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On the number of fixed points of a sofic shift-flip system

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Abstract. If X is a sofic shift and $\varphi : X \to X$ is a homeomorphism such that $\varphi^2 = \mathrm{id}_X$ and $\varphi\sigma_X = \sigma_X^{-1}\varphi$, the number of points in X that are fixed by σ_X^m and $\sigma_X^n\varphi$, $m = 1, 2, \ldots, n \in \mathbb{Z}$, is expressed in terms of a finite number of square matrices: the matrices are obtained from Krieger's joint state chain of a sofic shift which is conjugate to X.

1. *Introduction*

Let (X, T) be an invertible topological dynamical system. A homeomorphism $F : X \to X$ is called a *flip map* (or simply a *flip*) for (X, T) if

$$F^2 = \mathrm{id}_X \quad \text{and} \quad FT = T^{-1}F. \tag{1.1}$$

In this case, we call the triple (X, T, F) a *flip system*. If X is a shift space and φ is a flip for (X, σ_X), the flip system (X, σ_X, φ) will be called a *shift-flip system*; if X is of finite type, it will be called a *shift-flip system of finite type*; and if X is sofic, it will be called a *sofic shift-flip system*. Two flip systems (X, T, F) and (X', T', F') are said to be *conjugate* if there is a homeomorphism $\theta : X \to X'$ such that

$$\theta \circ T = T' \circ \theta \quad \text{and} \quad \theta \circ F = F' \circ \theta.$$

We call the homeomorphism θ a *conjugacy* from (X, T, F) to (X', T', F').

If m is a positive integer, the number of periodic points in (X, T) of period m is denoted by $p_m(T)$:

$$p_m(T) = |\{x \in X : T^m(x) = x\}|.$$

Similarly, if F is a flip for (X, T), m is a positive integer and n is an integer, then $p_{m,n}(T, F)$ will denote the number of points in X that are fixed by T^m and T^nF:

$$p_{m,n}(T, F) = |\{x \in X : T^m(x) = T^nF(x) = x\}|.$$

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It is evident that if the flip systems (X, T, F) and (X', T', F') are conjugate, then $p_m(T) = p_m(T')$ and $p_{m,n}(T, F) = p_{m,n}(T', F')$ for all m and n. From (1.1), it follows that

$$p_{m,n}(T, F) = p_{m,n+m}(T, F) = p_{m,n+2}(T, F).$$

Hence $p_{m,n}(T, F) = p_{m,0}(T, F)$ if m and n are even; $p_{m,n}(T, F) = p_{m,1}(T, F)$ if m is even and n is odd; and $p_{m,n}(T, F) = p_{m,0}(T, F)$ for all n if m is odd. Thus we need only consider the numbers $p_{2m,0}(T, F)$, $p_{2m,1}(T, F)$ and $p_{2m-1,0}(T, F)$, $m = 1, 2, \ldots$.

It is well known that if X is a shift of finite type, then there is a square matrix A whose entries are non-negative integers such that

$$p_m(\sigma_X) = \operatorname{tr}(A^m) \quad (m = 1, 2, \ldots). \tag{1.2}$$

It is also well known that if X is a sofic shift, then there are square matrices $A_1, A_2, \ldots, A_r$ whose entries are integers such that

$$p_m(\sigma_X) = \sum_{k=1}^{r} (-1)^{k+1} \operatorname{tr}(A_k^m) \quad (m = 1, 2, \ldots).$$

Suppose that the sequence $\{(p_m(T))^{1/m}\}$ is bounded. Then the Artin–Mazur zeta function ζ_T of (X, T), found in [**1**], is defined by

$$\zeta_T(t) = \exp\left(\sum_{m=1}^{\infty} \frac{p_m(T)}{m} t^m \right),$$

and the results mentioned above imply that if X is a shift of finite type, then ζ_{σ_X} is the reciprocal of a polynomial, and if X is a sofic shift, then ζ_{σ_X} is a rational function [**3, 9–11**].

Suppose that G is a group, M is a set and $\alpha : G \times M \to M$ is a G-action on M. Let $\mathcal{F}$ denote the set of finite-index subgroups (subgroups H such that $|G/H| < \infty$), and for each $H \in \mathcal{F}$ set

$$p_H(\alpha) = |\{x \in M : \forall h \in H \ \alpha(h, x) = x\}|.$$

The Lind zeta function ζ_α of the action α is defined by

$$\zeta_\alpha(t) = \exp\left(\sum_{H \in \mathcal{F}} \frac{p_H(\alpha)}{|G/H|} t^{|G/H|} \right)$$

(actually, Lind introduced this zeta function in the special case where $G = \mathbb{Z}^d$ [**8**]). It is clear that if $\alpha : \mathbb{Z} \times X \to X$ is given by $\alpha(m, x) = T^m x$, then the Lind zeta function ζ_α becomes the Artin–Mazur zeta function ζ_T. Let D_∞ be the infinite dihedral group, that is, D_∞ is the group with presentation $(a, b : b^2 = 1, \ ba = a^{-1}b)$. From (1.1), it follows that to each flip F for (X, T) there corresponds a unique D_∞-action $\alpha = \alpha_{T,F}$ on X such that $\alpha(a, x) = Tx$ and $\alpha(b, x) = Fx$ for all $x \in X$. In [**7**] it is shown that the Lind zeta function of the D_∞-action $\alpha = \alpha_{T,F}$ satisfies

$$\zeta_\alpha(t) = \zeta_T(t^2)^{1/2} \exp(G_{T,F}(t)),$$

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where

$$G_{T,F}(t) = \sum_{m=1}^{\infty} \left(p_{2m-1,0}(T, F)t^{2m-1} + \frac{p_{2m,0}(T, F) + p_{2m,1}(T, F)}{2} t^{2m} \right).$$

In §4 of this paper we will show that $p_{2m,0}(T, F) + p_{2m,1}(T, F)$ is even for every m. Thus the coefficients of $G_{T,F}$ are all non-negative integers. The function $G_{T,F}$ will be called the *generating function* of (X, T, F).

