stone duality. We begin by describing the two categories that are in duality.

A *restriction monoid* S satisfies Axioms (A1)–(A6). The set of projections of S is denoted by $\mathsf{Proj}(S)$; all projections are idempotents, but not every idempotent is a projection. If S is a restriction monoid, define the natural partial order $a \leq b$ if and only if $a = ba^*$. Say that a and b are *compatible*, denoted by $a \sim b$, if and only if $ab^* = ba^*$ and $a^+b = b^+a$. A *Boolean restriction monoid* is a restriction monoid in which every pair of compatible elements has a join, multiplication distributes over such joins (from both sides) and the set of projections forms a Boolean algebra under the natural partial order. A *homomorphism* of Boolean restriction monoids is a monoid homomorphism of restriction monoids that maps zero to zero and preserves binary joins. Restriction monoids and their homomorphisms form a category which we denote by $\mathcal{R}$.

Apart from categories of structures, all categories will be small and treated as algebraic generalisations of monoids; thus, we identify objects with identities. If C is a category, then its set of identities will be denoted by C_o. If $a \in C$, we denote the unique right identity of a by $\mathbf{d}(a)$ and the unique left identity by $\mathbf{r}(a)$. The product $\exists ab$ precisely when $\mathbf{d}(a) = \mathbf{r}(b)$. The partial multiplication in a category is denoted by $\mathbf{m}$. If the category C is a subcategory of the category D so that $C_o = D_o$, then we say that C is a *wide* subcategory of D. A *topological category* C is one equipped with a topology so that the maps $\mathbf{d}$, $\mathbf{r}$ and $\mathbf{m}$ are continuous. A *cancellative category* is one in which if $ab = ac$ is defined, then $b = c$ and if $ba = ca$ is defined, then $b = c$. A *groupoid* is a category such that for every element g, there is a (perforce) unique element g^{-1} such that $g^{-1}g = \mathbf{d}(g)$ and $gg^{-1} = \mathbf{r}(g)$. A topological category is said to be *étale* if the maps $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms. A *Boolean category* is an étale topological category whose space of identities is a Boolean space. We need maps a bit different from functors between Boolean categories. Let C and D be categories. A binary relation $\rho \subseteq C \times D$, which we regard as an arrow from D to C, is called a *relational functor* if it satisfies the following three conditions.

(RF1) For each $e \in D_o$, there is a unique $f \in C_o$ such that $(f, e) \in \rho$.
(RF2) If $(c, d) \in \rho$, then $(\mathbf{d}(c), \mathbf{d}(d)) \in \rho$ and $(\mathbf{r}(c), \mathbf{r}(d)) \in \rho$.
(RF3) If $(a, b), (c, d) \in \rho$ and $\exists ac$ and $\exists bd$, then $(ac, bd) \in \rho$.

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(Note that a relational functor is not a functor.) We say that such a relational functor is a *covering, relational functor* if the following two conditions hold.

(CRF1) If $(a,b),(a,b') \in \rho$ and $\mathbf{d}(b) = \mathbf{d}(b')$, then $b = b'$, and dually.
(CRF2) If e and f are identities such that $(f,e) \in \rho$ and $\mathbf{d}(a) = f$, then there exists b such that $\mathbf{d}(b) = e$ and $(a,b) \in \rho$, and dually.

If C and D are topological categories and ρ is a relational functor from D to C, then we say that ρ is *continuous* if for every open set U in C, the set

$$\rho^{-1}(U) = \{d \in D\colon \exists c \in U, \text{ such that } (c,d) \in \rho\}$$

is open. We say that ρ is *proper* if U compact implies that $\rho^{-1}(U)$ is compact. Boolean categories together with the proper, continuous, covering, relational functors form a category which we denote by $\mathcal{C}$. We have the following theorem.

THEOREM 2.1. *The category $\mathcal{R}$ is dually equivalent to the category $\mathcal{C}$.*

We now sketch out how this theorem is proved.

