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Elementary symplectic orbits and improved K_1-stability

by

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Abstract

It is shown that the set of orbits of the action of the elementary symplectic group on all unimodular rows over a commutative ring of characteristic not 2 is identical with the set of orbits of the action of the corresponding elementary general linear group. This result is used to improve injective stability for K_1 of the symplectic group over non-singular affine algebras.

Key Words: unimodular rows, elementary symplectic group

Mathematics Subject Classification 2000: 13C10, 13H05, 19B14, 19B99, 55R50

1. Introduction

We shall always work over a commutative ring R with 1. L.N. Vaserstein showed in [18] that if v is the first row of an elementary matrix of even size then it is also the first row of an elementary symplectic matrix. This led us to query whether the orbit space of unimodular rows under the action of the elementary subgroup is in bijective correspondence with the orbit space of unimodular rows under the action of the elementary symplectic group. In Section 4, we prove that this is so, and also establish the relative version, $v\mathrm{E}_{2n}(R,I) = v\mathrm{ESp}_{2n}(R,I)$, for an ideal I of R.

As a consequence of [8] and [9], it follows that the orbit space (both in the absolute and relative case) of unimodular rows under the action of the elementary symplectic group has an abelian group structure when the rows have size $\geq d/2+2$, where d is the stable dimension of the ring.

The main theorem (Theorem 5.6) enables us to recapture the main theorem in [6] *though only in the free case*, viz. a stably elementary symplectic $\sigma \in \mathrm{Sp}_{d+1}(R)$ is (symplectic) homotopic to the identity, when R is an affine algebra over a perfect C_1-field of even dimension d. The approach here enables us to improve the estimate further by one in the case when R is a finitely generated algebra over $\mathbb{Z}, F_p$ or the algebraic closure of F_p, where F_p is a finite field of characteristic p not equal to 2 and R is of even dimension.

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2. Preliminaries

In this section we recall a few definitions, state some results and fix a few notations which will be used throughout this paper.

A row $v = (v_1,\dots,v_n) \in R^n$ is said to be *unimodular* if there are elements $w_1,\dots,w_n$ in R such that $v_1w_1 + \cdots + v_nw_n = 1$. $\mathrm{Um}_n(R)$ will denote the set of all unimodular rows $v \in R^n$. Let I be an ideal in R. We denote by $\mathrm{Um}_n(R,I)$ the set of all unimodular rows of length n which are congruent to $e_1 = (1,0,\dots,0)$ modulo I. (If $I = R$, then $\mathrm{Um}_n(R,I)$ is $\mathrm{Um}_n(R)$).

The group $\mathrm{GL}_n(R)$ of invertible matrices acts on R^n in a natural way: $v \longrightarrow v\sigma$, if $v \in R^n$, $\sigma \in \mathrm{GL}_n(R)$. This map preserves $\mathrm{Um}_n(R)$, so $\mathrm{GL}_n(R)$ acts on $\mathrm{Um}_n(R)$. Note that any subgroup G of $\mathrm{GL}_n(R)$ also acts on $\mathrm{Um}_n(R)$. Let $v,w \in \mathrm{Um}_n(R)$, we denote $v \sim_{\mathrm{G}} w$ or $v \in w\mathrm{G}$ if there is a $g \in \mathrm{G}$ such that $v = wg$.

Let $\mathrm{E}_n(R)$ denote the subgroup of $\mathrm{SL}_n(R)$ consisting of all *elementary* matrices, i.e. those matrices which are a finite product of the *elementary generators* $\mathrm{E}_{ij}(\lambda) = I_n + e_{ij}(\lambda)$, $1 \le i \ne j \le n$, $\lambda \in R$, where $e_{ij}(\lambda) \in \mathrm{M}_n(R)$ has at most one non-zero entry λ in its (i,j)-th position.

In the sequel, if α denotes an $m \times n$ matrix, then we let α^t denote its *transpose* matrix. This is of course an $n \times m$ matrix. However, we will mostly be working with square matrices, or rows and columns.

Definition 2.1 The Relative Groups $\mathrm{E}_n(I)$, $\mathrm{E}_n(R,I)$: Let I be an ideal of R. The relative elementary group $\mathrm{E}_n(I)$ is the subgroup of $\mathrm{E}_n(R)$ generated as a group by the elements $\mathrm{E}_{ij}(x)$, $x \in I$, $1 \le i \ne j \le n$.

The relative elementary group $\mathrm{E}_n(R,I)$ is the normal closure of $\mathrm{E}_n(I)$ in $\mathrm{E}_n(R)$.

(Equivalently, $\mathrm{E}_n(R,I)$ is generated as a group by $\mathrm{E}_{ij}(a)\mathrm{E}_{ji}(x)\mathrm{E}_{ij}(-a)$, with $a \in R$, $x \in I$, $i \ne j$, provided $n \ge 3$ (see [24], Lemma 8)).

An important theorem of A.A. Suslin is that $\mathrm{E}_n(R,I)$ is a normal subgroup of $\mathrm{GL}_n(R)$, for $n \ge 3$.(see [20], Corollary 1.4).

Theorem 2.2 ([16], Theorem 2.3) *Let $n \ge 3$ and $v(X) = (v_1(X),\dots,v_n(X)) \in \mathrm{Um}_n(R[X])$. If $v(X) \in v(0)\mathrm{E}_n(R_{\mathfrak{m}}[X])$ for all maximal ideals $\mathfrak{m} \subset R$, then $v(X) \in v(0)\mathrm{E}_n(R[X])$.* □

Definition 2.3 Symplectic Group $\mathrm{Sp}_{2n}(R)$: This is the isotropy group of the standard symplectic form $\psi_n = \sum_{i=1}^{n} e_{2i-1,2i} - \sum_{i=1}^{n} e_{2i,2i-1}$. In other words, $\mathrm{Sp}_{2n}(R) = \{\alpha \in \mathrm{GL}_{2n}(R) \mid \alpha^t\psi_n\alpha = \psi_n\}$.

