Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2018.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

102.E (Martin Lukarevski)

Let a, b, c be the lengths of the sides of triangle ABC and let w_a, w_b, w_c be the lengths of the corresponding angle bisectors. Prove that

$$\frac{a^2}{w_a^2} + \frac{b^2}{w_b^2} + \frac{c^2}{w_c^2} \geqslant 4.$$

When does equality hold?

102.F (Michał Kremzer)

Find all pairs of positive real numbers (a, b) such that $\ln(\lfloor a + b \rfloor) = \lfloor a \rfloor + \lfloor b \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

102.G (John Melville)

Show that $\displaystyle\int_0^1 \frac{\ln\left(x + \sqrt{1 - x^2}\right)}{x}\, dx = \frac{\pi^2}{16}$.

102.H (Chris Starr)

Consider the ellipse given by the parametric equations $x = a\cos\theta$, $y = b\sin\theta$ and let $P(p, q)$ be a point not on the ellipse with $p, q \neq 0$.

(a) Show the value of θ giving the minimum distance from P to the ellipse is given by a quartic in $t = \tan\frac{1}{2}\theta$.

(b) Show that rational, non-zero values of a, b, p, q can be found such that the quartic in part (a) factorises as the product of two quadratics with rational coefficients.

(c) Find the exact value of the minimum distance from $P\left(-\frac{15}{4}, 1\right)$ to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

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Solutions and comments on **101.I, 101.J, 101.K, 101.L** (November 2017).

101.I (Michel Bataille)

Let a, b, c be the side-lengths of a triangle ABC, r its inradius and R its circumradius. Prove that

$$\sqrt{\frac{2r}{R}} \leqslant \frac{\sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)}}{a+b+c} \leqslant 1.$$

This refinement of Euler's inequality $R \geqslant 2r$ was clearly enjoyed by solvers. The right-hand inequality is quick to establish: the AM-GM inequality shows that

$$\begin{aligned}\sqrt{a(b+c-a)} + \sqrt{b(c+a-b)} + \sqrt{c(a+b-c)} &\leqslant \tfrac{1}{2}(b+c+c+a+a+b)\\ &= a+b+c,\end{aligned}$$

as required.

Solvers used a variety of methods for the harder left-hand inequality. The shortest proofs were trigonometric ones along the following lines. From the half-angle formula $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} = \sqrt{\frac{a(s-a)}{4Rr}}$, since $abc = 4Rrs$. The left-hand inequality is thus equivalent to

$$\frac{\sqrt{2Rr}}{s}\left(\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2}\right) \geqslant \sqrt{\frac{2r}{R}}$$

or $\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geqslant \frac{s}{R}$.

This result can be found in the literature. Martin Lukarevski proved it as follows:

By the AM-GM inequality

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geqslant 3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = 3\sqrt[3]{\frac{s}{4R}},$$

since $s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

It then suffices to show that $3\sqrt[3]{\frac{s}{4R}} \geqslant \frac{s}{R}$ or $\frac{s}{R} \leqslant \frac{3\sqrt{3}}{2}$; but this is immediate from Jensen's inequality:

$$\frac{s}{R} = \sin A + \sin B + \sin C \leqslant 3\sin\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{2}.$$

Correct solutions were received from: S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. T. Q. Hoare, G. Howlett, A. Li, M. Lukarevski (2 solutions), J. A. Mundie, P. Nüesch, V. Schindler, I. D. Sfikas, G. B. Trustrum, L. Wimmer (2 solutions) and the proposer Michel Bataille.

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101.J (Isaac Sofair)

Squares are described externally on the sides of a convex quadrilateral. By Van Aubel's theorem, the line segments joining the centres of opposite squares are equal and perpendicular to each other.

(a) Prove that the length of each line segment is $\sqrt{2S + \frac{1}{2}(x^2 + y^2)}$, where S is the area of the quadrilateral and x, y are the lengths of its diagonals.

(b) Show that $\dfrac{A}{S} \geqslant 2$, where A is the area of the quadrilateral whose vertices are the centres of the squares. When does equality hold?

The perpendicular line segments in the configuration attracted many elegant solutions using coordinate geometry, matrices, complex numbers and vectors. We illustrate this diversity with Geoff Strickland's proof of part (a).

(a) Let A, B, C, D be the (clockwise) vertices of the convex quadrilateral and E, F, G, H the respective centres of the squares on sides AB, BC, CD, DA. Let $\mathbf{a}, \ldots, \mathbf{h}$ be the position vectors of $A, \ldots, H$. Let $\mathbf{x} = \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ and $\mathbf{y} = \overrightarrow{BD} = \mathbf{d} - \mathbf{b}$ with θ the angle between AC and BD so that $S = \frac{1}{2}xy \sin\theta$. Also define $\mathbf{n}$ to be a unit vector normal to the plane of $ABCD$ such that $\mathbf{n} \times \mathbf{u}$ rotates $\mathbf{u}$ by $+90°$.