We begin by defining the contravariant functor from $\mathcal{R}$ to $\mathcal{C}$. We describe it on objects first. Let S be a Boolean restriction monoid. A *filter* in S is a subset A such that $A = A^{\uparrow}$ and for all $a, a' \in A$, there exists $a'' \in A$ such that $a'' \leq a', a$. The filter A is said to be *proper* if $0 \notin A$. A maximal proper filter is called an *ultrafilter*. A proper filter A is said to be *prime* if $a \vee b \in A$ implies $a \in A$ or $b \in A$. Just as with Boolean algebras, prime filters and ultrafilters are the same thing. Let $\mathsf{C}(S)$ be the set of prime filters of S. If X is any subset, define $X^* = \{a^*\colon a \in A\}$ and $A^+ = \{a^+\colon a \in A\}$. Define $\mathbf{d}(A) = (A^*)^{\uparrow}$ and $\mathbf{r}(A) = (A^+)^{\uparrow}$. If A is a prime filter, then $\mathbf{d}(A)$ and $\mathbf{r}(A)$ are both prime filters. If A and B are both prime filters, and $\mathbf{d}(A) = \mathbf{r}(B)$, define

$$A \cdot B = (AB)^{\uparrow},$$

also a prime filter. Then, with respect to the partially defined operation $\cdot$, the set $\mathsf{C}(S)$ is a category. The identities of this category are precisely the prime filters containing projections. A remarkable feature of prime filters is that they are essentially 'cosets' of prime filters that contain projections: if A is a prime filter, then

$$A = (a\mathbf{d}(A))^{\uparrow},$$

where $a \in A$. This result immediately implies that if A and B are (prime) filters such that $\mathbf{d}(A) = \mathbf{d}(B)$ and $A \cap B \neq \emptyset$, then $A = B$, and dually. We can relate prime filters in S with prime filters in $\mathsf{Proj}(S)$: if E is a prime filter in $\mathsf{Proj}(S)$, then $A = (aE)^{\uparrow}$, where $a^* \in E$, is a prime filter in S containing a and every prime filter in S is of this form, and dually. We now need to equip the category $\mathsf{C}(S)$ with a topology. For each $a \in S$, let X_a be the set of all prime filters in S that contain a. Let β be the set of all such X_a. Then, β is the basis for a topology on $\mathsf{C}(S)$, with respect to which, $\mathsf{C}(S)$ is an étale topological category. The space $\mathsf{C}(S)_o$ is homeomorphic to the Stone space of $\mathsf{Proj}(S)$. It follows that $\mathsf{C}(S)$ is a Boolean category. We now need to describe the effect of our contravariant functor on homomorphisms. Let $\theta\colon S \to T$ be a homomorphism

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between Boolean restriction monoids and let A be a prime filter in T. It could happen that $\theta^{-1}(A)$ is empty. If it is not empty, then Ganna Kudryavtseva discovered that $\theta^{-1}(A)$ is a disjoint union of prime filters: the easiest way to see this is to define the relation $\approx$ on $\theta^{-1}(A)$ by $a \approx b$ if and only if there is $c \in \theta^{-1}(A)$ such that $c \leq a, b$. This is an equivalence relation and the equivalence classes are prime filters. Define the arrow ρ_θ from $\mathsf{C}(T)$ to $\mathsf{C}(S)$ by $(A, B) \in \rho_\theta$ if and only if $A \subseteq \theta^{-1}(B)$, where $B \in \mathsf{C}(T)$. Then, ρ_θ is a proper, continuous, covering, relational functor. Our contravariant functor therefore does the following: on objects S, it does $S \mapsto \mathsf{C}(S)$ and on homomorphisms $\theta\colon S \to T$, it does $\theta \mapsto \rho_\theta$.

We now define the contravariant functor from $\mathcal{C}$ to $\mathcal{R}$. We describe it on objects first. Let C be a Boolean category. A subset $X \subseteq C$ is called a *local bisection* if $a, b \in X$ and $\mathbf{d}(a) = \mathbf{d}(b)$ implies that $a = b$, and dually. Under subset multiplication, if A and B are compact-open local bisections, then AB is a compact-open local bisection. Denote by $\mathsf{KB}(C)$ the set of all compact-open local bisections on C. If $A \in \mathsf{KB}(C)$, define $A^* = \{\mathbf{d}(a)\colon a \in A\}$ and $A^+ = \{\mathbf{r}(a)\colon a \in A\}$. Then, $\mathsf{KB}(C)$ is a Boolean restriction monoid whose set of projections is just the set of all clopen subsets of the Boolean space C_o. Before we consider the effect of our contravariant functor on relational functors, we need the following result.

THEOREM 2.2.