In the case where (X, σ_X, φ) is a shift-flip system of finite type, a formula for $p_{m,n}(\sigma_X, \varphi)$ which is similar to (1.2) is known [**7**]. In order to present it, we need some notation: if M is a matrix, $\mathcal{S}[M]$ will denote the sum of the entries of M, that is,

$$\mathcal{S}[M] = \sum_{I,J} M_{IJ},$$

and M^{Δ} will denote the matrix whose diagonal entries are identical with those of M but whose other entries are equal to 0, that is,

$$(M^{\Delta})_{IJ} = \begin{cases} M_{IJ} & \text{if } I = J, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, $\mathrm{tr}(M) = \mathcal{S}[M^{\Delta}]$ whenever M is a square matrix. The following is proved in [**7**, p. 298].

THEOREM A. *If (X, σ_X, φ) is a shift-flip system of finite type, then there are zero–one square matrices A and J such that*

$$\begin{aligned} p_m(\sigma_X) &= \mathrm{tr}(A^m), \\ p_{2m,0}(\sigma_X, \varphi) &= \mathcal{S}[J^{\Delta} A^m J^{\Delta}], \\ p_{2m,1}(\sigma_X, \varphi) &= \mathcal{S}[(JA)^{\Delta} A^{m-1} (AJ)^{\Delta}] \quad \textit{and} \\ p_{2m-1,0}(\sigma_X, \varphi) &= \mathcal{S}[J^{\Delta} A^{m-1} (AJ)^{\Delta}] \quad (m = 1, 2, \ldots). \end{aligned}$$

Remark. As is proved in [**7**], Theorem A implies that if (X, σ_X, φ) is a shift-flip system of finite type, then its generating function is a rational function.

In this paper, we extend Theorem A to the sofic case.

THEOREM B. *If (X, σ_X, φ) is a sofic shift-flip system, then there are square matrices A_k, B_k and J_k, $k = 1, 2, \ldots, r$, whose entries are integers such that*

$$\begin{aligned} p_m(\sigma_X) &= \sum_{k=1}^{r} (-1)^{k+1} \mathrm{tr}(A_k^m), \\ p_{2m,0}(\sigma_X, \varphi) &= \sum_{k=1}^{r} (-1)^{k+1} \mathcal{S}[J_k^{\Delta} B_k^m J_k^{\Delta}], \\ p_{2m,1}(\sigma_X, \varphi) &= \sum_{k=1}^{r} (-1)^{k+1} \mathcal{S}[(J_k A_k)^{\Delta} B_k^{m-1} (A_k J_k)^{\Delta}] \quad \textit{and} \\ p_{2m-1,0}(\sigma_X, \varphi) &= \sum_{k=1}^{r} (-1)^{k+1} \mathcal{S}[J_k^{\Delta} B_k^{m-1} (A_k J_k)^{\Delta}] \quad (m = 1, 2, \ldots). \end{aligned}$$

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The same method as in [**7**, proof of Theorem 3.2] gives the following corollary.

COROLLARY. *The generating function of a sofic shift-flip system is a rational function.*

Remark. (Suggested by a referee) As is mentioned above, the coefficients of the generating function are all non-negative integers. This fact and well-known theorems of Berstel and Soittola [**4, 5, 12**] imply that the generating function of a sofic shift-flip system is $\mathbb{N}$-rational. See §4 of this paper.

We prove Theorem B in §2. In our proof of the theorem, Krieger's joint state chain plays a crucial role, and it is described in §3. We conclude this paper with some remarks and examples (§4).

2. *Proof of Theorem B*

We start with some preliminaries. A flip φ for a shift dynamical system (X, σ_X) is said to be a *one-block flip* if

$$x, x' \in X \quad \text{and} \quad x_0 = x'_0 \Rightarrow \varphi(x)_0 = \varphi(x')_0.$$

In this case, there is a unique map $\tau : \mathcal{B}_1(X) \to \mathcal{B}_1(X)$ such that $\tau^2 = \mathrm{id}_{\mathcal{B}_1(X)}$ and that

$$\varphi(x)_i = \tau(x_{-i}) \quad (x \in X, i \in \mathbb{Z}).$$

We call τ the *symbol map* of φ.

LEMMA 2.1. *Let (X, σ_X, φ) be a shift-flip system. Then there is a shift-flip system (Y, σ_Y, ψ) such that it is conjugate to (X, σ_X, φ) and ψ is one-block.*

Proof. If we define $\mathcal{A}$, $\tau : \mathcal{A} \to \mathcal{A}$, $\Psi : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ and $\theta : X \to \mathcal{A}^{\mathbb{Z}}$ by

$$\mathcal{A} = \{(a, b) : \exists x \in X \text{ s.t. } a = x_0 \text{ and } b = \varphi(x)_0\},$$
$$\tau(a, b) = (b, a) \quad ((a, b) \in \mathcal{A}),$$
$$\Psi(y)_i = \tau(y_{-i}) \quad (y \in \mathcal{A}^{\mathbb{Z}}, i \in \mathbb{Z}),$$

and

$$\theta(x)_i = (x_i, \varphi(x)_{-i}) \quad (x \in X, i \in \mathbb{Z}),$$

then $\mathcal{A}$ is a finite set, $\tau^2 = \mathrm{id}_{\mathcal{A}}$, Ψ is a one-block flip for $(\mathcal{A}^{\mathbb{Z}}, \sigma)$, θ is one-to-one and continuous, and

$$\theta \circ \sigma_X = \sigma \circ \theta \quad \text{and} \quad \theta \circ \varphi = \Psi \circ \theta;$$

hence we obtain the desired result by setting $Y = \theta(X)$ and $\psi = \Psi|_Y$. □

If $\mathcal{A}$ is a finite set and A is a zero–one $\mathcal{A} \times \mathcal{A}$ matrix, then X_A will denote the topological Markov chain determined by A:

$$\mathsf{X}_A = \{x \in \mathcal{A}^{\mathbb{Z}} : \forall i \in \mathbb{Z}\ A(x_i, x_{i+1}) = 1\}.$$

