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Constructive complete distributivity. I

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0. *Introduction*

The relationships, in many cases equivalences, between lattice distributivity, adjunction and continuity have been studied by many authors, for example [**1, 3–8, 12, 13, 15, 17–20, 22, 23**]. Very roughly, we refer to the following circle of ideas. Let L be an ordered set, and $\mathscr{D}L$ a class of subsets of L, and suppose that L has a supremum for each element in $\mathscr{D}L$. We might say that L has $\mathscr{D}$-sups. The 'distributivity' we refer to is that of infs over $\mathscr{D}$-sups. The 'adjunction' is that given by a left adjoint to the map $\vee : \mathscr{D}L \to L$. Now the latter has a left adjoint if and only if it preserves infs, and this means roughly that the $\mathscr{D}$-sup of an intersection is an inf of $\mathscr{D}$-sups. When one does succeed in identifying the $\mathscr{D}$-sup of an intersection as a $\mathscr{D}$-sup of infs, one has an instance of distributivity.

We will not deal with 'continuity' in this paper except to give now a similarly coarse description in terms of adjunction. Write $\Downarrow : L \to \mathscr{D}L$ for a left adjoint to $\vee$ (actually the left adjoint). Write $b \ll a$ to indicate that $b \in \Downarrow a$ and read it as 'b is way below a'. Taking account of the meaning of the adjunction $(\forall a \in L)(\forall S \in \mathscr{D}L)(a \leqslant \vee S \Leftrightarrow \Downarrow a \subseteq S)$, one finds that $b \ll a$ is equivalent to $(\forall S \in \mathscr{D}L)(a \leqslant \vee S \Rightarrow b \in S)$. One can take the latter as the definition of 'way below' and say that L is 'continuous' if every element in L is the $\mathscr{D}$-sup of the elements way below it.

So far, we have been deliberately vague about $\mathscr{D}L$. For the existence of an adjunction $\Downarrow \dashv \vee$ to be a reasonable condition on L, it is important that the elements of $\mathscr{D}L$ be down-closed. That is, if $b \leqslant a \in S \in \mathscr{D}L$ then $b \in S$. However, it may well be the case that such $\mathscr{D}$-sups imply the existence of some non-down-closed sups and that distributivity holds for the wider class. Indeed, and this is almost our focus, the general form of complete distributivity for (a complete lattice) L, in which one asks for distributivity of arbitrary infs over arbitrary sups, is equivalent to the existence of a left adjoint to $\vee : \mathscr{D}L \to L$, where $\mathscr{D}L$ is the class of all down-closed subsets.

At least, that last assertion holds in the presence of the axiom of choice. It in turn implies the axiom of choice as well. For take L to be a power set $\mathscr{P}X$ for an arbitrary set X. The existence of $\Downarrow \dashv \vee$ in this case follows from entirely elementary considerations and yet, complete distributivity for power sets is equivalent to choice. ([**10**], p. 25; [**21**]). One seems to need choice to prove the key results about complete distributivity. For this reason [**13**] is a particularly useful reference since results depending on choice are flagged there with an asterisk.

In this paper we pursue the existence of $\Downarrow \dashv \vee$ as a definition and work without the axiom of choice. We regard the principle embodied in this definition as constructive complete distributivity. Recall that a complete lattice L is a locale (or frame or complete Heyting algebra) if and only if $\vee : \mathscr{D}L \to L$ (where $\mathscr{D}L =$ all down-closed

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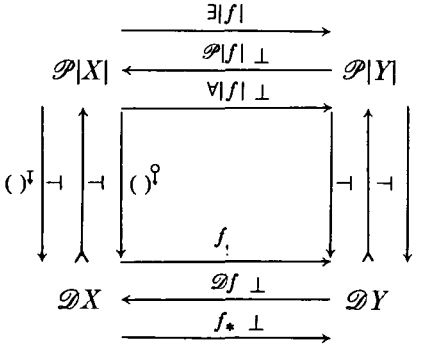
subsets) preserves finite infs. As we remarked above, the existence of $\Downarrow$ is equivalent to the preservation of all infs by $\vee$. So we view our subject as a specialization of locale theory and in [11] we consider questions about change of base topos. Two basic results about complete distributivity which require choice are replaced by choice-free results about constructive complete distributivity. For these we do require that the base topos is Boolean, i.e. that power sets are Boolean algebras, but we also show by counter-example, in one case, that the result is quite good. Precisely, we show that constructive complete distributivity is constructively self-dual in a Boolean topos and that it need not be so in a general topos. We extend Johnstone's asterisk convention by marking with an obelisk those results whose proofs depend on Booleaness of power sets. (See [2], pp. 247–256; see also [9].)