(1) *Let S be a Boolean restriction monoid. Then, there is an isomorphism $S \cong \mathsf{KB}(\mathsf{C}(S))$ given by the map $a \mapsto X_a$.*
(2) *Let C be a Boolean category. Then, there is an isomorphism $C \cong \mathsf{C}(\mathsf{KB}(C))$ of Boolean categories given by the map $x \mapsto F_x$, where F_x is the set of all compact-open local bisections containing x.*

Let $\rho\colon \mathsf{C}(T) \to \mathsf{C}(S)$ be a proper, continuous, covering, relational functor. A corresponding homomorphism $\theta_\rho\colon \mathsf{KB}(\mathsf{C}(S)) \to \mathsf{KB}(\mathsf{C}(T))$ is defined by $\rho^{-1}(X_s) = X_{\theta_\rho(s)}$. (This set is compact and open since ρ is proper and continuous. The set $\rho^{-1}(X_s)$ is a local bisection since its elements are prime filters.) Our contravariant functor therefore does the following: on objects C, it does $C \mapsto \mathsf{KB}(C)$ and on a proper, continuous, covering, relational functor $\rho\colon C \to B$, it does $\rho \mapsto \theta_\rho$. That we really have a contravariant functor in this direction depends on the following result, which is easy to prove.

LEMMA 2.3. *Let ρ be a covering relational functor. If $(ab, d) \in \rho$, then there exists d_1 and d_2 such that $d = d_1 d_2$ and $(a, d_1), (b, d_2) \in \rho$.*

Parts (1) and (2) of the following result are proved in [6], but we give a direct proof here.

PROPOSITION 2.4.

(1) *If S is a Boolean ample monoid, then $\mathsf{C}(S)$ is cancellative.*
(2) *If C is a Boolean cancellative category, then $\mathsf{KB}(C)$ is ample.*

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(3) *If S is a Boolean inverse monoid, then $\mathsf{C}(S)$ is a groupoid.*
(4) *If C is a Boolean groupoid, then $\mathsf{KB}(C)$ is inverse.*

PROOF. (1) Suppose that $A \cdot B = A \cdot C$, where A, B, C are prime filters. We prove that $B = C$. The dual result follows by symmetry. In fact, we prove $B \subseteq C$. The reverse inclusion then follows by symmetry. Let $b \in B$. Then, $b^+ \in \mathbf{r}(B)$. So, $b^+ \in \mathbf{d}(A)$, since $A \cdot B$ is defined in the category. There is therefore $a \in A$ such that $a^* \leq b^+$. However, $ab \in A \cdot B$ and so $ab \in A \cdot C$. Therefore, $a_1 c \leq ab$, where $a_1 \in A$ and $c \in C$. Since $a, a_1 \in A$, there exists $a_2 \leq a, a_1$, where $a_2 \in A$. Thus, $a_2 c \leq ab$. However, $a_2 = aa_2^*$. Thus, $aa_2^* c = ab(a_2 c)^*$. We are working in an ample semigroup, so $a^* a_2^* c = a^* b(a_2 c)^*$. Now, $a^*, a_2^* \in \mathbf{r}(C)$. Thus, $a^* a_2^* c \in \mathbf{r}(C)\mathbf{r}(C)C \subseteq C$. It follows that $a^* b(a_2 c)^* \in C$. Thus, $b \in C$, as required.

(2) It is enough to check that Axiom (A7) holds. Suppose that $AB = AC$, where A, B, C are compact-open local bisections. We prove that $A^*B = A^*C$. Let $\mathbf{d}(a)b \in A^*B$, where $a \in A$. Then, $ab \in AB$. By assumption, $ab = a_1 c_1$, where $a_1 \in A$ and $c_1 \in C$. Clearly, $\mathbf{r}(a) = \mathbf{r}(a_1)$. However, A is a local bisection. Thus, $a = a_1$. Therefore, $ab = ac_1$. The category C is cancellative; thus, $b = c_1$. It follows that $\mathbf{d}(a)b \in A^*C$, as required.

The proofs of parts (3) and (4) may be found in [7, Ch. 9]. □

3. Etale groupoids of fractions

We say that a category C is *right reversible* if for all elements $a, b \in C$ such that $\mathbf{d}(a) = \mathbf{d}(b)$, we have $Ca \cap Cb \neq \emptyset$; we are using the same terminology as [2] since we regard small categories as generalisations of monoids. The following result is proved in [4].