In this case, we denote the restriction of the shift map of $\mathcal{A}^{\mathbb{Z}}$ to X_A by σ_A. Suppose that $\mathcal{A}$ is a finite set, and A, J are zero–one $\mathcal{A} \times \mathcal{A}$ matrices such that

$$J^2 = I \quad \text{and} \quad JA = A^{\mathsf{T}}J. \tag{2.1}$$

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Since J is zero–one and $J^2 = I$, it follows that there is a unique map $\tau_J : \mathcal{A} \to \mathcal{A}$ such that $\tau_J^2 = \mathrm{id}_{\mathcal{A}}$, and that for $a, b \in \mathcal{A}$ we have $J(a, b) = 1$ if and only if $\tau_J(a) = b$; and since $JA = A^{\mathsf{T}}J$,

$$A(a, b) = A(\tau_J(b), \tau_J(a)) \quad (a, b \in \mathcal{A}). \tag{2.2}$$

We define $\varphi_J : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by

$$\varphi_J(x)_i = \tau_J(x_{-i}) \quad (x \in \mathcal{A}^{\mathbb{Z}}, i \in \mathbb{Z}).$$

Since $\tau_J^2 = \mathrm{id}_{\mathcal{A}}$, it follows that φ_J is a one-block flip for $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ whose symbol map is τ_J; and (2.2) implies that $\varphi_J(\mathsf{X}_A) = \mathsf{X}_A$. Thus the restriction of φ_J to X_A is a flip for (X_A, σ_A). We denote the restriction by $\varphi_{J,A}$.

The following proposition will be proved in the next section.

PROPOSITION 2.2. *Suppose that X is a sofic shift and φ is a one-block flip for (X, σ_X) with symbol map τ. Then there are a finite set $\mathcal{A}$, a map (labeling) $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$ and zero–one $\mathcal{A} \times \mathcal{A}$ matrices A and J having the following properties.*

(1) *$\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring.*
(2) *$\mathcal{L}_\infty$ has no graph diamonds.*
(3) *A and J satisfy (2.1).*
(4) *$\mathcal{L} \circ \tau_J = \tau \circ \mathcal{L}$.*
(5) *If $\delta \in \{0, 1\}$, $x \in X$ and $\sigma_X^\delta \varphi(x) = x$, then there is a $y \in \mathsf{X}_A$ such that $\mathcal{L}_\infty(y) = x$ and $\sigma_A^\delta \varphi_{J,A}(y) = y$.*

Remarks. (a) Recall that a graph diamond for $\mathcal{L}_\infty$ is a pair $(a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_n)$ of distinct blocks in $\mathcal{B}(\mathsf{X}_A)$ such that $a_1 = b_1$, $a_n = b_n$ and $\mathcal{L}(a_i) = \mathcal{L}(b_i)$ for all i. (b) By (4), we have $\mathcal{L}_\infty \circ \varphi_{J,A} = \varphi \circ \mathcal{L}_\infty$.

To prove Theorem B, suppose that (X, σ_X, φ) is a sofic shift-flip system. By Lemma 2.1, we may assume that φ is a one-block flip. Let τ denote the symbol map of φ, and let $\mathcal{A}$, $\mathcal{L}$, A and J be as in Proposition 2.2.

We denote the cardinality of $\mathcal{A}$ by r, and for each integer j we denote the projection $\mathsf{X}_A \ni y \mapsto y_j \in \mathcal{A}$ by P_j. The following is trivially proved.

LEMMA 2.3. *We have $1 \le |\mathcal{L}_\infty^{-1}(x)| \le r^2$ for every $x \in X$. If $x \in X$ is periodic, then P_j is one-to-one on $\mathcal{L}_\infty^{-1}(x)$ for every j. As a consequence, $1 \le |\mathcal{L}_\infty^{-1}(x)| \le r$ whenever $x \in X$ is periodic.*

If π is a permutation of a finite set, its sign will be denoted by $\mathrm{sgn}(\pi)$. When $<$ is a linear order of $\mathcal{A}$ and f is a one-to-one function from a subset S of $\mathcal{A}$ into $\mathcal{A}$, we set

$$\mathrm{sgn}_<(f) = (-1)^{N(<, f)},$$

where

$$N(<, f) = |\{(a, b) \in S \times S : a < b \text{ and } f(b) < f(a)\}|.$$

If $g : f(S) \to \mathcal{A}$ is one-to-one again, then

$$\mathrm{sgn}_<(g \circ f) = \mathrm{sgn}_<(g)\, \mathrm{sgn}_<(f).$$

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We also have $\mathrm{sgn}_<(f) = \mathrm{sgn}(f)$ whenever f is a permutation of a subset of $\mathcal{A}$. From here on, we fix a linear order $<$ of $\mathcal{A}$ and write

$$\mathrm{sgn}(f) = \mathrm{sgn}_<(f)$$

for an arbitrary one-to-one function f from a subset of $\mathcal{A}$ into $\mathcal{A}$.

If $S \subset \mathcal{A}$, the set $\tau_J(S)$ will be denoted by S^*. It is obvious that $(S^*)^* = S$. When $S, T \subset \mathcal{A}$ and $f : S \to T$ is a one-to-one correspondence, we denote the map

$$T^* \ni \tau_J(f(a)) \mapsto \tau_J(a) \quad (a \in S)$$

by f^*. Thus $f^* = (\tau_J|_S) \circ f^{-1} \circ (\tau_J|_{T^*}) : T^* \to S^*$ is a one-to-one correspondence and

$$\mathrm{sgn}(f^*) = \mathrm{sgn}(\tau_J|_S)\mathrm{sgn}(f^{-1})\mathrm{sgn}(\tau_J|_{T^*}). \tag{2.3}$$

It is clear that $(f^*)^* = f$.

Let k be a positive integer $\leq r(= |\mathcal{A}|)$. We set

$$\mathcal{A}_k = \{S \subset \mathcal{A} : |S| = k \text{ and } |\mathcal{L}(S)| = 1\};$$

for $S_1, S_2 \in \mathcal{A}_k$ we denote by $F(S_1, S_2)$ the set of one-to-one functions $f : S_1 \to S_2$ such that $A(a, f(a)) = 1$ for all $a \in S_1$. By (4) in Proposition 2.2, we have $S^* \in \mathcal{A}_k$ whenever $S \in \mathcal{A}_k$ and, by (2.2), we have $f^* \in F(S_2^*, S_1^*)$ whenever $f \in F(S_1, S_2)$. It is clear that

$$F(S_1, S_2) \ni f \mapsto f^* \in F(S_2^*, S_1^*)$$

is a one-to-one correspondence.