1. *Preliminaries*

An ordered set $(X, \leqslant)$ is a set X together with a binary relation $\leqslant$ on X which is reflexive and transitive. We have no need of antisymmetry but the reader may wish to assume it in order that our terminology agrees with common practice. We write **ord** = **ord** (**set**) for the 2-category of ordered sets, order-preserving functions and (point-wise) inequalities. By **set** we understand the usual base topos. Except where choice or Booleaness are noted, what we have to say is valid in any topos **E** or, more precisely, in the corresponding 2-category **ord** (**E**).

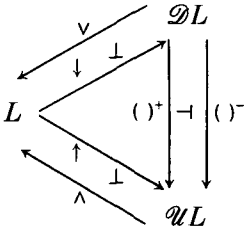
If $x \leqslant y$ and $y \leqslant x$ in X in **ord**, we write $x \cong y$. Similarly, if we are given $f: X \rightleftarrows Y: g$ in **ord** with $gf \cong 1X$ and $fg \cong 1Y$, then we can say that f and g provide an equivalence. Of course if both X and Y are antisymmetric, we can replace $\cong$ by $=$ and equivalences become (order) isomorphisms. We tend to write $X \rightarrowtail Y$ whenever an arrow i is such that $(i(x) \leqslant i(y) \Rightarrow x \leqslant y)$.

For X in **ord**, $\mathscr{D}X$ (henceforth) denotes the (antisymmetric) ordered set of all down-closed subsets of X. (Recall that for any topos **E**, $\mathscr{D}X$ is isomorphic to **ord** $(\mathbf{E})(X^{\mathrm{op}}, \Omega)$, where here $(\ )^{\mathrm{op}}$ denotes reversal of order and Ω is the subobject classifier of **E**, with its usual order.) We regard $\mathscr{D}$ as a 2-functor, $\mathscr{D}: \mathbf{ord}^{\mathrm{coop}} \to \mathbf{ord}$ via inverse image. (Here '$(\ )^{\mathrm{coop}}$' indicates that $\mathscr{D}$ reverses inequalities ($(\ )^{\mathrm{co}}$), and the direction of arrows ($(\ )^{\mathrm{op}}$).) So if $f \dashv u$ in **ord** then $\mathscr{D}f \dashv \mathscr{D}u$. For $f: X \to Y$ in **ord**, $\mathscr{D}f$ has a left adjoint, $f_!$, and a right adjoint, f_*. Now if we write the underlying set functor as $|-|: \mathbf{ord} \to \mathbf{set}$, then $f_!$ and f_* appear in what might be called the Doctrinal Diagram, where $\mathscr{P}$ denotes the power set functor (cf. [2], p. 194; [14], p. 32).



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Thus, for example, $f_!$ is given by applying the ordinary direct image $\exists|f|$ followed by down-closure, $(\ )\downarrow$. We read $(\ )\Downarrow$ *down-interior*. We write $\downarrow = \downarrow X : X \rightarrowtail \mathscr{D}X$ for the function which sends x to $\downarrow x = \{y \in X \mid y \leqslant x\}$. An object L in **ord** is said to be (co)complete if $\downarrow L$ has a left adjoint. It was shown in [**16**] that L complete implies L^{op} complete in any topos. It is sometimes convenient to write $\mathscr{U}L$ for $(\mathscr{D}L^{\mathrm{op}})^{\mathrm{op}}$ and speak



of up notions. The situation is summarized by the above diagram, where the meaning of $\uparrow$ is obvious and S^+ denotes the set of upper bounds for S. Thus for any L, $(\downarrow x)^+ = \uparrow x$ and $(\uparrow x)^- = \downarrow x$. If $\vee$ exists then $\wedge T = \vee(T^-)$, etc. For any L, the adjunction

$$(\ )^c : (\mathscr{P}|L|)^{\mathrm{op}} \underset{\leftarrow}{\overset{\rightarrow}{\top}} \mathscr{P}|L| : ((\ )^c)^{\mathrm{op}}$$