THEOREM 3.1. *Let C be a right reversible cancellative category. Then, there is a groupoid G such that C is embedded in G by a functor ι such that $\iota(C)$ is a wide subcategory of G and $\iota(C)^{-1}\iota(C) = G$. The groupoid G is unique with these properties.*

The groupoid G is called a *groupoid of fractions* of C. For the remainder of this section, to ease notation, we always regard C as a subcategory of G.

Having dealt with matters algebraical, we now deal with matters topological. The first result is immediate because $\mathbf{d}$ and $\mathbf{r}$ are local homeomorphisms.

LEMMA 3.2. *In any étale category, the space of identities is open.*

Our next result tells us that we can choose the basis of an étale category to have a special form. We refine this result later when we deal with Boolean categories.

LEMMA 3.3. *Let C be an étale category. Then, the open local bisections form a basis for the topology on C.*

PROOF. Let U be any nonempty open set in C and let $a \in U$. Then, there is an open neighbourhood V of a on which $\mathbf{d}$ is a homeomorphism. Likewise, there is an open neighbourhood W of a on which $\mathbf{r}$ is a homeomorphism. The set $U \cap V \cap W$ is an

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open set containing a inside U, which is a local bisection. This proves that the open local bisections form a basis for C. □

If C is a category and $X \subseteq C$, then we write $\mathbf{d}(X) = \{\mathbf{d}(x) \colon x \in X\}$, and dually.

LEMMA 3.4. *Let C be an étale topological category. Then, the open local bisection A is compact if and only if $\mathbf{d}(A)$ is compact, and dually.*

PROOF. Suppose that A is an open local section and that $\mathbf{d}(A)$ is compact. We prove that A is compact. Suppose that $A \subseteq \bigcup_{i\in I} U_i$, where U_i are open sets of C. Then, $A = \bigcup_{i\in I} U_i \cap A$. The sets $U_i \cap A$ are open and they are local bisections since they are contained in a local bisection. It follows that $\mathbf{d}(A) = \bigcup_{i\in I} \mathbf{d}(U_i \cap A)$. By assumption, $\mathbf{d}(A)$ is compact. We may therefore write $\mathbf{d}(A) = \bigcup_{j=1}^{m} \mathbf{d}(U_j \cap A)$, relabelling if necessary. We claim that $A \subseteq \bigcup_{i=1}^{m} U_i$. Let $x \in A$. Then, $\mathbf{d}(x) \in \mathbf{d}(A)$ and so $\mathbf{d}(x) \in \mathbf{d}(U_j \cap A)$ for some j. However, $U_j \cap A$ is a local section. Thus, there is a unique element $y \in U_j \cap A$ such that $\mathbf{d}(y) = \mathbf{d}(x)$. However, $x, y \in A$, which is a local bisection. It follows that $x = y$ and so $x \in U_j \cap A \subseteq U_j$. The proof of the converse follows from the fact that C is étale □

We may now prove the following result about Boolean categories; this is the refinement we promised earlier.

PROPOSITION 3.5. *Let C be a Boolean category. Then, the compact-open local bisections form a basis for the topology on C.*

PROOF. By Lemma 3.3, it is enough to consider the case where U is an open local bisection in C. Let $x \in U$. There is an open set V containing x such that V and $\mathbf{d}(V)$ are homeomorphic by $\mathbf{d}$. Thus, $U \cap V$ contains a and is homeomorphic to $\mathbf{d}(U \cap V)$ under $\mathbf{d}$. However, the clopen subsets of the space of identities form a basis for the topology on the space of identities. Thus, there is an open set $Y \subseteq U \cap V$ containing x such that $\mathbf{d}(Y)$ is clopen. However, the closed subsets of Hausdorff spaces are compact and so Y is compact by Lemma 3.4. □

We can now state the main theorem we prove in this section; we prove it later.

THEOREM 3.6. *Let C be a right reversible Boolean cancellative category. Then, its groupoid of fractions is a Boolean groupoid. In addition, if β is the basis for C consisting of all compact-open local bisections, then $\beta^{-1}\beta$ is a basis for the topology on the groupoid of fractions and consists of compact-open local bisections.*

We now work towards proving the above theorem.