For notational convenience, we adopt the convention that an empty sum is 0 and an empty product is 1:

$$\sum_{j \in \varnothing} x_j = 0 \quad \text{and} \quad \prod_{j \in \varnothing} x_j = 1.$$

Define the $\mathcal{A}_k \times \mathcal{A}_k$ matrices A_k, B_k and J_k by

$$A_k(S_1, S_2) = \sum_{f \in F(S_1, S_2)} \mathrm{sgn}(f),$$

$$B_k(S_1, S_2) = \begin{cases} 1 & \text{if } F(S_1, S_2) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$J_k(S_1, S_2) = \begin{cases} \mathrm{sgn}(\tau_J|_{S_1}) & \text{if } \tau_J(S_1) = S_2, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that J_k is symmetric and satisfies $J_k^2 = I$. A simple calculation using (2.3) and the fact that $F(S_1, S_2) \ni f \mapsto f^* \in F(S_2^*, S_1^*)$ is a one-to-one correspondence gives

$$A_k(S_2^*, S_1^*) = J_k(S_1^*, S_1)A_k(S_1, S_2)J_k(S_2, S_2^*) \quad (S_1, S_2 \in \mathcal{A}_k), \tag{2.4}$$

which may be expressed as $A_k^{\mathsf{T}} = J_k A_k J_k$ or, equivalently, as $J_k A_k = A_k^{\mathsf{T}} J_k$. Similarly, we have $B_k(S_1, S_2) = B_k(S_2^*, S_1^*)$.

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It is well known [**2, 9, 10**] that

$$p_m(\sigma_X) = \sum_{k=1}^{r} (-1)^{k+1} \operatorname{tr}(A_k^m) \quad (m = 1, 2, \ldots). \tag{2.5}$$

Hence the first formula in Theorem B holds. A proof of (2.5) is found in [**9**, §6.4] and our proof of the other formulas in Theorem B is a 'modification' of it. In order to explain the modification, we introduce some notation and give an outline of the proof of (2.5).

Let m be a positive integer. For $k = 1, 2, \ldots, r$ we denote by $\mathcal{I}_k(m)$ the set of all $(m+1)$-blocks $S_0S_1 \cdots S_m$ over the alphabet $\mathcal{A}_k$ such that $S_0 = S_m$ and $\Pi_{j=0}^{m-1} A_k(S_j, S_{j+1}) \neq 0$, so that

$$\operatorname{tr}(A_k^m) = \sum_{S_0S_1\cdots S_m \in \mathcal{I}_k(m)} \prod_{j=0}^{m-1} A_k(S_j, S_{j+1}).$$

LEMMA 2.4. *Suppose that $S_0S_1 \cdots S_m \in \mathcal{A}_k^{m+1}$ and $S_0 = S_m$. Then each of the following implies the other three:*

(1) $S_0S_1 \cdots S_m \in \mathcal{I}_k(m)$;
(2) $B_k(S_j, S_{j+1}) = 1$ *for* $0 \le j \le m-1$;
(3) $|F(S_j, S_{j+1})| \ge 1$ *for* $0 \le j \le m-1$;
(4) $|F(S_j, S_{j+1})| = 1$ *for* $0 \le j \le m-1$.

Proof. It is obvious that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1). If $|F(S_j, S_{j+1})| \ge 2$ for some j, then an elementary argument shows that there is a graph diamond for $\mathcal{L}_\infty$. Hence (3) implies (4). □

Let $F(m)$ be the set of points in X that are fixed by σ_X^m:

$$F(m) = \{x \in X : \sigma_X^m(x) = x\}.$$

Thus $p_m(\sigma_X) = |F(m)|$. Suppose that $x \in F(m)$. By Lemma 2.3, we have $1 \le |\mathcal{L}_\infty^{-1}(x)| \le r$. Since $\sigma_X^m(x) = x$, it follows that $\sigma_A^m(\mathcal{L}_\infty^{-1}(x)) = \mathcal{L}_\infty^{-1}(x)$. Thus the restriction of σ_A^m to $\mathcal{L}_\infty^{-1}(x)$ is a permutation of $\mathcal{L}_\infty^{-1}(x)$. Let $\mathcal{C}(m; x)$ denote the set of all σ_A^m-invariant subsets of $\mathcal{L}_\infty^{-1}(x)$:

$$\mathcal{C}(m; x) = \{E \subset \mathcal{L}_\infty^{-1}(x) : \sigma_A^m(E) = E\}.$$

The following lemma is proved in [**9**, §6.4].

LEMMA 2.5. *Let π be a permutation of a non-empty finite set S and $\mathcal{C} = \{E \subset S : \pi(E) = E\}$. Then*

$$\sum_{E \in \mathcal{C}\setminus\{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\pi|_E) = 1.$$

With the help of this lemma, we obtain

$$p_m(\sigma_X) = \sum_{x \in F(m)} \sum_{E \in \mathcal{C}(m;x)\setminus\{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\sigma_A^m|_E).$$

We set $\mathcal{C}(m) = \bigcup_{x \in F(m)} \mathcal{C}(m; x)$ and

$$\mathcal{C}_k(m) = \{E \in \mathcal{C}(m) : |E| = k\} \quad (k = 1, 2, \ldots, r).$$

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If $x, x' \in F(m)$ and $x \neq x'$, then $\mathcal{C}(m; x) \cap \mathcal{C}(m; x') = \{\varnothing\}$; hence

$$p_m(\sigma_X) = \sum_{E \in \mathcal{C}(m)\setminus\{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\sigma_A^m|_E).$$

It is obvious that $\mathcal{C}(m)\setminus\{\varnothing\} = \bigcup_{k=1}^{r} \mathcal{C}_k(m)$ and the sets $\mathcal{C}_1(m), \mathcal{C}_2(m), \ldots, \mathcal{C}_r(m)$ are mutually disjoint. Hence, we obtain

$$p_m(\sigma_X) = \sum_{k=1}^{r} (-1)^{k+1} \sum_{E \in \mathcal{C}_k(m)} \operatorname{sgn}(\sigma_A^m|_E).$$

Now (2.5) is an immediate consequence of the following lemma.