Then $\mathbf{e} = \frac{1}{2}\big[\mathbf{a} + \mathbf{b} + \mathbf{n} \times (\mathbf{b} - \mathbf{a})\big]$, $\mathbf{g} = \frac{1}{2}\big[\mathbf{c} + \mathbf{d} + \mathbf{n} \times (\mathbf{d} - \mathbf{c})\big]$ so

$$\begin{aligned}\overrightarrow{EG} &= \tfrac{1}{2}\big[\mathbf{c} - \mathbf{a} + \mathbf{d} - \mathbf{b} + \mathbf{n} \times (\mathbf{d} - \mathbf{b} + \mathbf{a} - \mathbf{c})\big] \\ &= \tfrac{1}{2}\big[\mathbf{x} + \mathbf{y} + \mathbf{n} \times (\mathbf{y} - \mathbf{x})\big]\end{aligned}$$

and

$$\left|\overrightarrow{EG}\right|^2 = \tfrac{1}{4}\big[(\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) + (\mathbf{y}-\mathbf{x}).(\mathbf{y}-\mathbf{x}) + 2\mathbf{n}.(\mathbf{x}+\mathbf{y}) \times (\mathbf{y}-\mathbf{x})\big]$$

since $\mathbf{n} \times (\mathbf{y} - \mathbf{x}) \cdot \mathbf{n} \times (\mathbf{y} - \mathbf{x}) = (\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$.

Hence

$$\begin{aligned}\left|\overrightarrow{EG}\right|^2 &= \tfrac{1}{4}\big[2x^2 + 2y^2 + 4\mathbf{n}.(\mathbf{x} \times \mathbf{y})\big] \\ &= \tfrac{1}{2}\big(x^2 + y^2\big) + xy \sin\theta \\ &= 2S + \tfrac{1}{2}\big(x^2 + y^2\big).\end{aligned}$$

(b) The area of quadrilateral $EFGH$ is given by

$$\Delta = \tfrac{1}{2}\left|\overrightarrow{EG}\right|\left|\overrightarrow{HF}\right| \sin 90° = \tfrac{1}{2}\left|\overrightarrow{EG}\right|^2.$$

Then

$$\begin{aligned}\Delta &= S + \tfrac{1}{4}\big(x^2 + y^2\big) \\ &\geqslant S + \tfrac{1}{2}xy, \text{ by the AM-GM inequality}\end{aligned}$$

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$$\geqslant S + \tfrac{1}{2}xy \sin\theta$$

$$= 2S.$$

There is equality if, and only if, $x = y$ and $\theta = 90°$ so the diagonals of the original quadrilateral are equal and perpendicular just like the Van Aubel line segments.

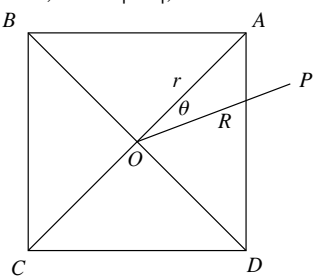
Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, A. Li, P. Nüesch, I. D. Sfikas, C. Starr, G. Strickland, L. Wimmer and the proposer Isaac Sofair.

101.K (Abdurrahim Yilmaz)

Consider a square $ABCD$ with centre O, circumradius r and a point P in the plane of the square with $|OP| = R$. Let $\alpha, \beta, \gamma, \delta$ denote the areas of the triangles with side-lengths $|PA|, |PB|, |PC|$; $|PA|, |PB|, |PD|$; $|PA|, |PC|, |PD|$ and $|PB|, |PC|, |PD|$ respectively, whenever these triangles exist. Prove that

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = \tfrac{1}{4}\left[3\left(R^2 + r^2\right)^2 - 10R^2r^2\right].$$

All respondents deftly handled the potentially heavy algebra in this problem and most solutions went along the following lines. With the notation as in the figure and writing $a = |PA|, \ldots, d = |PD|$, we have



$$a^2 = R^2 + r^2 - 2Rr\cos\theta$$

$$b^2 = R^2 + r^2 + 2Rr\sin\theta$$

$$c^2 = R^2 + r^2 + 2Rr\cos\theta$$

$$d^2 = R^2 + r^2 - 2Rr\sin\theta$$

from which

$$\sum a^4 = 4\left(R^2 + r^2\right)^2 + 8R^2r^2$$

and

$$\sum a^2b^2 = \frac{1}{2}\left[\left(\sum a^2\right)^2 - \sum a^4\right]$$

$$= \frac{1}{2}\left[16\left(R^2 + r^2\right)^2 - 4\left(R^2 + r^2\right)^2 - 8R^2r^2\right]$$

$$= 6\left(R^2 + r^2\right)^2 - 4R^2r^2.$$

Heron's formula gives

$$\alpha^2 = s(s-a)(s-b)(s-c)$$

$$= \tfrac{1}{16}\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)$$

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and so

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= \tfrac{1}{16}\left(4\sum a^2b^2 - 3\sum a^4\right)\\ &= \tfrac{1}{16}\left[24\left(R^2+r^2\right)^2 - 16R^2r^2 - 12\left(R^2+r^2\right)^2 - 24R^2r^2\right]\\ &= \tfrac{1}{4}\left[3\left(R^2+r^2\right)^2 - 10R^2r^2\right].\end{aligned}$$

James Mundie provided a graph showing the region where one or more of the triangles with sides $|PA|$, $|PB|$, $|PC|$, etc. does not exist which he described as a 'cuspy annulus' containing the points A, B, C, D. And the proposer, Abdurrahim Yilmaz noted that $\dfrac{\alpha^2+\beta^2+\gamma^2+\delta^2}{r^4}$ attains a minimum value of $\frac{5}{12}$ when $R = \sqrt{\frac{2}{3}}r$.