LEMMA 3.7. *Let C be an étale topological cancellative right reversible category with groupoid of fractions G, where β is the set of all open local bisections of C. Then, $\beta^{-1}\beta$ is a basis for a topology on G. In addition, each element of $\beta^{-1}\beta$ is a local bisection.*

PROOF. We regard C as a subcategory of G in what follows to ease notation. We use the fact that a basis β for the topology in C consists of open local bisections.

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An element g of G can be written $a^{-1}b$, where $a, b \in C$. By assumption, $a \in U \in \beta$ and $b \in V \in \beta$ for some U and V. It follows that $g \in U^{-1}V$. Thus, every element of G is in some element of $\beta^{-1}\beta$.

Suppose that $g \in U_1^{-1}V_1 \cap U_2^{-1}V_2$, where $U_1, U_2, V_1, V_2 \in \beta$. Then, $g = a_1^{-1}b_1 = a_2^{-1}b_2$, where $a_1 \in U_1$, $a_2 \in U_2$, $b_1 \in V_1$ and $b_2 \in V_2$. Observe that $\mathbf{d}(b_1) = \mathbf{d}(b_2)$ and $\mathbf{d}(a_1) = \mathbf{d}(a_2)$. Since C is right reversible, we may find elements $a, a' \in C$ such that $ab_1 = a'b_2 = r$ (say) and $aa_1 = a'a_2 = s$ (say). (That the elements a, a' are the same in each case follows from the two different ways that g can be written). Choose $A, B \in \beta$ such that A contains a and B contains a'. Using the fact that the product of open local bisections is an open local bisection, $AV_1 \cap BV_2 = X$ is an open local bisection containing r and $AU_1 \cap BU_2 = Y$ is an open local bisection containing s. Thus, X, $Y \in \beta$. Observe that $s^{-1}r \in Y^{-1}X$, which simplifies to g. We have shown that $g \in Y^{-1}X$. Now, let $h \in Y^{-1}X$ be any element. Then, we can write $h = (au)^{-1}(a_1v)$, where $a \in A$, $u \in U_1$, $a_1 \in A$ and $v \in V_1$. Thus, $h = u^{-1}a^{-1}a_1v$. However, $a, a_1 \in A$, a local bisection, and $\mathbf{r}(a) = \mathbf{r}(a_1)$. It follows that $a = a_1$. We deduce that $h \in U_1^{-1}V_1$. We may similarly show that $h \in U_2^{-1}V_2$. We have therefore proved that $g \in Y^{-1}X \subseteq U_1^{-1}V_1 \cap U_2^{-1}V_2$.

We have therefore proved that $\beta^{-1}\beta$ really is a basis.

We now prove that each element of $\beta^{-1}\beta$ is a local bisection. Let $g, h \in U^{-1}V$ such that $\mathbf{d}(g) = \mathbf{d}(h)$. We may write $g = u_1^{-1}v_1$ and $h = u_2^{-1}v_2$. Thus, $\mathbf{d}(v_1) = \mathbf{d}(v_2)$. It follows that $v_1 = v_2$. From this, we deduce that $\mathbf{r}(u_1) = \mathbf{r}(u_2)$ and so $u_1 = u_2$. It follows that $g = h$. Symmetry delivers the result. □

The following result reassures us that we do get an étale groupoid.

PROPOSITION 3.8. *Let C be an étale topological cancellative right reversible category whose groupoid of fractions is G. Let β be a basis for the topology on C consisting of all open local bisections of C. Then, G is an étale groupoid with respect to the topology with basis $\beta^{-1}\beta$. In addition, C is an open subset of G.*