where $(\ )^c$ is the pseudocomplement, restricts to

$$(\ )^c : \mathscr{U}L \underset{\leftarrow}{\overset{\rightarrow}{\top}} \mathscr{D}L : ((\ )^c)^{\mathrm{op}},$$

and this adjunction gives an isomorphism if the topos is Boolean. Dualizing an object of **ord** gives a 2-functor $(\ )^{\mathrm{op}} : \mathbf{ord}^{\mathrm{co}} \to \mathbf{ord}$ by $L \mapsto L^{\mathrm{op}}$. So now if $f \dashv u$ then $u^{\mathrm{op}} \dashv f^{\mathrm{op}}$. For $\mathscr{S} \subseteq \mathscr{P}L$ we abbreviate the set of choice functions from $\mathscr{S}$ to the union of its members by $\Pi\mathscr{S}$.

2. *Constructive complete distributivity*

From now on, let L denote a complete object of **ord**, i.e. essentially a complete lattice. Complete distributivity for L, written (CD) for L, is given by: $(\forall \mathscr{S} \subseteq \mathscr{P}L)(\wedge\{\vee S \mid S \in \mathscr{S}\} = \vee\{\wedge\{T(S) \mid S \in \mathscr{S}\} \mid T \in \Pi\mathscr{S}\})$. We define constructive complete distributivity for L, written (ÇCD) for L, to mean the same equation with the quantification given by $(\forall \mathscr{S} \subseteq \mathscr{D}L)$.

LEMMA 1. *For L complete and $\mathscr{S} \subseteq \mathscr{D}L$, we have* $\cap \mathscr{S} = \{\wedge\{T(S) \mid S \in \mathscr{S}\} \mid T \in \Pi\mathscr{S}\}$.

Proof. Let $x \in \cap \mathscr{S}$. Define $\hat{x} \in \Pi\mathscr{S}$ by $\hat{x}(S) = x$, for all $S \in \mathscr{S}$. Now $\wedge\{\hat{x}(S) \mid S \in \mathscr{S}\} = \wedge\{x\} = x$ so $\cap \mathscr{S} \subseteq \{\wedge\{T(S) \mid S \in \mathscr{S}\} \mid T \in \Pi\mathscr{S}\}$. Let $T \in \Pi\mathscr{S}$. For all $S \in \mathscr{S}$ we have $\wedge\{T(S) \mid S \in \mathscr{S}\} \leqslant T(S) \in S$. It follows from down-closedness that $\wedge\{T(S) \mid S \in \mathscr{S}\} \in \cap \mathscr{S}$.

The proof of the Lemma shows that the right side of the (CDD) equation is much simpler than that of the (CD) equation; only constant choice functions need to be considered.

THEOREM 2. (CCD) *for L (complete) is equivalent to the existence of a left adjoint* $\Downarrow$, *to* $\vee : \mathscr{D}L \to L$.

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Proof. $\vee$ has a left adjoint if and only if $\wedge\{\vee S \mid S \in \mathscr{S}\} = \vee \cap \mathscr{S}$ for all $\mathscr{S} \subseteq \mathscr{D}L$. Now by an application of Lemma 1, it follows that $\vee \cap \mathscr{S} = \vee(\wedge\{T(S) \mid S \in \mathscr{S}) \mid T \in \Pi\mathscr{S}\}$.

Thus, as mentioned in the Introduction, we may take the existence of $\Downarrow \dashv \vee$ as the definition of (CCD).

THEOREM 3. *For any X in* **ord**, $\mathscr{D}X$ *is* (CCD).

Proof. Consider the adjoint sequence resulting, via $\mathscr{D}$, from $\downarrow X : X \to \mathscr{D}X$, namely

$$\mathscr{D}X \underset{\mathscr{D}\downarrow X \ \perp \ (\downarrow X)_*}{\overset{(\downarrow X)_!}{\rightleftarrows}} \mathscr{D}\mathscr{D}X.$$