Let σ denote the permutation of the natural numbers given by $\sigma(2i) = 2i-1$ and $\sigma(2i-1) = 2i$.

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Definition 2.4 Elementary Symplectic Group $\mathrm{ESp}_{2n}(R)$: We define for $z \in R$, $1 \leq i \neq j \leq 2n$,

$$se_{ij}(z) = \begin{cases} 1_{2n} + ze_{ij} & \text{if } i = \sigma(j), \\ 1_{2n} + ze_{ij} - (-1)^{i+j} ze_{\sigma(j)\sigma(i)} & \text{if } i \neq \sigma(j) \text{ and } i < j. \end{cases}$$

(It is easy to check that all these generators belong to $\mathrm{Sp}_{2n}(R)$).

We call them the elementary symplectic matrices over R and the subgroup of $\mathrm{Sp}_{2n}(R)$ generated by them is called the elementary symplectic group $\mathrm{ESp}_{2n}(R)$.

Definition 2.5 The Relative Group $\mathrm{ESp}_{2n}(I)$, $\mathrm{ESp}_{2n}(R, I)$: Let I be an ideal of R. The relative elementary group $\mathrm{ESp}_{2n}(I)$ is the subgroup of $\mathrm{ESp}_{2n}(R)$ generated as a group by the elements $se_{ij}(x)$, $x \in I$ and $1 \leq i \neq j \leq 2n$.

The relative elementary group $\mathrm{ESp}_{2n}(R, I)$ is the normal closure of $\mathrm{ESp}_{2n}(I)$ in $\mathrm{ESp}_{2n}(R)$.

Lemma 2.6 ([10], Lemma 1.5) *Let $n \geq 2$, and I be an ideal of R. Let $a \in I, v \in R^{2n}$, or $a \in R, v \in I^{2n}$. Then $I_{2n} + av^t\widetilde{v} \in \mathrm{ESp}_{2n}(R, I)$, where $\widetilde{v} = v\psi_n$.* □

Lemma 2.7 ([10], Lemma 1.10) *Let $n \geq 2$, and I be an ideal in R. Let $w \in \mathrm{Um}_{2n}(R)$, and $v \in I^{2n}$ ($\subset R^{2n}$) be such that $\widetilde{v}w^t = 0$. Then $I_{2n} + v^t\widetilde{w} + w^t\widetilde{v} \in \mathrm{ESp}_{2n}(R, I)$.* □

Theorem 2.8 ([10], Theorem 3.6) *Let $n \geq 2$ and let $\alpha(X) \in \mathrm{Sp}_{2n}(R[X])$, with $\alpha(0) = I_{2n}$. Then $\alpha(X) \in \mathrm{ESp}_{2n}(R[X])$ if and only if for any maximal ideal $\mathfrak{m} \subset R$, the canonical image of $\alpha(X)$ in $\mathrm{Sp}_{2n}(R_{\mathfrak{m}}[X])$ lies in $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X])$.*

(For more details see [5]). □

Lemma 2.9 ([18], Lemma 2.7) *Let R be an associative ring with 1, I be a two-sided ideal in R, and $n \geq 3$ be a natural number. Then $\mathrm{E}_n(R, I^2) \subset \mathrm{E}_n(I)$, where I^2 is a two sided ideal of R consisting of sums of elements of the form ab where $a, b \in I$.* □

Definition 2.10 Alternating Matrix: A matrix from $\mathrm{M}_n(R)$ is said to be *alternating* if it has the form $v - v^t$, where $v \in \mathrm{M}_n(R)$. (It follows that its diagonal elements are zeros.)

Lemma 2.11 ([18], Lemma 5.4) *Let n be a natural number and φ be an alternating matrix from $\mathrm{GL}_{2n}(R)$. Then for any v from R^{2n-1} there exists α, β in $\mathrm{SL}_{2n-1}(R)$ such that $\left(\begin{smallmatrix} 1 & 0 \\ \alpha v^t & \alpha \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & v \\ 0 & \beta \end{smallmatrix}\right)$ belong to $\mathrm{E}_{2n}(R) \cap \mathrm{Sp}_{\varphi}(R)$, where $\mathrm{Sp}_{\varphi}(R)$ denotes the isotropy group of φ, i.e.*

$$\mathrm{Sp}_{\varphi}(R) = \{\alpha \in \mathrm{SL}_{2n}(R) | \alpha^t \varphi \alpha = \varphi\}.$$ □

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Lemma 2.12 ([18], Lemma 5.5) *For any natural number $n \geq 2$ and any alternating matrix φ from* $\mathrm{GL}_{2n}(R), e_1(\mathrm{E}_{2n}(R)) = e_1(\mathrm{E}_{2n}(R) \cap \mathrm{Sp}_\varphi(R))$. □

We will only use Lemma 2.12 in the special case when $\varphi = \psi_n$. In this special case the proof is much easier to establish. In this case one could show the following.

Lemma 2.13 *For any natural number $n \geq 2$, $e_1\mathrm{E}_{2n}(R) = e_1\mathrm{ESp}_{2n}(R)$.*

Lemma 2.14 *Let R be a commutative ring and let $\varepsilon \in \mathrm{E}_{2n}(R, I)$, $n \geq 2$. Then there exists $\rho \in \mathrm{E}_{2n-1}(R, I)$ such that $\varepsilon(1 \perp \rho) \in \mathrm{ESp}_{2n}(R, I)$.*

The proof is similar, via ([18], Lemma 5.5) and its symplectic analogue, and is left to the reader. □

3. A Local Global Principle

In a letter to H. Bass, L.N. Vaserstein gave a short proof of an "action version" of D. Quillen's well known Local Global Principle (see [15]), viz. if $v(X) \in \mathrm{Um}_n(R[X]), n \geq 3$, and $v(X) \sim_{\mathrm{GL}_n(R_{\mathfrak{m}}[X])} v(0)$, for all maximal ideals $\mathfrak{m}$ of R, then $v(X) \sim_{\mathrm{GL}_n R([X])} v(0)$. (See [11], Chapter III for more details.) We prove a relative (w.r.t. an extended ideal) elementary symplectic action version of this result below.