LEMMA 2.6. *If $E \in \mathcal{C}_k(m)$ and*

$$S_j = P_j(E) \quad (0 \le j \le m), \tag{2.6}$$

then $S_0 S_1 \cdots S_m \in \mathcal{I}_k(m)$, $A_k(S_j, S_{j+1}) \in \{-1, 1\}$ for $0 \le j \le m-1$, and

$$\operatorname{sgn}(\sigma_A^m|_E) = \prod_{j=0}^{m-1} A_k(S_j, S_{j+1}).$$

Conversely, if $S_0 S_1 \cdots S_m \in \mathcal{I}_k(m)$, then there is a unique $E \in \mathcal{C}_k(m)$ such that (2.6) holds.

Proof. Suppose that $E \in \mathcal{C}_k(m)$ and the sets $S_0, S_1, \ldots, S_m$ are given by (2.6). Let $x \in F(m)$ be such that $E \subset \mathcal{L}_\infty^{-1}(x)$. Then

$$\mathcal{L}(S_j) = \{\mathcal{L}(y_j) : y \in E\} = \{x_j\} \quad (j = 0, 1, \ldots, m) \tag{2.7}$$

and Lemma 2.3 implies that $P_j|_E : E \to S_j$ is a one-to-one correspondence for all j. In particular,

$$|S_j| = |E| = k \quad (j = 0, 1, \ldots, m).$$

Thus $S_0, S_1, \ldots, S_m \in \mathcal{A}_k$; and since $\sigma_A^m(E) = E$, we have $S_0 = S_m$.

If we set $f_j = P_{j+1} \circ (P_j|_E)^{-1}$ for $j = 0, 1, \ldots, m-1$, then $f_j \in F(S_j, S_{j+1})$ for all j; hence Lemma 2.4 implies that $S_0 S_1 \cdots S_m \in \mathcal{I}_k(m)$. By the same lemma, we also have $F(S_j, S_{j+1}) = \{f_j\}$ for all j, and this implies that $A_k(S_j, S_{j+1}) \in \{-1, 1\}$ for all j, and that

$$\prod_{j=0}^{m-1} A_k(S_j, S_{j+1}) = \prod_{j=0}^{m-1} \operatorname{sgn}(f_j) = \operatorname{sgn}(f_{m-1} \circ \cdots \circ f_1 \circ f_0) = \operatorname{sgn}(P_m \circ (P_0|E)^{-1}).$$

Since $P_m(y) = P_0(\sigma_A^m(y))$ for all $y \in \mathsf{X}_A$,

$$\operatorname{sgn}(\sigma_A^m|_E) = \operatorname{sgn}(P_0 \circ \sigma_A^m \circ (P_0|_E)^{-1}) = \operatorname{sgn}(P_m \circ (P_0|_E)^{-1}),$$

and this proves the first assertion.

To prove the second assertion, suppose that $S_0 S_1 \cdots S_m \in \mathcal{I}_k(m)$. By Lemma 2.4, there are functions $f_0, f_1, \ldots, f_{m-1}$ such that $F(S_j, S_{j+1}) = \{f_j\}$ for all j. If E denotes the set of all $y \in \mathsf{X}_A$ such that

$$l \in \mathbb{Z} \text{ and } 0 \le j \le m-1 \quad \Rightarrow \quad y_{lm+j} \in S_j \text{ and } y_{lm+j+1} = f_j(y_{lm+j}),$$

then it is easy to see that $E \in \mathcal{C}_k(m)$ and (2.6) holds.

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It remains to prove the uniqueness of E, and to do this, suppose that $x, x' \in F(m)$, $E, E' \in \mathcal{C}_k(m)$, $E \subset \mathcal{L}_\infty^{-1}(x)$, $E' \subset \mathcal{L}_\infty^{-1}(x')$ and that $P_j(E) = P_j(E')$ for $0 \le j \le m-1$. Then (2.7) implies that $x = x'$. Hence we have $E = E'$, by Lemma 2.3. This completes the proof. □

In our proof of the other formulas in Theorem B, we need the following modification of Lemma 2.5.

LEMMA 2.7. *Let π be a permutation of a finite set S. Suppose that $\mathcal{C} \subset 2^S$ and $S_0 \subset S$ satisfy the following.*

(1) *$\pi(E) = E$ for every $E \in \mathcal{C}$.*
(2) *If $E_1, E_2 \in \mathcal{C}$, then $E_1 \cup E_2, E_1 \setminus E_2 \in \mathcal{C}$.*
(3) *$S_0 \neq \varnothing$ and $\pi(S_0) = S_0$.*
(4) *If $E \subset S_0$ and $\pi(E) = E$, then $E \in \mathcal{C}$.*

Then

$$\sum_{E \in \mathcal{C} \setminus \{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\pi|_E) = 1. \tag{2.8}$$

Proof. The assumptions imply that $\mathcal{C}$ has at least two distinct elements, namely S_0 and $\varnothing$. For convenience, we set $\operatorname{sgn}(\pi|_\varnothing) = 1$. With this notation, (2.8) is equivalent to

$$\sum_{E \in \mathcal{C}} (-1)^{|E|} \operatorname{sgn}(\pi|_E) = 0.$$

If we put

$$\mathcal{N} = \{E \in \mathcal{C} : E \subset S_0\} \text{ and } \mathcal{R} = \{E \in \mathcal{C} : E \cap S_0 = \varnothing\},$$

then (2) implies that

$$\mathcal{N} \times \mathcal{R} \ni (E_1, E_2) \mapsto E_1 \cup E_2 \in \mathcal{C}$$

is a one-to-one correspondence. If $(E_1, E_2) \in \mathcal{N} \times \mathcal{R}$, then $E_1 \cap E_2 = \varnothing$, and hence

$$(-1)^{|E_1 \cup E_2|} \operatorname{sgn}(\pi|_{E_1 \cup E_2}) = (-1)^{|E_1|} \operatorname{sgn}(\pi|_{E_1}) \, (-1)^{|E_2|} \operatorname{sgn}(\pi|_{E_2}).$$