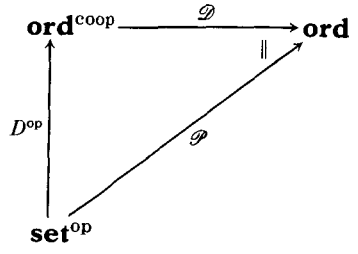
For $T, S \in \mathscr{D}X$, we have $T \in (\downarrow X)_*(S)$ if and only if $\downarrow\mathscr{D}X(T) \subseteq (\downarrow X)_*(S)$ if and only if $\mathscr{D}\downarrow X(\downarrow\mathscr{D}X(T)) \subseteq S$. However, we have

$$\mathscr{D}\downarrow X(\downarrow\mathscr{D}X(T)) = \{x \in X \mid \downarrow X(x) \in \downarrow\mathscr{D}X(T)\} = \{x \in X \mid \downarrow X(x) \subseteq T\} = T.$$

From $T \in (\downarrow X)_*(S)$ if and only if $T \subseteq S$, we are able to conclude that $(\downarrow X)_* = \downarrow\mathscr{D}X$. By uniqueness of adjoints it follows that $\mathscr{D}\downarrow X = \vee (= \cup)$ and $(\downarrow X)_! = \Downarrow$. (See also [**24**].)

COROLLARY 4. *For any X in* **set**, $\mathscr{P}X$ *is* (CCD). *In particular* $\Omega = \mathscr{P}1$ *is* (CCD).

Proof.



where D, discrete order, is left adjoint to $|-|$: **ord** $\to$ **set**.

*THEOREM 5. (CCD) *is equivalent to* (CD).

Proof. Trivially (CD) implies (CCD), so let $\mathscr{S} \subseteq \mathscr{P}L$ and assume that (CCD) holds. With these assumptions we are now able to conclude that

$$\begin{aligned}\wedge\{\vee S \mid S \in \mathscr{S}\} &= \wedge\{\vee(S\downarrow) \mid S \in \mathscr{S}\} = \vee\{\wedge\{T(S\downarrow) \mid S \in \mathscr{S}\} \mid T \in \Pi\{S\downarrow \mid S \in \mathscr{S}\}\} \\ &= \vee \cap \{S\downarrow \mid S \in \mathscr{S}\}.\end{aligned}$$

(The third equality is an instance of Lemma 1.) To finish, it suffices to show that

$$\cap\{S\downarrow \mid S \in \mathscr{S}\} = \{\wedge\{T(S) \mid S \in \mathscr{S}\} \mid T \in \Pi\mathscr{S}\}\downarrow.$$

Let $T \in \Pi\mathscr{S}$. For all $S \in \mathscr{S}$, we have $\wedge\{T(S) \mid S \in \mathscr{S}\} \leqslant T(S) \in S \subseteq S\downarrow$. Since the intersection is down-closed we have $\{\wedge\{T(S) \mid S \in \mathscr{S}\} \mid T \in \Pi\mathscr{S}\}\downarrow \subseteq \cap\{S\downarrow \mid S \in \mathscr{S}\}$. Now let $x \in \cap\{S\downarrow \mid S \in \mathscr{S}\}$. Then $(\forall S \in \mathscr{S})(x \in S\downarrow)$. An equivalent statement is $(\forall S \in \mathscr{S})(\exists y \in S)(x \leqslant y)$, which implies, in the presence of the Axiom of Choice,

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$(\exists T\in\Pi\mathscr{S})(\forall S\in\mathscr{S})(x\leqslant T(S))$ which gives $(\exists T\in\Pi\mathscr{S})(x\leqslant\wedge\{T(S)\,|\,S\in\mathscr{S}\})$ and finally, $x\in\{\wedge\{T(S)\,|\,S\in\mathscr{S}\}\,|\,T\in\Pi\mathscr{S}\}\downarrow$.

THEOREM 6. *The statement of Theorem* 5 *is equivalent to the Axiom of Choice.*

Proof. From the remarks in the Introduction, it suffices to give a constructive demonstration that power sets are (CCD). This we have done in Theorem 3 and Corollary 4.

It is instructive to compare the generality of (CCD) with complete lattices of the form $\mathscr{D}X$. For any (CCD) object L we have $\Downarrow\subseteq\downarrow$ as arrows from L to $\mathscr{D}L$. (This follows from adjointness and $\vee\downarrow x\cong x$ for all $x\in L$.)

For any $f\leqslant g\colon L\to M$ in **ord** a bi-inverter for the inequality is a diagram

$$I\xrightarrow{\ i\ }L\underset{g}{\overset{f}{\rightrightarrows}}M$$

for which

(i) $fi\cong gi$;

(ii) $(\forall T$ in **ord**$)(\forall a,b;T\rightrightarrows I)(ia\leqslant ib\Rightarrow a\leqslant b)$;

(iii) $(\forall T$ in **ord**$)(\forall h\colon T\to L$ with $fh\cong gh)(\exists k\colon T\to I$ with $ik\cong h)$.