We begin with a couple of preliminary lemmas.

Lemma 3.1 *Let $n \geq 2$. Let I be an ideal, S be a multiplicatively closed set in R. Let $\alpha(X) \in \mathrm{ESp}_{2n}(I_S[X])$, with $\alpha(0) = I_{2n}$. Then there exists $\alpha^*(X) \in \mathrm{ESp}_{2n}(R[X], I[X])$ such that $\alpha^*(X)$ localizes to $\alpha(sX)$, for some $s \in S$, with $\alpha^*(0) = I_{2n}$.*

Proof: Since there are only finitely many denominators involved, there exists $t \in S$ such that $\alpha(X) \in \mathrm{ESp}_{2n}(I_t[X])$. Therefore, as $\alpha(0) = I_{2n}$,

$$\alpha(X) = \prod_{k=1}^{r} \gamma_k se_{i_k j_k}(Xf_k(X)/t^m)\gamma_k^{-1},$$

where $\gamma_k \in \mathrm{ESp}_{2n}(I_t)$ and $f_k(X) \in I[X]$.

Let σ be the permutation defined before Definition 2.4. For any row vector v we define $\widetilde{v} = v\psi_n$. Let $v^t_{i_k}$ and $v^t_{\sigma(j_k)}$ be the i_k-th and $\sigma(j_k)$-th columns of γ_k respectively. Then $\widetilde{v}_{i_k}$ and $\widetilde{v}_{\sigma(j_k)}$ are the $\sigma(i_k)$-th and j_k-th rows of γ_k^{-1} respectively. If $i_k = \sigma(j_k)$ then,

$$\gamma_k se_{i_k j_k}(Xf_k(X)/t^m)\gamma_k^{-1} = I_{2n} + (Xf_k(X)/t^m)v^t_{i_k}\widetilde{v}_{i_k} \in \mathrm{ESp}_{2n}(I_t[X]).$$

If $i_k \neq \sigma(j_k)$ and $i_k < j_k$ then, $\gamma_k se_{i_k j_k}(Xf_k(X)/t^m)\gamma_k^{-1} =$

$$I_{2n} + (Xf_k(X)/t^m)v^t_{\sigma(j_k)}\widetilde{v}_{i_k} + (Xf_k(X)/t^m)v^t_{i_k}\widetilde{v}_{\sigma(j_k)} \in \mathrm{ESp}_{2n}(I_t[X]).$$

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If $v \in R_t^{2n}$, then $v = (1/t^m)v^*$, for some $m \geq 0$, with $v^* \in R^{2n}$. Let us define,

$$a_k = \begin{cases} I_{2n} + (Xf_k(X)/t^{3m})v_{i_k}^{*t}\widetilde{v}_{i_k}^*, & \text{if } i_k = \sigma(j_k), \\ I_{2n}, & \text{otherwise.} \end{cases}$$

$$b_k = \begin{cases} I_{2n} + (Xf_k(X)/t^{3m})v_{\sigma(j_k)}^{*t}\widetilde{v}_{i_k}^* \\ +(Xf_k(X)/t^{3m})v_{i_k}^{*t}\widetilde{v}_{\sigma(j_k)}^*, & \text{if } i_k \neq \sigma(j_k), i_k < j_k, \\ I_{2n}, & \text{otherwise.} \end{cases}$$

Now we have, $\alpha(X) = \prod_{k=1}^{r} a_k b_k$. Let us take $N = 3m$ and define,

$$\widetilde{a}_k = \begin{cases} I_{2n} + (Xf_k(t^N X))v_{i_k}^{*t}\widetilde{v}_{i_k}^*, & \text{if } i_k = \sigma(j_k), \\ I_{2n}, & \text{otherwise.} \end{cases}$$

$$\widetilde{b}_k = \begin{cases} I_{2n} + (Xf_k(t^N X))v_{\sigma(j_k)}^{*t}\widetilde{v}_{i_k}^* \\ +(Xf_k(t^N X))v_{i_k}^{*t}\widetilde{v}_{\sigma(j_k)}^*, & \text{if } i_k \neq \sigma(j_k), i_k < j_k, \\ I_{2n}, & \text{otherwise.} \end{cases}$$

Define $\alpha^*(X) = \prod_{k=1}^{r} \widetilde{a}_k \widetilde{b}_k$. $\alpha^*(X) \in \mathrm{ESp}_{2n}(R[X], I[X])$ since $f_k(t^N X) \in I[X]$ (see Lemma 2.6 and Lemma 2.7). Then obviously $\alpha^*(X)$ localises to $\alpha(t^N X)$. □

The following argument is standard (see [11], Chapter III, Proposition 2.3):

Lemma 3.2 *Let $n \geq 2$. Let I be an ideal, S be a multiplicatively closed set in R. Let $v(X) \in \mathrm{Um}_{2n}(R[X])$ and let $v(X) \in v(0)\mathrm{ESp}_{2n}(I_S[X])$. Then there is an s in S such that for any a in R, $v(X + asT) \in v(X)\mathrm{ESp}_{2n}(R[X,T], I[X,T])$.*

Proof: Let $\alpha(X) \in \mathrm{ESp}_{2n}(I_S[X])$ such that $v(X)\alpha(X) = v(0)$. Let us set $\beta(X,T) = \alpha(X+T)\alpha(X)^{-1} \in \mathrm{ESp}_{2n}(I_S[X,T])$. Then

$$\begin{aligned} v(X+T)\beta(X,T) &= v(X+T)\alpha(X+T)\alpha(X)^{-1} \\ &= v(0)\alpha(X)^{-1} \\ &= v(X) \text{ in } R_S[X,T]^{2n}. \end{aligned}$$