From (3) and Lemma 2.5, we have

$$\sum_{E \subset S_0, \pi(E) = E} (-1)^{|E|} \operatorname{sgn}(\pi|_E) = 0,$$

and from (1) and (4), we have

$$\{E \subset S_0 : \pi(E) = E\} = \mathcal{N}.$$

Therefore,

$$\begin{aligned} \sum_{E \in \mathcal{C}} (-1)^{|E|} \operatorname{sgn}(\pi|_E) &= \sum_{(E_1, E_2) \in \mathcal{N} \times \mathcal{R}} (-1)^{|E_1 \cup E_2|} \operatorname{sgn}(\pi|_{E_1 \cup E_2}) \\ &= \sum_{E_1 \in \mathcal{N}} (-1)^{|E_1|} \operatorname{sgn}(\pi|_{E_1}) \sum_{E_2 \in \mathcal{R}} (-1)^{|E_2|} \operatorname{sgn}(\pi|_{E_2}) \\ &= 0. \end{aligned}$$

□

Remark. The condition that $S_0 \neq \varnothing$ is crucial. Otherwise we cannot apply Lemma 2.5.

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Now, suppose that N is a positive integer and $\delta \in \{0, 1\}$. We put

$$F(N, \delta) = \{x \in X : \sigma_X^N(x) = \sigma_X^\delta \varphi(x) = x\},$$

so that $p_{N,\delta}(\sigma_X, \varphi) = |F(N, \delta)|$; for $x \in F(N, \delta)$ we put

$$\mathcal{C}(N, \delta; x) = \{E \subset \mathcal{L}_\infty^{-1}(x) : \sigma_A^N(E) = \sigma_A^\delta \varphi_{J,A}(E) = E\}.$$

It is clear that $F(N, \delta) \subset F(N)$ and that $\mathcal{C}(N, \delta;\ x) \subset \mathcal{C}(N; x)$ for each $x \in F(N, \delta)$.

LEMMA 2.8. *We have*

$$p_{N,\delta}(\sigma_X, \varphi) = \sum_{x \in F(N,\delta)} \sum_{E \in \mathcal{C}(N,\delta;x)\setminus\{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\sigma_A^N|_E).$$

Proof. Let $x \in F(N, \delta)$ and put

$$S_0(x) = \{y \in \mathcal{L}_\infty^{-1}(x) : \exists\, n \in \mathbb{Z}\ \sigma_A^\delta \varphi_{J,A}(y) = \sigma_A^{nN}(y)\}.$$

It follows from (5) in Proposition 2.2 that $S_0(x) \neq \varnothing$. It is then straightforward to verify that the conditions in Lemma 2.7 are satisfied with $S = \mathcal{L}_\infty^{-1}(x)$, $\pi = \sigma_A^N|_{\mathcal{L}_\infty^{-1}(x)}$, $\mathcal{C} = \mathcal{C}(N, \delta; x)$ and $S_0 = S_0(x)$. Hence we obtain

$$\sum_{E \in \mathcal{C}(N,\delta;x)\setminus\{\varnothing\}} (-1)^{|E|+1} \operatorname{sgn}(\sigma_A^N|_E) = 1 \quad (x \in P(N, \delta)),$$

and the result follows. □

If we set

$$\mathcal{C}(N, \delta) = \bigcup_{x \in F(N,\delta)} \mathcal{C}(N, \delta; x)$$

and

$$\mathcal{C}_k(N, \delta) = \{E \in \mathcal{C}(N, \delta) : |E| = k\} \quad (k = 1, 2, \ldots, r),$$

then, as in the proof of (2.5),

$$p_{N,\delta}(\sigma_X, \varphi) = \sum_{k=1}^{r} (-1)^{k+1} \sum_{E \in \mathcal{C}_k(N,\delta)} \operatorname{sgn}(\sigma_A^N|_E).$$

Therefore, Theorem B will follow, once we prove that

$$\sum_{E \in \mathcal{C}_k(2m,0)} \operatorname{sgn}(\sigma_A^{2m}|_E) = \mathcal{S}[J_k^\Delta B_k^m J_k^\Delta], \tag{2.9}$$

$$\sum_{E \in \mathcal{C}_k(2m,1)} \operatorname{sgn}(\sigma_A^{2m}|_E) = \mathcal{S}[(J_k A_k)^\Delta B_k^{m-1} (A_k J_k)^\Delta], \tag{2.10}$$

and

$$\sum_{E \in \mathcal{C}_k(2m-1,0)} \operatorname{sgn}(\sigma_A^{2m-1}|_E) = \mathcal{S}[J_k^\Delta B_k^{m-1} (A_k J_k)^\Delta] \tag{2.11}$$

for $m = 1, 2, \ldots$ and for $k = 1, 2, \ldots, r$.

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We prove (2.11) only. The other results are proved similarly. Let m and k be given. First of all, it is easy to prove that

$$\mathcal{C}_k(2m-1,0)=\{E\in\mathcal{C}_k(2m-1):P_{m+j}(E)=P_{m-j-1}(E)^*\ (0\le j\le m-1)\}. \quad (2.12)$$

In particular, every $E\in\mathcal{C}_k(2m-1,0)$ is completely determined by $P_0(E),\dots,P_{m-1}(E)$. A simple calculation gives

$$\begin{aligned}&\mathcal{S}[J_k^{\Delta}B_k^{m-1}(A_kJ_k)^{\Delta}]\\&\quad=\sum_{S_0\cdots S_{m-1}\in\mathcal{A}_k^m}J_k(S_0,S_0)\Bigg[\prod_{j=0}^{m-2}B_k(S_j,S_{j+1})\Bigg]A_k(S_{m-1},S_{m-1}^*)J_k(S_{m-1}^*,S_{m-1}).\end{aligned}$$