Bi-inverters are unique up to equivalence. If M is antisymmetric then $E=\{x\in L\,|\,fx=gx\}$, with the order induced from L, is an instance of a bi-inverter for $f\leqslant g$.

PROPOSITION 7. *For any* X *in* **ord**

$$X\xrightarrow{\ \downarrow X\ }\mathscr{D}X\underset{(\downarrow X)_*}{\overset{(\downarrow X)_!}{\rightrightarrows}}\mathscr{D}\mathscr{D}X$$

is a bi-inverter diagram.

Proof. We have already shown that $(\downarrow X)_*(S)=\{T\in\mathscr{D}X\,|\,T\subseteq S\}$. From the definition of $(\ )_!$ we find that $(\downarrow X)_!(S)=\{T\in\mathscr{D}X\,|\,(\exists x\in X)(T\subseteq\downarrow x\subseteq S)\}$. Taking $S=\downarrow x=(\downarrow X)(x)$ in the above equations allows us to verify (i) in the definition of bi-inverter. We get (ii) from $(\downarrow x\subseteq\downarrow y\Rightarrow x\leqslant y)$. Finally, assume that $(\downarrow X)_*(S)\subseteq(\downarrow X)_!(S)$ and consider $S\in(\downarrow X)_*(S)$. It follows that for some $x\in X$, $S\subseteq\downarrow x\subseteq S$. This gives (iii). Note that in general we do not have unique existence for x so that $\downarrow X$ may fail to be the equalizer of $(\downarrow X)_!$ and $(\downarrow X)_*$.

It follows that if a (CCD) object L is equivalent to a lattice of the form $\mathscr{D}X$ for some X in **ord**, then X is determined, up to equivalence, as the inverter of $\Downarrow\subseteq\downarrow$. For definiteness, we write $X(L)$ for $\{x\in L\,|\,\Downarrow x=\downarrow x\}$.

For any $i\colon X\to L$ in **ord**, i is said to be dense if $1L\colon L\to L$ is the left Kan extension of i along i. In terms of elements this means that for all $a\in L$ we have $a\cong\vee\{i(x)\,|\,x\in X$ and $i(x)\leqslant a\}$. Clearly $\downarrow X\colon X\to\mathscr{D}X$ is dense.

THEOREM 8. *For* L *a* (CCD) *object,* L *is equivalent to a lattice of the form* $\mathscr{D}X$ *if and only if the inclusion* $i\colon X(L)\mapsto L$ *is dense.*

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Proof. We have observed the necessity of the condition. Define $g: L \to \mathscr{D}X$, $X = X(L)$ by $g(a) = \cup\{\downarrow X(x) \mid x \in X \text{ and } ix \leqslant a\}$. ($g$ is the left Kan extension of $\downarrow X$ along i.) Since L is complete, it is a simple formality that g has a left adjoint f, given by

$$f(S) = \vee\{ix \mid x \in X \text{ and } \downarrow X(x) \subseteq S\} = \vee\{ix \mid x \in S\}.$$

Thus $S \subseteq g(f(S))$, for all $S \in \mathscr{D}X$, and $f(g(a)) \leqslant a$, for all $a \in L$.

While S is assumed to be down-closed in X we do not have S down-closed in L. Thus $(\ )\downarrow$ below refers to L. Similarly $\downarrow x = \downarrow L(x)$, which may properly contain $\downarrow X(x)$. To show $g(f(S)) \subseteq S$, let

$$x \in g(f(S)) = \cup\{\downarrow X(x) \mid x \in X \text{ and } ix \leqslant \vee\{iy \mid y \in S\}\}.$$

That is, we assume $x \in X$ and $x \leqslant \vee S$. We must show $x \in S$. But $x \leqslant \vee S = \vee(S\downarrow)$ gives $\downarrow x = \Downarrow x \subseteq S\downarrow$. So $x \leqslant y$ for some $y \in S$. Hence $x \in S$.

Finally, it is standard that the remaining inequality $a \leqslant f(g(a))$ is given by density, so that f and g provide an equivalence of L and $\mathscr{D}X$.