Correct solutions were received from: M. Bataille S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, M. Lukarevski, J. A. Mundie, G. Strickland, L. Wimmer and the proposer Abdurrahim Yilmaz.

101.L (Finbarr Holland)

Suppose G and K denote the centroid and the Lemoine point of a triangle ABC. Prove that, unless ABC is equilateral, at least one of the ratios

$$\frac{|AG|}{|AK|}, \frac{|BG|}{|BK|}, \frac{|CG|}{|CK|}$$

exceeds 1.

Vectors, complex numbers and trilinear/areal coordinates all featured in the solutions to this problem.

If $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{g}, \mathbf{k}$ are the position vectors of A, B, C, G, K and if $a = |BC|$, etc. with the usual triangle notation, then, by standard results, we have

$$3\mathbf{g} = \mathbf{p} + \mathbf{q} + \mathbf{r} \qquad \text{and} \qquad \left(a^2+b^2+c^2\right)\mathbf{k} = a^2\mathbf{p} + b^2\mathbf{q} + c^2\mathbf{r}.$$

Then

$$\begin{aligned}9\left|\overrightarrow{AG}\right|^2 &= (\mathbf{p}+\mathbf{q}+\mathbf{r}-3\mathbf{p})^2\\ &= (\mathbf{q}-\mathbf{p}+\mathbf{r}-\mathbf{p})^2\\ &= c^2 + b^2 + 2(\mathbf{q}-\mathbf{p})\cdot(\mathbf{r}-\mathbf{p})\\ &= c^2 + b^2 + b^2 + c^2 - a^2, \text{ by the cosine rule}\\ &= 2b^2 + 2c^2 - a^2.\end{aligned}$$

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Similarly,

$$\begin{aligned}\left(a^2 + b^2 + c^2\right)^2 \left|\overrightarrow{AK}\right|^2 &= \left[b^2(\mathbf{q} - \mathbf{p}) + c^2(\mathbf{r} - \mathbf{p})\right]^2 \\ &= b^4c^2 + c^4b^2 + 2b^2c^2\left(b^2 + c^2 - a^2\right) \\ &= b^2c^2\left(2b^2 + 2c^2 - a^2\right)\end{aligned}$$

so that

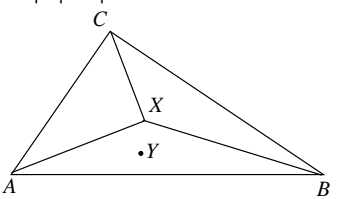
$$\frac{|AG|}{|AK|} = \frac{a^2 + b^2 + c^2}{3bc}.$$

Then $\dfrac{|AG|}{|AK|} \cdot \dfrac{|BG|}{|BK|} \cdot \dfrac{|CG|}{|CK|} = \dfrac{\left(a^2 + b^2 + c^2\right)^3}{27a^2b^2c^2} \geqslant 1$ by the AM-GM inequality with equality if, and only if, $a = b = c$. Thus, unless ABC is equilateral, at least one of the ratios $\dfrac{|AG|}{|AK|}, \dfrac{|BG|}{|BK|}, \dfrac{|CG|}{|CK|}$ exceeds 1.

Several respondents wondered about how special the pair of points G, K were in this problem. Indeed, M. G. Elliott, Graham Howlett and Stan Dolan (below) showed that for *any* two distinct internal points X, Y of triangle ABC, at least one of the ratios $\dfrac{|AX|}{|AY|}, \dfrac{|BX|}{|BY|}, \dfrac{|CX|}{|CY|}$ exceeds 1. For Y is in one of the three subtriangles shown in the figure, say AXB. Then

$$|AY| + |YB| < |AX| + |XB|$$

and therefore at least one of the ratios $\dfrac{|AX|}{|AY|}$ and $\dfrac{|BX|}{|AY|}$ exceeds 1.



Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, M. Lukarevski, J. A. Mundie, V. Schindler, I. D. Sfikas, G. Strickland, L. Wimmer and the proposer Finbarr Holland.

10.1017/mag.2018.81 N.J.L.

An appreciation of the legacy of John Chick

As readers will recall, the November 2017 Problem Corner ended with the sad news of the death John Chick in June 2017, together with an appreciation of his many contributions to the work of the *Gazette*. Recently, The Mathematical Association received a substantial legacy from John's estate and here records its gratitude for this act of generosity from a much loved and greatly missed stalwart of the Association.

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