PROOF. We proved in Lemma 3.7 that $\beta^{-1}\beta$ is a basis for a topology on G. However, the topology on G has, as a basis, local bisections. From this, it follows that G will be étale. It is clear that inversion is a homeomorphism. It remains to show that the partial multiplication in G is continuous. To that end, let $g_1h_1 \in U^{-1}V$ be any element where $U, V \in \beta$. Let $g_1 = a_1^{-1}b_1$ and $h_1 = a_2^{-1}b_2$, where $a_1, b_1, a_2, b_2 \in C$. By right reversibilty, let p and q be such that $pb_1 = qa_2 = r$. Then, $g_1h_1 = (pa_1)^{-1}(qb_2)$. By assumption, $g_1h_1 = u^{-1}v$, where $u \in U$ and $v \in V$. From the fact that we are dealing with a groupoid of fractions, it follows that there are elements $a, b \in C$ such that $apa_1 = bu$ and $aqb_2 = bv$. Let Z and B be open local bisections of C such that $apb_1 \in Z$ and $b \in B$. Consider, first, the set $(BU)^{-1}Z$. We have $apa_1 \in BU$. Thus, $a_1^{-1}p^{-1}a^{-1} \in (BU)^{-1}$. By definition, $apb_1 \in Z$. It follows that $g_1 \in (BU)^{-1}Z$. Now, we consider $Z^{-1}BV$. By definition, $apb_1 \in Z$. However, $pb_1 = qa_2$. Thus, $aqa_2 \in Z$. Whence, $a_2^{-1}q^{-1}a^{-1} \in Z^{-1}$. However, $aqb_2 \in BV$. It follows that $h_1 \in Z^{-1}BV$. We now calculate the product $[(BU)^{-1}Z][Z^{-1}BV]$. This is contained in $U^{-1}V$ because if X is a

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local bisection, both XX^{-1} and $X^{-1}X$ consist of identities. This is enough to show that the partial multiplication in the groupoid G is continuous.

It only remains to show that C is an open subset of G. The set of identities C_o is an open set in C. It is therefore an open local bisection. Thus, $\beta \subseteq \beta\beta^{-1}$. This means that C is an open subset of its groupoid of fractions. □

PROOF OF THEOREM 3.6. Let C be a Boolean cancellative right reversible category whose groupoid of fractions is G. We have seen that G is an étale category. It is Boolean because C is a wide subcategory of G. Because C is open in G, it follows that a compact subset of C is also compact in G. Let β be a basis for the topology on C consisting of all compact-open local bisections of C. The elements of $\beta^{-1}\beta$ are compact-open local bisections of G. It only remains to show that $\beta^{-1}\beta$ is a basis for the topology on G. We have proved that a basis for the topology on G consists of the subsets of the form $U^{-1}V$, where U and V are open local bisections of C. However, β is a basis for the topology on C. Thus, U is a union of elements of β. Likewise, V is a union of elements of β. It follows that $U^{-1}V$ is a union of elements of $\beta^{-1}\beta$. This proves that $\beta^{-1}\beta$ is a basis for the topology on G. □

The following is a converse to what we have found.

THEOREM 3.9. *Let G be an étale topological groupoid. Suppose that C is an open, wide subcategory of G which is right reversible and such that $CC^{-1} = G$. Then, C is an étale topological cancellative category and if α is a basis of C consisting of open local bisections, then $\alpha\alpha^{-1}$ is a basis for the topology on G.*

PROOF. It is immediate that C is cancellative and, because C is open in G, it inherits the main features of the topology of G. Thus, C is an étale topological category. Let X be an open local bisection of G containing the element g. Let $g = a^{-1}b$, where $a, b \in C$. Let A be any open local bisection of C that contains a. Then, $b \in AX$. Thus, $Y = AX \cap C$ is nonempty, since it contains b, and is an open local bisection of C. Thus, $g \in A^{-1}Y$. We prove that $A^{-1}Y \subseteq X$. Let $h \in A^{-1}Y$. Then, $h = a_1^{-1}a_2x$, where $a_1, a_2 \in A$ and $x \in X$. However, A is a local bisection and so $a_1 = a_2$. It follows that $h \in X$. We have proved that each open local bisection of G is a union of open local bisections of the form $U^{-1}V$, where U and V are open local bisections of C. □

4. The embedding theorem

Let S be an ample monoid. From Section 2, its associated category of prime filters $\mathsf{C}(S)$ is cancellative. The first question therefore is: what condition on S is equivalent to $\mathsf{C}(S)$ being right reversible? This is answered by the following theorem. To state it succinctly, we introduce some notation. Let F be a prime filter in $\mathsf{E}(S)$. We write $x \leq_F y$ if and only if $x^*, y^* \in F$ and $x \leq y$; we say that this is the natural partial order *relativised to* F. Let S be an ample monoid. Then, S satisfies *condition (C)* if the following holds: for any prime filter $F \subseteq \mathsf{E}(S)$ and elements $a, b \in S$ such that $a^*b^* \in F$, there exist nonzero elements $c, d \in S$ such that the following two conditions hold.