It is well known that the closed unit interval [0, 1] is a completely distributive lattice which is not of the form $\mathscr{D}X$; so density of $X(L)$ in L is not automatic. Nevertheless, this example is interesting when viewed in the spirit of this paper.

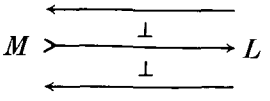
†*Example* 9. Consider [0, 1] as an object of **ord** (**set**). A subset of [0, 1] is down-closed if and only if it is of the form [0, a] or [0, a) for some $a \in [0, 1]$. (See Lemma 14 in the next section.) As usual, $\downarrow a = [0, a]$ but it is also trivial to verify that [0, 1] is (CCD) via $\Downarrow a = [0, a)$. In this example

$$X = X([0, 1]) = \{a \in [0, 1] \mid [0, a) = [0, a]\} = \varnothing.$$

Of course, $\varnothing$ is not dense in [0, 1]. Stated otherwise: $\mathscr{D}(\varnothing) = \{\varnothing\}$ is not equivalent to [0, 1].

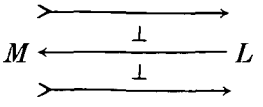
The following two general results about (CCD) enable us to give a simple proof of a classical result of Raney which was also established independently by Büchi (see [7] and [**18–20**]; see also [**3–5**]. We are grateful to Saunders MacLane for correspondence on this matter).

PROPOSITION 10. *If L is* (CCD) *and*



then M is (CCD).

PROPOSITION 11. *If L is* (CCD) *and*



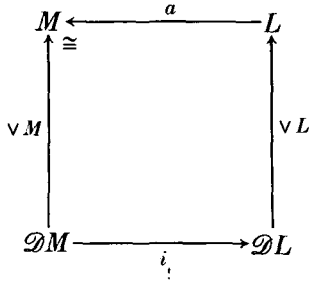
then M is (CCD).

Remark 12. Write **comp** for the sub 2-category of **ord** determined by the complete objects and morphisms which preserve all sups and all infs. Then in **comp**, (CCD) is

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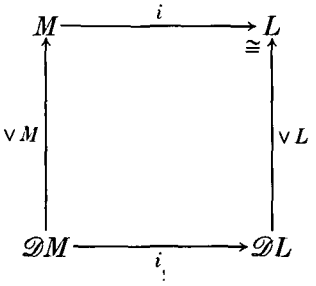
closed with respect to subobjects (Proposition 10) and quotient objects (Proposition 11).

Proofs (of 10 and 11). Write $i : M \overset{\leftarrow}{\perp} L : a$ and recall that in this situation $\vee M$ is given, up to $\cong$, by the following diagram:



If we have $j \dashv a$, the situation in Proposition 11, then it follows immediately that $\Downarrow M \cong a_!(\Downarrow L)j$.

On the other hand, if we have $i \dashv r$, as in Proposition 10, then we have the following $\cong$:



from which it is easy to calculate that $\Downarrow M \cong a_!(\Downarrow L)i$.

*THEOREM 13 (Raney–Büchi). *A complete lattice is completely distributive if and only if it is a (complete) homomorphic image of a complete ring of sets.*

Proof. Let L be a complete lattice. To say that R is a complete ring of sets is to say that $R \rightarrowtail \mathscr{P}X$, for some set X with adjoints as in Proposition 10. To say that L is a homomorphic image of R is to say that $L \leftarrow R$ with adjoints as in Proposition 11. Thus by Corollary 4, and by Propositions 10 and 11, a homomorphic image of a complete ring of sets is (CCD). Conversely if L is (CCD), then $\vee : L \leftarrow \mathscr{D}L \rightarrowtail \mathscr{P}L$ exhibits L as a homomorphic image of a complete ring of sets. Finally, we recover the original Raney–Büchi Theorem by an application of Theorem 5.

3. *Boolean topos results*

Our first result generalizes the construction of $\Downarrow$ in Example 7. We write $y < x$ as an abbreviation for ($y \leqslant x$ and $x \not\leqslant y$) and ${}_{\downarrow}x = \{y \in X \mid y < x\}$. We say that X in **ord** is linear if $x \not\leqslant y$ implies $y \leqslant x$.