If $S_0\cdots S_{m-1}\in\mathcal{A}_k^m$, then the corresponding term on the right-hand side is different from zero if and only if

$$S_0=S_0^*,\ B_k(S_j,S_{j+1})=1\quad(0\le j\le m-2)\quad\text{and}\quad A_k(S_{m-1},S_{m-1}^*)\ne0. \quad (2.13)$$

In this case, the term reduces to

$$J_k(S_0,S_0)A_k(S_{m-1},S_{m-1}^*)J_k(S_{m-1}^*,S_{m-1}).$$

If we denote the set of all m-blocks $S_0\cdots S_{m-1}$ over $\mathcal{A}_k$ that satisfy (2.13) by $\mathcal{J}_k(2m-1,0)$, then

$$\mathcal{S}[J_k^{\Delta}B_k^{m-1}(A_kJ_k)^{\Delta}]=\sum_{S_0\cdots S_{m-1}\in\mathcal{J}_k(2m-1,0)}J_k(S_0,S_0)A_k(S_{m-1},S_{m-1}^*)J_k(S_{m-1}^*,S_{m-1}).$$

Thus (2.11) is a consequence of the following lemma.

LEMMA 2.9. *If $E\in\mathcal{C}_k(2m-1,0)$ and*

$$S_j=P_j(E)\quad(0\le j\le m-1), \quad (2.14)$$

then $S_0\cdots S_{m-1}\in\mathcal{J}_k(2m-1,0)$ and

$$\operatorname{sgn}(\sigma_A^{2m-1}|_E)=J_k(S_0,S_0)A_k(S_{m-1},S_{m-1}^*)J_k(S_{m-1}^*,S_{m-1}). \quad (2.15)$$

Conversely, if $S_0\cdots S_{m-1}\in\mathcal{J}_k(2m-1,0)$, then there is a unique $E\in\mathcal{C}_k(2m-1,0)$ such that (2.14) holds.

Proof. Suppose that $E\in\mathcal{C}_k(2m-1,0)$ and $S_0,S_1,\dots,S_{m-1}$ are given by (2.14). Then $E\in\mathcal{C}_k(2m-1)$ and Lemma 2.6 implies that

$$P_0(E)P_1(E)\cdots P_{2m-1}(E)\in\mathcal{I}_k(2m-1),$$

$A_k(P_j(E),P_{j+1}(E))\in\{1,-1\}$ for $0\le j\le2m-2$, and

$$\operatorname{sgn}(\sigma_A^{2m-1}|_E)=\prod_{j=0}^{2m-2}A_k(P_j(E),P_{j+1}(E)).$$

By (2.12) and (2.14),

$$P_0(E)P_1(E)\cdots P_{2m-1}(E)=S_0\cdots S_{m-1}S_{m-1}^*\cdots S_0^*.$$

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Hence $S_0 \cdots S_{m-1} \in \mathcal{J}_k(2m-1, 0)$ and

$$\operatorname{sgn}(\sigma_A^{2m-1}|_E) = \left[\prod_{j=0}^{m-2} A_k(S_j, S_{j+1})\right] A_k(S_{m-1}, S_{m-1}^*) \left[\prod_{j=0}^{m-2} A_k(S_{j+1}^*, S_j^*)\right]. \quad (2.16)$$

If $m = 1$, the right-hand side of (2.16) is equal to $A_k(S_0, S_0^*)$ and that of (2.15) is equal to $J_k(S_0, S_0)A_k(S_0, S_0^*)J_k(S_0^*, S_0)$, hence (2.15) holds, because $S_0 = S_0^*$ and $J_k(S, S^*) = J_k(S^*, S) \in \{-1, 1\}$ for all $S \in \mathcal{A}_k$. Now suppose that $m \geq 2$. Then (2.4) implies that

$$\prod_{j=0}^{m-2} A_k(S_{j+1}^*, S_j^*) = \prod_{j=0}^{m-2} (J_k(S_j^*, S_j)A_k(S_j, S_{j+1})J_k(S_{j+1}, S_{j+1}^*))$$

and the right-hand side reduces to

$$J_k(S_0^*, S_0)\left[\prod_{j=0}^{m-2} A_k(S_j, S_{j+1})\right] J_k(S_{m-1}, S_{m-1}^*),$$

because $J_k(S, S^*) = J_k(S^*, S) \in \{-1, 1\}$ for all $S \in \mathcal{A}_k$. Hence the right-hand side of (2.16) is equal to that of (2.15), because $A_k(S_j, S_{j+1}) \in \{-1, 1\}$ for all j. This proves (2.15).

Conversely, suppose that $S_0 \cdots S_{m-1} \in \mathcal{J}_k(2m-1, 0)$. Then

$$S_0 \cdots S_{m-1} S_{m-1}^* \cdots S_0^* \in \mathcal{I}_k(2m-1),$$

by Lemma 2.4, and Lemma 2.6 implies that there is an $E \in \mathcal{C}_k(2m-1)$ such that

$$P_0(E)P_1(E) \cdots P_{2m-1}(E) = S_0 \cdots S_{m-1} S_{m-1}^* \cdots S_0^*.$$

Thus $E \in \mathcal{C}_k(2m-1, 0)$ by (2.12). Finally, the uniqueness is obvious, because every $E \in \mathcal{C}_k(2m-1, 0)$ is completely determined by $P_0(E) \cdots P_{m-1}(E)$. □

3. *Proof of Proposition 2.2 (Krieger's joint state chain)*

Suppose that X is a sofic shift and φ is a one-block flip for (X, σ_X) with symbol map τ. We prove Proposition 2.2 by showing that Krieger's joint state chain of X has a natural one-block flip and the resulting shift-flip system has the desired properties. We start with the construction of Krieger's joint state chain.