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(1) There exists $a_1 \leq_F a$ such that $a_1^+ \leq c^*$; there exists $b_1 \leq_F b$ such that $b_1^+ \leq d^*$.
(2) There is an element $e \in F$ such that $cae = dbe$.

PROPOSITION 4.1. *Let S be an ample monoid. Then, $\mathsf{C}(S)$ is right reversible if and only if condition (C) holds for S.*

PROOF. Suppose first that $\mathsf{C}(S)$ is right reversible. We show that condition (C) holds. Let $F \subseteq \mathsf{E}(S)$ be a prime filter and let $a, b \in S$ such that $a^*b^* \in F$. Put $A = (aF)^\uparrow$ and $B = (bF)^\uparrow$. Both A and B are prime filters with $\mathbf{d}(A) = F^\uparrow = \mathbf{d}(B)$. Thus, by right reversibility, there are prime filters C and D such that $\exists C \cdot A$ and $\exists B \cdot D$ and $C \cdot A = D \cdot B$. We may write $C = (c\mathbf{r}(A))^\uparrow$, where $c^* \in \mathbf{r}(A)$, and $D = (d\mathbf{r}(B))^\uparrow$, where $d^* \in \mathbf{r}(B)$. Then, $\mathbf{r}(A) = ((aF)^+)^\uparrow$ and $\mathbf{r}(B) = ((bF)^+)^\uparrow$. It follows that $(ae)^+ \leq c^*$ for some $e \in F$ and $(bf)^+ \leq d^*$ for some $f \in F$. We may therefore put $a_1 = ae$ and $b_1 = bf$. We can write $C \cdot A = (caF)^\uparrow$ and $D \cdot B = (dbF)^\uparrow$. Since $\mathbf{d}(C \cdot A) = \mathbf{d}(D \cdot B)$, the equality $C \cdot A = D \cdot B$ follows from the fact that $C \cdot A \cap D \cdot B \neq \emptyset$. This means that $cae = dbe$ for some $e \in F$.

We now prove the converse. Suppose that condition (C) holds. We prove that $\mathsf{C}(S)$ is right reversible. Let A and B be prime filters such that $\mathbf{d}(A) = \mathbf{d}(B)$. Then, $A = (aF)^\uparrow$, where $a^* \in F$, and $B = (bF)^\uparrow$, where $b^* \in F$. We now invoke condition (C). There are elements c and d having various properties. Put $C = (c\mathbf{r}(A))^\uparrow$ and $D = (d\mathbf{r}(B))^\uparrow$. Now, $\mathbf{r}(A) = ((aF)^+)^\uparrow$. However, $a_1 = ae$, where $e \in F$. Thus, $a_1^+ \in \mathbf{r}(A)$. It follows that $c^* \in \mathbf{r}(A)$. Thus, $c \in C$. Similarly, $d \in D$. Thus, $C \cdot A = (ca\mathbf{d}(A))^\uparrow$ and $D \cdot B = (db\mathbf{d}(A))$. There is some $e \in F$ such that $cae = dbe$. It follows that $C \cdot A \cap D \cdot B \neq \emptyset$. Thus, in fact, $C \cdot A = D \cdot B$ and so $\mathsf{C}(S)$ is right reversible. □

Condition (C) is not particularly nice, but it does have a pleasant consequence.

COROLLARY 4.2. *Let S be a Boolean ample monoid satisfying condition (C). Then, for all elements a and b such that $a^*b^* \neq 0$, we have $Sa \cap Sb \neq \{0\}$.*

PROOF. Because $a^*b^* \neq 0$, there is a prime filter F in $\mathsf{E}(S)$ containing a^*b^*. Invoking condition (C), there are elements c and d, and an idempotent $e \in F$ such that $cae = dbe$. We now apply Axiom (A6) to get $(c(ae)^+)a = (d(be)^+)b$, which is nonzero since it lives in a prime filter. □

Condition (C) does have a natural interpretation.