†LEMMA 14. *If L is linear and complete, then $\mathscr{D}L = \{{}_{\downarrow}a \mid a \in L\} \cup \{\downarrow a \mid a \in L\}$. For $a < b$ in such L we have ${}_{\downarrow}a \subset \downarrow a \subseteq {}_{\downarrow}b$.*

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Proof. We have $\downarrow b \in \mathscr{D}L$ since $a' \leqslant a < b$ implies $a' < b$. Of course this also shows $(a < b \Rightarrow \downarrow a \subseteq \downarrow b)$.

Let $S \in \mathscr{D}L$ and $s = \vee S$. It is always true that $S \subseteq \downarrow s$ and for L linear, $\downarrow s \subseteq S$. To see the latter let $x < s$ and suppose $(\exists t \in S)(x \leqslant t)$. Then $x \in S$, but if $(\forall t \in S)(x \nleqslant t)$, then $(\forall t \in S)(t \leqslant x)$ which implies that $\vee S = s \leqslant x$, a contradiction. Now, if $s \in S$ then $\downarrow s \subseteq S$, hence $S = \downarrow s$. If $s \notin S$ and $x \in S$ then $x \leqslant s$ and $s \nleqslant x$ (by down-closedness of S), which shows $S \subseteq \downarrow s$ and hence $S = \downarrow s$ in this case.

We always have $\vee \downarrow a \leqslant a$. Call a a limit element if also $a \leqslant \vee \downarrow a$.

*THEOREM 15. *If L is linear and complete then L is* (CD).

†THEOREM 16. *If L is linear and complete then L is* (CCD).

Proof. We show that $\Downarrow : L \to \mathscr{D}L$ is given by $\Downarrow a = \downarrow a$, if a is a limit element, and $\Downarrow a = \downarrow a$ otherwise. In either event, we have $a \leqslant \vee \Downarrow a$, so we have only to show $\Downarrow \vee S \subseteq S$ for $S \in \mathscr{D}L$. Now for $S = \downarrow a$ we have $\Downarrow \vee S = \Downarrow \vee \downarrow a = \Downarrow a \subseteq \downarrow a$. For $S = \downarrow a$ with a a limit element, $\Downarrow \vee S = \Downarrow \vee \downarrow a = \downarrow a$. For $S = \downarrow a$ with a not a limit element, we have $\vee \downarrow a = b < a$ in which case $\Downarrow \vee S = \Downarrow \vee \downarrow a = \Downarrow b \subseteq \downarrow a$ as in the second statement of Lemma 14.

We included the classical Theorem 15 for comparison. Of course it now follows from Theorem 16 and Theorem 5.

*THEOREM 17. *If L is* (CD) *then L^{op} is* (CD).

†THEOREM 18. *If L is* (CCD) *then L^{op} is* (CCD).

Proof. Consider

$$L \underset{\vee L \quad \bot}{\overset{\Downarrow L}{\underset{\downarrow L \quad \bot}{\overset{\longrightarrow}{\longleftarrow}}}} \mathscr{D}L.$$

Applying $(\ )^{\mathrm{op}} : \mathbf{ord}^{\mathrm{co}} \to \mathbf{ord}$ we get

$$L^{\mathrm{op}} \underset{(\vee L)^{\mathrm{op}} \quad \bot}{\overset{(\downarrow L)^{\mathrm{op}}}{\underset{(\Downarrow L)^{\mathrm{op}} \quad \bot}{\overset{\longrightarrow}{\longleftarrow}}}} (\mathscr{D}L)^{\mathrm{op}}.$$

By Booleaness $(\mathscr{D}L)^{\mathrm{op}}$ is isomorphic via $(\ )^{c}$ to $\mathscr{D}(L^{\mathrm{op}})$, which is (CCD) by Theorem 3. By Proposition 11, L^{op} is (CCD).

To see that Theorem 18 does not hold in an arbitrary topos, begin by letting L be a locale for which L^{op} is not a Heyting algebra. For example, take L to be the lattice of open subsets of the reals. Let $\mathbf{E} = \mathbf{sh}(L)$, take $\Gamma : \mathbf{E} \to \mathbf{set}$ global sections and Ω the subobject classifier of $\mathbf{E}$. We know that $\Gamma\Omega$ is isomorphic to L. If Theorem 18 is valid in $\mathbf{E}$ then Ω^{op} is a Heyting algebra in $\mathbf{E}$. However, Γ preserves Heyting algebras and opposites and this immediately provides a contradiction.

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