Since $\beta(X,0) = I_{2n}$, using Lemma 3.1 we can find $\beta^*(X,T) \in \mathrm{ESp}_{2n}(R[X,T], I[X,T])$ which localises to $\beta(X,sT)$, for some $s \in S$, with $\beta^*(X,0) = I_{2n}$. Then in $R[X,T]^{2n}$ we have, $v(X+sT)\beta^*(X,T) - v(X) = Tw(X,T)$, for some $w(X,T)$ which localises to 0. Thus for some $s^* \in S$, and

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for all $a \in R$, we get, $v(X + ass^*T)\beta^*(X,as^*T) - v(X) = Tas^*w(X,as^*T) = 0$. □

Proposition 3.3 (Local Global Principle w.r.t. an extended ideal)

Let $n \geq 2$. Let I be an ideal of R and $v(X) \in \mathrm{Um}_{2n}(R[X], I[X])$. If for all prime (or even maximal) ideals $\wp$ of R, $v(X)_\wp \in v(0)_\wp \mathrm{ESp}_{2n}(I_\wp[X])$, then

$$v(X) \in v(0)\mathrm{ESp}_{2n}(R[X], I[X]).$$

Proof: By assumption $v(X)_\wp \in v(0)_\wp \mathrm{ESp}_{2n}(I_\wp[X])$, for all prime (or all maximal) ideals $\wp$ of R. Using Lemma 3.2 it follows that, for each prime ideal $\wp$ of R, there exists $s_k \in R \setminus \wp$ such that for all $a \in R$,

$$v(X + as_kT) \in v(X)\mathrm{ESp}_{2n}(R[X,T], I[X,T]). \qquad (\star)$$

Note that the ideal generated by $s_k's$ is the whole ring R. Therefore there exists $s_{k_1},\ldots,s_{k_r}$ such that $a_1s_{k_1} + \cdots + a_rs_{k_r} = 1$ where $a_1,\ldots,a_r \in R$.

In $(\star)$ replacing as_kT by $a_1s_{k_1}X$ and replacing X by $a_2s_{k_2}X + \cdots + a_rs_{k_r}X$ we get, $v(X) = v(a_1s_{k_1}X + a_2s_{k_2}X + \cdots + a_rs_{k_r}X) \in v(a_2s_{k_2}X + \cdots + a_rs_{k_r}X)\mathrm{ESp}_{2n}(R[X], I[X])$. Again in $(\star)$ replacing as_kT by $a_2s_{k_2}X$ and X by $a_3s_{k_3}X + \cdots + a_rs_{k_r}X$ we get,

$$v(a_2s_{k_2}X + \cdots + a_rs_{k_r}X) \in v(a_3s_{k_3}X + \cdots + a_rs_{k_r}X)\mathrm{ESp}_{2n}(R[X], I[X]).$$

Continuing in this way we get, $v(a_rs_{k_r}X + 0) \in v(0)\mathrm{ESp}_{2n}(R[X], I[X])$. Combining all these we get,

$$v(X) \in v(0)\mathrm{ESp}_{2n}(R[X], I[X]).$$

□

4. The absolute case

In this section we prove that the set of orbits of the action of the elementary symplectic group on all unimodular rows is the same as the set of orbits of the action of the elementary general linear group on all unimodular rows.

Theorem 4.1 *Let $v \in \mathrm{Um}_{2n}(R)$. Then $v\mathrm{E}_{2n}(R) = v\mathrm{ESp}_{2n}(R)$, for $n \geq 2$.*

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Proof: Let $v^*_{ij}(X) = v\mathrm{E}_{ij}(X)$. Let $\wp \in \mathrm{Spec}(R)$. It is easy to see by Lemma 2.13 that $v^*_{ij}(X) \sim_{\mathrm{ESp}_{2n}(R_\wp[X])} v^*_{ij}(0)$. By Proposition 3.3 (when $I = R$)(or see the main theorem in [5]), $v^*_{ij}(X) \sim_{\mathrm{ESp}_{2n}(R[X])} v^*_{ij}(0)$; whence also to $v^*_{ij}(\lambda)$, for any $\lambda \in R$. The result follows. □

Theorem 4.2 *The natural map* $\mathrm{Um}_{2n}(R)/\mathrm{ESp}_{2n}(R) \longrightarrow \mathrm{Um}_{2n}(R)/\mathrm{E}_{2n}(R)$ *is bijective for* $n \geq 2$.

Proof: The proof follows from Theorem 4.1. □

5. The relative case

In general, there is a 'double ring' method, first introduced by J. Milnor, to transfer 'relative problems' to 'absolute problems'. The reader can find the latest known results of such type in the reference ([1], §3). Surprisingly, we were not able to settle the relative case of our problem by such a technique; but conceivably there should be such an argument. Such a method would have served to deduce Theorem 5.6 below from the absolute case proved in Theorem 4.2.

We begin with a few preliminary remarks before establishing the relative case of Theorem 4.2.

Lemma 5.1 *Let I be an ideal of R. Let $v = (1 + x_1, x_2, \ldots, x_{2n}) \in \mathrm{Um}_{2n}(R, I)$. Let $1 + x_1$ be a unit in R. Then there exists $g \in \mathrm{ESp}_{2n}(R, I)$ such that $v = e_1 g$.*

Proof: Since $1 + x_1 = u$ is a unit in the ring R, it is easy to show that there exists $g^* \in \mathrm{ESp}_{2n}(R, I)$ such that $vg^* = (u, 0, \ldots, 0)$. We have

$$\begin{aligned}(u,0,\ldots,0) &= e_1 \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & I_{2n-2} \end{pmatrix} \\ &= e_1 \mathrm{E}_{21}(u^{-1} - 1)\mathrm{E}_{12}(1)\mathrm{E}_{21}(u-1)\mathrm{E}_{12}(-1)\mathrm{E}_{12}(1-u^{-1}),\end{aligned}$$

(by Whitehead's lemma), where $\mathrm{E}_{21}(u^{-1} - 1)\mathrm{E}_{12}(1)\mathrm{E}_{21}(u - 1)\mathrm{E}_{12}(-1)\mathrm{E}_{12}(1 - u^{-1}) \in \mathrm{ESp}_{2n}(R, I)$. Take $g = \mathrm{E}_{21}(u^{-1} - 1)\mathrm{E}_{12}(1)\mathrm{E}_{21}(u - 1)\mathrm{E}_{12}(-1)\mathrm{E}_{12}(1 - u^{-1})(g^*)^{-1}$. Clearly $g \in \mathrm{ESp}_{2n}(R, I)$, and $v = e_1 g$. □

The following lemma is proved in a similar way as Lemma 2.9 was proven. We include its proof for completeness.