PROPOSITION 4.3. *Let S be a full submonoid of a Boolean inverse monoid T. We suppose also that S is a Boolean ample monoid. Then, $\mathsf{C}(T)$ is a groupoid of fractions of $\mathsf{C}(S)$ if and only if each nonzero element of T can be written as a finite join of elements of the form $a^{-1}b$, where $a, b \in S$.*

PROOF. We define first a functor $\iota\colon \mathsf{C}(S) \to \mathsf{C}(T)$, which is an embedding where $\iota(\mathsf{C}(S))$ is a wide subcategory of $\mathsf{C}(T)$. This will depend only on the fact that S is a full submonoid of T. This will involve comparing prime filters in S with prime filters

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in T. Let A be a prime filter of S. Then,

$$A^u = \{t \in T \colon a \leq t \text{ for some } a \in A\}$$

is a prime filter in T that contains elements of S. Conversely, let B be a prime filter in T that contains elements of S. Then,

$$B^d = B \cap S$$

is a prime filter in S. Observe that $(A^u)^d = A$ and $(B^d)^u = B$. We have therefore defined a bijection between the prime filters in S and the prime filters in T that contain elements of S. We therefore define the functor ι by $A \mapsto A^u$. We may now prove the proposition.

Suppose that each nonzero element of T can be written as a finite join of elements of the form $a^{-1}b$, where $a, b \in S$. Let X be any prime filter of T. Let $t \in X$ be any element, necessarily nonzero. Then, from the fact that we are dealing with a prime filter and given how t can be written, there exists $a^{-1}b \in X$ for some $a, b \in S$. Then, $X = A \cdot B$, where $a^{-1} \in A$ and $b \in B$. We may therefore write $X = ((A)^{-1})^{-1} \cdot B$, where $A = (a^{-1}\mathbf{r}(B))^{\uparrow}$ and $B = (b\mathbf{d}(X))^{\uparrow}$. Observe that $a \in A^{-1}$ and so both A^{-1} and B contain elements of S. It follows that both A^{-1} and B are in the image of ι.

To prove the converse, let t be any nonzero element of T. By assumption, any prime filter X of T containing t must have the form $A^{-1} \cdot B$, where A and B are prime filters of T that have nonempty intersections with S. Let $a \in A \cap S$ and let $b \in B \cap S$. Thus, $a^{-1}b \in A^{-1} \cdot B = X$. Now, X is a prime filter that contains both t and $a^{-1}b$. However, any element less than $a^{-1}b$ has the form $a_1^{-1}b_1$, where $a_1, b_1 \in S$; this follows from the fact that S is an order ideal of T. Whence, $X_t = \bigcup_{a^{-1}b \leq t, a,b \in S} X_{a^{-1}b}$. By compactness, $t = \bigvee_{i=1}^{n} a_i^{-1}b_i$, where $a_i, b_i \in S$. □

We can now prove our main theorem. We use noncommuative Stone duality on objects only; we could use the full version of noncommutative Stone duality, but it is easier to construct embeddings directly.

THEOREM 4.4. *Let S be a Boolean ample monoid that satisfies condition (C). Then, S can be embedded as a full submonoid of a Boolean inverse monoid T in such a way that every nonzero element of T is a finite join of elements of the form $a^{-1}b$, where $a, b \in S$.*

PROOF. Let S be a Boolean ample monoid satisfying condition (C). By Theorem 2.1 and part (1) of Proposition 2.4, the category $\mathsf{C}(S)$ is cancellative. By Proposition 4.1, the cancellative category $\mathsf{C}(S)$ is right reversible because condition (C) holds. Thus, by Theorem 3.1, there is a groupoid G which contains $\mathsf{C}(S)$ as a wide subcategory. However, by Theorem 3.6, G is, in fact, a Boolean groupoid. Thus, $\mathsf{C}(S) \subseteq G$ as an open and wide subcategory; we proved openness in Proposition 3.8. It follows that every compact-open local bisection of $\mathsf{C}(S)$ is a compact-open local bisection of G. Thus, $\mathsf{KB}(\mathsf{C}(S))$ is a full submonoid of $\mathsf{KB}(G)$; the fact that we have a full submonoid is a result of the fact that the category $\mathsf{C}(S)$ is wide in G. By part (1) of Theorem 2.2, S is isomorphic to $\mathsf{KB}(\mathsf{C}(S))$ and we identify these two monoids.

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By part (4) of Proposition 2.4, $\mathsf{KB}(G)$ is a Boolean inverse monoid T, say. We have therefore shown that S is a full submonoid of the Boolean inverse monoid T. Finally, by Proposition 4.3, every nonzero element of T is a finite join of elements of the form $a^{-1}b$, where $a, b \in S$. □

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