Lemma 5.2 *Let I be an ideal of R. Assume that $2R = R$. Then* $\mathrm{ESp}_{2n}(R, I^2) \subset \mathrm{ESp}_{2n}(I)$, *for* $n \geq 3$.

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Proof: Let $z^* = \sum a_t b_t$ with $a_t, b_t \in I$. Let $\beta = se_{ij}(z^*) \in \mathrm{ESp}_{2n}(I^2)$ and $\alpha = se_{kl}(z) \in \mathrm{ESp}_{2n}(R)$ for some $z \in R$. It suffices to show that $\alpha\beta\alpha^{-1} \in \mathrm{ESp}_{2n}(I)$. If $(i,j) \neq (l,k)$ and $(i,j) \neq (\sigma(k),\sigma(l))$, then the matrix $\alpha\beta\alpha^{-1}$ splits into a product of elementary matrices from $\mathrm{ESp}_{2n}(I)$.

When $(i,j) = (l,k)$ or $(i,j) = (\sigma(k),\sigma(l))$ we need to consider the following two cases:

Case 1. In this case $i \neq \sigma(j)$. We can choose $r \leq 2n$ different from $\sigma(i), \sigma(j)$ and j. $\beta = se_{ij}(z^*) = \prod_t [se_{ir}(a_t), se_{rj}(b_t)]$ and

$$\alpha\beta\alpha^{-1} = \prod_t [\alpha se_{ir}(a_t)\alpha^{-1}, \alpha se_{rj}(b_t)\alpha^{-1}] \in \mathrm{ESp}_{2n}(I).$$

Case 2. In this case $i = \sigma(j)$. We can choose $r \leq 2n$ different from $\sigma(i)$ and $\sigma(j)$. $\beta = se_{i\sigma(i)}(z^*) = \prod_t [se_{ir}(a_t/2), se_{r\sigma(i)}(b_t)]$ and

$$\alpha\beta\alpha^{-1} = \prod_t [\alpha se_{ir}(a_t/2)\alpha^{-1}, \alpha se_{r\sigma(i)}(b_t)\alpha^{-1}] \in \mathrm{ESp}_{2n}(I).$$

□

Lemma 5.3 *Let I be an ideal of R and $v \in \mathrm{Um}_{2n}(R,I)$. Then there exists $g \in \mathrm{ESp}_{2n}(R,I)$ such that $vg \in \mathrm{Um}_{2n}(R,I^k)$ for any positive integer k.*

Proof: Let $v = (1+i_1, i_2, \ldots, i_{2n})$ and $\overline{v} = (\overline{1}+\overline{i}_1, \overline{i}_2, \ldots, \overline{i}_{2n}) \in \mathrm{Um}_{2n}(R/I^k, I/I^k)$. Here 'bar' means reduction modulo I^k. As $\overline{i}_1$ is nilpotent in R/I^k, $\overline{1} + \overline{i}_1$ is unit in R/I^k. By Lemma 5.1 there exists $\varepsilon \in \mathrm{ESp}_{2n}(R/I^4, I/I^4)$ such that $\overline{v}\varepsilon = (\overline{1}, \overline{0}, \ldots, \overline{0})$ in $(R/I^4)^{2n}$. Note that there exists $\varepsilon_0 \in \mathrm{ESp}_{2n}(R,I)$ such that $\overline{\varepsilon_0} = \varepsilon$ and $\overline{v\varepsilon_0} = (\overline{1}, \overline{0}, \ldots, \overline{0})$ in $(R/I^k)^{2n}$. So $v\varepsilon_0 = (1 + x_1, x_2, \ldots, x_{2n})$, where $x_1, \ldots, x_{2n} \in I^k$. □

Lemma 5.4 *Let I be an ideal of R and $v \in \mathrm{Um}_{2n}(R, I^2)$. Assume $2R = R$. Then for $\rho \in \mathrm{E}_{2n-1}(R,I)$, we have $v(1 \perp \rho) \in v\mathrm{ESp}_{2n}(R,I)$.*

Proof: Let $\rho(X) \in \mathrm{E}_{2n-1}(R[X], I[X])$, with $\rho(1) = \rho$ and $\rho(0) = I_{2n-1}$. Let $v = (1 + a_1, a_2, \ldots, a_{2n})$, with $a_i \in I^2$, for $1 \leq i \leq 2n$. Let $V(X) = v(1 \perp \rho(X))$. Note that $\langle e_1, V(X)\rangle = 1 + a_1$.

Let $\mathfrak{m}$ be a maximal ideal of R. If $I \subset \mathfrak{m}$, then $(1 + a_1)_{\mathfrak{m}}$ is unit in $R_{\mathfrak{m}}$. Using Lemma 5.1 we get $g(\mathfrak{m})(X) \in \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]^2)$ such that $V(X)_{\mathfrak{m}} = e_1 g(\mathfrak{m})(X)$. If $I \not\subseteq \mathfrak{m}$, then either $(1 + a_1)_{\mathfrak{m}}$ is a unit or for some i_0, $1 < i_0 \leq 2n$, $a_{i_0} \notin \mathfrak{m}$. In either case, since $I_{\mathfrak{m}} = R_{\mathfrak{m}}$, by Lemma 2.13 we have,

$$V(X)_{\mathfrak{m}} \in e_1 \mathrm{E}_{2n}(R_{\mathfrak{m}}[X]) = e_1 \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X]) = e_1 \mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X], I_{\mathfrak{m}}[X]^2).$$

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Therefore $V(X)_{\mathfrak{m}} \sim_{\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X],I_{\mathfrak{m}}[X]^2)} e_1$, for each maximal ideal $\mathfrak{m}$ of R. Also we will have $g(\mathfrak{m})_0$ from $\mathrm{ESp}_{2n}(R_{\mathfrak{m}}, I_{\mathfrak{m}}^2)$ such that $V(0)_{\mathfrak{m}} g(\mathfrak{m})_0 = e_1$. Therefore $V(X)_{\mathfrak{m}} \sim_{\mathrm{ESp}_{2n}(R_{\mathfrak{m}}[X],I_{\mathfrak{m}}[X]^2)} V(0)_{\mathfrak{m}} g(\mathfrak{m})_0$, for each maximal ideal $\mathfrak{m}$ of R. Using Lemma 5.2 we get, $V(X)_{\mathfrak{m}} \sim_{\mathrm{ESp}_{2n}(I_{\mathfrak{m}}[X])} V(0)_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m}$ of R.

Using Proposition 3.3, we get $V(X) \sim_{\mathrm{ESp}_{2n}(R[X],I[X])} V(0)$. Substituting $X = 1$ we get $v(1 \perp \rho) \in v\mathrm{ESp}_{2n}(R, I)$. □

Theorem 5.5 *Let I be an ideal of R and $2R = R$. Let $v \in \mathrm{Um}_{2n}(R, I)$, then $v\mathrm{E}_{2n}(R, I) = v\mathrm{ESp}_{2n}(R, I)$, for $n \geq 3$.*

Proof: It suffices to show the left hand side is contained in the right hand side. The reverse inclusion is obvious. Let $\varepsilon \in \mathrm{E}_{2n}(R, I)$. Using Lemma 5.3 we get ε_1 from $\mathrm{ESp}_{2n}(R, I)$ such that $v\varepsilon\varepsilon_1 \in \mathrm{Um}_{2n}(R, I^2)$. Using Lemma 2.14 we get ρ in $\mathrm{E}_{2n-1}(R, I)$ with $\varepsilon\varepsilon_1(1 \perp \rho) \in \mathrm{ESp}_{2n}(R, I)$. Now $v\varepsilon = v\varepsilon\varepsilon_1(1 \perp \rho)(1 \perp \rho)^{-1}\varepsilon_1^{-1}$. We have $v\varepsilon\varepsilon_1(1 \perp \rho)$ is in $\mathrm{Um}_{2n}(R, I^2)$. Hence by Lemma 5.4, $[v\varepsilon\varepsilon_1(1 \perp \rho)](1 \perp \rho)^{-1} \in v\mathrm{ESp}_{2n}(R, I)$. Let $v\varepsilon\varepsilon_1(1 \perp \rho)(1 \perp \rho)^{-1} = v\beta$, where β is in $\mathrm{ESp}_{2n}(R, I)$. Therefore $v\varepsilon = v\beta\varepsilon_1^{-1} \in v\mathrm{ESp}_{2n}(R, I)$. □

Theorem 5.6 *Let I be an ideal of R. Assume $2R = R$. Then the natural map $\mathrm{Um}_{2n}(R, I)/\mathrm{ESp}_{2n}(R, I) \longrightarrow \mathrm{Um}_{2n}(R, I)/\mathrm{E}_{2n}(R, I)$ is bijective for $n \geq 3$.*

Proof: The proof follows from Theorem 5.5 □

6. Decrease in injective stability for $K_1\mathrm{Sp}(R)$

Recall that Bass-Milnor-Serre began the study of the stabilisation for the linear group $\mathrm{GL}_n(R)/\mathrm{E}_n(R)$, for $n \geq 3$, where R is a commutative ring with identity. In [4], they showed that $K_1(R) = \mathrm{GL}_{d+3}(R)/\mathrm{E}_{d+3}(R)$, where d is the dimension of the maximal spectrum under the Zariski topology. This is called the *Jacobson-Krull dimension*. They also showed that $K_1(R) = \mathrm{GL}_3(R)/\mathrm{E}_3(R)$, when $d = 1$.

In [3] Bass has proved that if A is an algebra, finite as a module over a commutative ring R, whose spectrum of maximal ideals is a noetherian space of dim d, then the stable range of the ring A is not more than $d + 1$. (See below for definition of stable range of a ring). In [3], §11 Bass conjectured that $K_1(R) = \mathrm{GL}_{d+2}(R)/\mathrm{E}_{d+2}(R)$, where d is the Jacobson-Krull dimension. In [22], L.N. Vaserstein proved this conjecture for an algebra A, which is finite as a module over a commutative ring R, and whose spectrum of maximal ideals is a noetherian space of dim d.

In [17], the second author and W. van der Kallen began the study of whether the stabilization bound above improves in the case of finitely generated algebras A over a field k. Such rings A are called *affine algebras*. Note that by Hilbert's Nullstellensatz, for such rings the *Krull dimension* and the *Jacobson-Krull*

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dimension (i.e. the dimension of the maximal ideal spectrum under the Zariski topology) coincide.

An affine algebra A will be called *non-singular* if $A_{\wp}$ is a regular local ring, for every prime ideal $\wp$ of A. The well-known Jacobian criterion gives an effective method to determine whether a given ideal $I = (f_1,\ldots,f_r)$ of $k[X_1,\ldots,X_n]$ gives rise to a non-singular algebra $k[X_1,\ldots,X_n]/I$.

In [17] it was shown that if A is a non-singular affine algebra of Krull dimension $d > 1$ over a perfect C_1-field, then the natural map

$$\frac{\mathrm{SL}_n(A)}{\mathrm{E}_n(A)} \longrightarrow \frac{\mathrm{SL}_{n+1}(A)}{\mathrm{E}_{n+1}(A)}$$

is injective for $n \geq d+1$. (We refer the reader to [17] for the definition of a C_1-field.)

Thus, the set of all non-singular affine algebras over a perfect C_1-field, of Krull dimension d, became an important subclass of the set of all commutative rings of Jacobson-Krull dimension d, over which one could seek improvement in K_0 and K_1 results. (The famous theorems of Suslin in [19], [21] first showed that stabilization results of H. Bass for K_0 improved over such rings. This is the key reason why one could expect improvement over K_1 for this class of rings.)

L.N. Vaserstein in [23] considered the symplectic, orthogonal and the unitary K_1-functors, and obtained stabilisation theorems for them. These results have been sharpened and extended to other groups in [2].

We restrict ourselves to the symplectic case here.

In ([6], Theorem 2) R. Basu and the second author showed, in particular, that if R is a non-singular affine algebra over a perfect C_1-field of odd Krull dimension $d \geq 2$ then the natural map $\varphi_{n,n+1} : \mathrm{Sp}_{2n}(R)/\mathrm{ESp}_{2n}(R) \longrightarrow \mathrm{Sp}_{2n+2}(R)/\mathrm{ESp}_{2n+2}(R)$ is an isomorphism when $2n \geq d+1$, where d is the Krull dimension of R.

In this section we are going to reprove this result. Moreover via our main result we show that there is a further decrease in the injective stabilisation bound (for the symplectic K_1) of a non-singular affine algebra over a finite field of characteristic not equal to 2 (or its algebraic closure). We show that if the field is a finite field of characteristic not equal to 2 (or its algebraic closure) then the bound improves to $2n \geq d$, provided d is even ≥ 4.

We would like to observe the surjective stability estimates first. First recall

Definition 6.1 Stable Range: Let R be a commutative ring. The following concept was introduced by H. Bass:

(R_m) for every $(a_1,\ldots,a_m) \in \mathrm{Um}_m(R)$, there are $x_i \in R$, for $1 \leq i \leq m-1$, such that $(a_1 + x_1a_m,\ldots,a_{m-1} + x_{m-1}a_m) \in \mathrm{Um}_{m-1}(R)$. The condition (R_m) implies (R_{m+1}), for every $m > 0$. Moreover, for any $n \geq m$, the condition (R_m) implies

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(R_n), with $x_i = 0$, for $i \geq m$. By stable range $\mathrm{sr}(R)$ of a ring R we mean the least n such that (R_n) holds.

Definition 6.2 Stable Dimension: The stable dimension s of a ring R is the integer one less than the stable range. It is denoted by $\mathrm{sdim}(R)$. If R is a Noetherian ring of Krull dimension d, then $\mathrm{sdim}(R) \leq d+1$.

The following is well-known:

Lemma 6.3 *Let I be an ideal of R and $\mathrm{sr}(R) \leq t$. Then $\mathrm{Um}_n(R,I) = e_1\mathrm{E}_n(R,I)$, if $t \geq 2$, and $n \geq t$.*

Proof: We recall its proof. Let $v = (a_1,\ldots,a_{n-1},a_n) \in \mathrm{Um}_n(R,I)$, then $w = (a_1,\ldots,a_{n-1},a_n^2) \in \mathrm{Um}_n(R,I)$. Since $\mathrm{sr}(R) \leq t$, there exists $b_i \in R$ such that $w^* = (a_1+b_1a_n^2,\ldots,a_{n-1}+b_{n-1}a_n^2) \in \mathrm{Um}_{n-1}(R,I)$. There exists $E_1 \in \mathrm{E}_n(R,I)$ such that $vE_1 = (w^*,a_n)$. Let us consider $(w^*,a_n) \in \mathrm{Um}_n(\mathbb{Z} \oplus I)$, with $w^* \in \mathrm{Um}_{n-1}(\mathbb{Z} \oplus I)$. Clearly there exists an elementary matrix $E_2 \in \mathrm{E}_n(\mathbb{Z} \oplus I)$ such that $(w^*,a_n)E_2 = e_1$. By van der Kallen's Excision theorem (see [8], Theorem 3.21) we get an $E_3 \in \mathrm{E}_n(R,I)$ such that $(w^*,a_n)E_3 = e_1$. Hence we have $vE_1E_3 = e_1$, where $E_1E_3 \in \mathrm{E}_n(R,I)$. □

Corollary 6.4 *Let R be a noetherian commutative ring of odd Krull dimension d. Assume that $2R = R$. Let $\sigma \in \mathrm{Sp}_{d+1}(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+3}(R)$. Then σ is symplectic homotopic to the identity, i.e. there exists a $\rho(X) \in \mathrm{Sp}_{d+1}(R[X])$, such that $\sigma = \rho(1)$, and $\rho(0) = I_{d+1}$.*

Proof: Let $\alpha(X) \in \mathrm{ESp}_{d+3}(R[X])$ be such that $(I_2 \perp \sigma) = \alpha(1)$ and $\alpha(0) = I_{d+3}$. Let $e_1\alpha(X) = v(X)$. Therefore $v(X) \in \mathrm{Um}_{d+3}(R[X],(X^2-X))$. The Krull dimension of $R[X]$ is $d+1$. Note that $R[X]$ is noetherian and hence $sdim(R[X]) \leq d+2$. This implies $sr(R[X]) \leq d+3$. Therefore $\mathrm{Um}_{d+3}(R[X],(X^2-X)) = e_1\mathrm{E}_{d+3}(R[X],(X^2-X)) = e_1\mathrm{ESp}_{d+3}(R[X],(X^2-X))$. (First equalities follow from Lemma 6.3 and second equality follows from Theorem 5.6.) Let $\varepsilon(X) \in \mathrm{ESp}_{d+3}(R[X],(X^2-X))$ be such that $v(X) = e_1\varepsilon(X)$. Let us define $\beta(X) = \alpha(X)\varepsilon(X)^{-1}$. Clearly $e_1\beta(X) = e_1$ and $\beta(X) \in \mathrm{ESp}_{d+3}(R[X])$. This implies $\beta(X)$ is of the form $\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \beta^*(X) \end{pmatrix}$, where $\beta^*(X) \in \mathrm{Sp}_{d+1}(R[X])$. Now $(I_2 \perp \sigma) = \alpha(1) = \beta(1)$ since $\varepsilon(1) = I_{d+3}$. Therefore $\beta^*(1) = \sigma, \beta^*(0) = I_d$; and hence σ is (symplectic) homotopic to the identity. □

The next result is proven in the linear case in [25] (Theorem 3.3), using H. Lindel's insight in [12], when R is a localisation of an affine algebra over a field k at a non-singular point. This can be extended to any regular ring $(R,\mathfrak{m})$ containing a field, or if characteristic $R/\mathfrak{m} \notin \mathfrak{m}^2$, using the deep approximation theorem of D. Popescu in [13]. (Also see [14], Corollary 4.4.) The argument imitated in the symplectic case is outlined in [6] and asserts:

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Theorem 6.5 ([6], Theorem 3.13) *Let $(R,\mathfrak{m})$ be a regular local ring. Assume that R contains a field, or characteristic $R/\mathfrak{m} \notin \mathfrak{m}^2$. Then $\mathrm{Sp}_{2n}(R[X]) = \mathrm{ESp}_{2n}(R[X])$, for $n \geq 2$.* □

Corollary 6.6 *Let R be a non-singular affine algebra of odd Krull dimension $d \geq 5$ over a field k of characteristic $\neq 2$. . Then $\mathrm{Sp}_{d+1}(R) \cap \mathrm{ESp}_{d+3}(R) = \mathrm{ESp}_{d+1}(R)$.*

Proof: It suffices to show that the left hand side is contained in the right hand side. The reverse inclusion is obvious. Let $\sigma \in \mathrm{Sp}_{d+1}(R) \cap \mathrm{ESp}_{d+3}(R)$. By Corollary 6.4, σ is symplectic homotopic to the identity, i.e. there exists $\alpha(X) \in \mathrm{Sp}_{d+1}(R[X])$ such that $\alpha(1) = \sigma$ and $\alpha(0) = I_{d+1}$. Now $\alpha(X)_{\wp} \in \mathrm{Sp}_{d+1}(R_{\wp}[X])$ for all prime ideals $\wp$ in R. By Theorem 6.5, $\mathrm{Sp}_{d+1}(R_{\wp}[X]) = \mathrm{ESp}_{d+1}(R_{\wp}[X])$ for all prime ideals $\wp$ in R . Therefore $\alpha(X)_p \in \mathrm{ESp}_{d+1}(R_{\wp}[X])$ for all prime ideals $\wp$ in R. Theorem 2.8 implies $\alpha(X) \in \mathrm{ESp}_{d+1}(R[X])$. We have $\sigma = \alpha(1) \in \mathrm{ESp}_{d+1}(R)$. □

Corollary 6.7 *Let R be a finitely generated algebra of even Krull dimension d over k, where $k = \mathbb{Z}$ or F or $\overline{F}$ and char(k)$\neq 2$. (Here F is a finite field and $\overline{F}$ is its algebraic closure.) Let $\sigma \in \mathrm{Sp}_d(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$. Then σ is (symplectic) homotopic to the identity. In fact, $\sigma = \rho(1)$ for some $\rho(X) \in \mathrm{Sp}_d(R[X]) \cap \mathrm{ESp}_{d+2}(R[X])$, with $\rho(0) = I_d$.*

Proof: Let $\alpha(X) \in \mathrm{ESp}_{d+2}(R[X])$ with $(I_2 \perp \sigma) = \alpha(1)$, $\alpha(0) = I_{d+2}$. Let $e_1\alpha(X) = v(X)$. Therefore $v(X) \in \mathrm{Um}_{d+2}(R[X],(X^2 - X))$. By ([18], Corollary 20.4),

$$\mathrm{Um}_{d+2}(R[X],(X^2-X)) = e_1\mathrm{E}_{d+2}(R[X],(X^2-X)) = e_1\mathrm{ESp}_{d+2}(R[X],(X^2-X)).$$

(The last equality follows from Theorem 5.6.) Let $\varepsilon(X) \in \mathrm{ESp}_{d+2}(R[X],(X^2 - X))$ be such that $v(X) = e_1\varepsilon(X)$. Let us define $\beta(X) = \alpha(X)\varepsilon(X)^{-1}$. Clearly $e_1\beta(X) = e_1$ and $\beta(X) \in \mathrm{ESp}_{d+2}(R[X])$. This implies $\beta(X)$ is of the form $\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \beta^*(X) \end{pmatrix}$, where $\beta^*(X) \in \mathrm{Sp}_d(R[X])$. Now $(I_2 \perp \sigma) = \alpha(1) = \beta(1)$ since $\varepsilon(1) = I_{d+2}$. Therefore $\beta^*(1) = \sigma, \beta^*(0) = I_d$; and hence σ is (symplectic) homotopic to identity. □

Corollary 6.8 *Let R be a finitely generated non-singular algebra of even Krull dimension $d \geq 4$ over k, where k is a finite field or the algebraic closure of a finite field and char(k)$\neq 2$. Let $\sigma \in \mathrm{Sp}_d(R)$ and $(I_2 \perp \sigma) \in \mathrm{ESp}_{d+2}(R)$. Then σ belongs to $\mathrm{ESp}_d(R)$.*

Proof: From the proof of Corollary 6.7 it follows that $\sigma = \beta^*(1)$ for some $\beta^*(X) \in \mathrm{Sp}_d(R[X])$, with $\beta^*(0) = I_d$. We know $\mathrm{Sp}_d(R_{\wp}[X]) = \mathrm{ESp}_d(R_{\wp}[X])$, for all prime ideals $\wp$ of R (see Theorem 6.5). This implies $\beta^*(X) \in \mathrm{ESp}_d(R_{\wp}[X])$, for all prime

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ideals $\wp$ in R. By Theorem 2.8, $\beta^*(X) \in \mathrm{ESp}_d(R[X])$. Hence $\sigma = \beta^*(1)$ belongs to $\mathrm{ESp}_d(R)$. □

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