Let X^+ and X^- denote the set of right-infinite sequences and left-infinite sequences that appear in X, respectively:

$$X^+ = \{x_{[0,\infty)} : x \in X\} \quad \text{and} \quad X^- = \{x_{(-\infty,0]} : x \in X\}.$$

If n is a positive integer, $u = u_0 \cdots u_{n-1} \in \mathcal{B}_n(X)$, $\rho \in X^+$ and $\lambda \in X^-$, we define the right-infinite sequence $u\rho$, the left-infinite sequence λu and the bi-infinite sequence $\lambda\rho$ in the usual way:

$$(u\rho)_i = \begin{cases} u_i, & 0 \leq i \leq n-1, \\ \rho_{i-n}, & i \geq n, \end{cases}$$

$$(\lambda u)_i = \begin{cases} u_{i+n-1}, & -n+1 \leq i \leq 0, \\ \lambda_{i+n}, & i \leq -n, \end{cases}$$

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and

$$(\lambda\rho)_i = \begin{cases} \rho_i, & i \geq 0, \\ \lambda_{i+1}, & i \leq -1. \end{cases}$$

A subset F of X^+ is said to be a *future* if there is a $\lambda \in X^-$ such that

$$F = \{\rho \in X^+ : \lambda\rho \in X\},$$

and a subset P of X^- is said to be a *past* if there is a $\rho \in X^+$ such that

$$P = \{\lambda \in X^- : \lambda\rho \in X\}.$$

In this case, F and P are called the *future of* λ and the *past of* ρ, respectively. It is well known that there are finitely many futures and pasts [**6, 9**]. If F is a future, P is a past and $a \in \mathcal{B}_1(X)$, we set

$$F(a) = \{\rho \in X^+ : a\rho \in F\} \quad \text{and} \quad P(a) = \{\lambda \in X^- : \lambda a \in P\}.$$

It is clear that $F(a)$ is a future whenever $F(a) \neq \varnothing$, and that $P(a)$ is a past whenever $P(a) \neq \varnothing$. A triple (F, a, P) is said to be a *joint state* if F is a future, $a \in \mathcal{B}_1(X)$, P is a past, $F(a) \neq \varnothing$ and $P(a) \neq \varnothing$. Let $\mathcal{A}$ denote the set of all joint states. It is obvious that $|\mathcal{A}| < \infty$. Define the labeling $\mathcal{L} : \mathcal{A} \to \mathcal{B}_1(X)$ and the zero–one $\mathcal{A} \times \mathcal{A}$ matrix A by

$$\mathcal{L}(F, a, P) = a$$

and

$$A((F_1, a_1, P_1), (F_2, a_2, P_2)) = 1 \Leftrightarrow F_1(a_1) = F_2 \quad \text{and} \quad P_1 = P_2(a_2).$$

The topological Markov chain X_A is called *Krieger's joint state chain* of X. It is well known and easy to prove that $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring and has no graph diamonds. Hence (1) and (2) in Proposition 2.2 hold.

If $a \in \mathcal{B}_1(X)$, $\rho = \rho_0\rho_1\rho_2 \cdots \in X^+$ and $\lambda = \cdots \lambda_{-2}\lambda_{-1}\lambda_0 \in X^-$, we define a^*, ρ^* and λ^* by $a^* = \tau(a)$, $\rho^* = \cdots \rho_2^*\rho_1^*\rho_0^*$ and $\lambda^* = \lambda_0^*\lambda_{-1}^*\lambda_{-2}^* \cdots$. It is obvious that $a^* \in \mathcal{B}_1(X)$, $\rho^* \in X^-$, $\lambda^* \in X^+$, $(a^*)^* = a$, $(\rho^*)^* = \rho$ and $(\lambda^*)^* = \lambda$. If $S \subset X^+$, we set $S^* = \{\rho^* : \rho \in S\}$ and if $S \subset X^-$, we set $S^* = \{\lambda^* : \lambda \in S\}$. It is clear that $(S^*)^* = S$, a subset F of X^+ is a future if and only if F^* is a past, and that a subset P of X^- is a past if and only if P^* is a future. We also have $(S(a))^* = S^*(a^*)$, whenever $a \in \mathcal{B}_1(X)$ and S is a future or a past. Thus (F, a, P) is a joint state if and only if (P^*, a^*, F^*) is a joint state. We define the zero–one $\mathcal{A} \times \mathcal{A}$ matrix J by

$$J((F_1, a_1, P_1), (F_2, a_2, P_2)) = 1 \Leftrightarrow (P_1^*, a_1^*, F_1^*) = (F_2, a_2, P_2).$$

It is then straightforward to verify that (3) and (4) in Proposition 2.2 hold.

To prove (5), suppose that $\delta \in \{0, 1\}$, $x \in X$ and $\sigma_X^\delta \varphi(x) = x$. Since $\mathcal{L}_\infty : \mathsf{X}_A \to X$ is a factoring, there is a $y \in \mathsf{X}_A$ such that $\mathcal{L}_\infty(y) = x$. If we write

$$y_i = (F_i, a_i, P_i) \quad (i \in \mathbb{Z}),$$

then $F_i(a_i) = F_{i+1}$, $P_i = P_{i+1}(a_{i+1})$ and $a_i = x_i$ for all $i \in \mathbb{Z}$. Suppose that $\delta = 0$. Then $x_i = (x_{-i})^*$ for all i, and hence

$$(F_{-i})^*(x_i) = (F_{-i})^*((x_{-i})^*) = (F_{-i}(x_{-i}))^* = (F_{-i}(a_{-i}))^* = (F_{-i+1})^*$$

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for all i. From this, we see that $(F_i, x_i, (F_{-i})^*) \in \mathcal{A}$ for all i and the point $y' \in \mathcal{A}^{\mathbb{Z}}$ defined by

$$y'_i = (F_i, x_i, (F_{-i})^*) \quad (i \in \mathbb{Z})$$

is actually an element of X_A. Furthermore, it is obvious that $\mathcal{L}_\infty(y') = x$ and $\varphi_{J,A}(y') = y'$.

Now suppose that $\delta = 1$. Then $x_i = (x_{-i-1})^*$ for all i and the same argument as above shows that the bi-infinite sequence y' defined by

$$y'_i = (F_i, x_i, (F_{-i-1})^*) \quad (i \in \mathbb{Z})$$

has the desired properties. This completes the proof of Proposition 2.2. □

4. *Some remarks and examples*

Suppose that (X, T, F) is a flip system, m is a positive integer and $p_{2m,0}(T, F) + p_{2m,1}(T, F) < \infty$. We will show that $p_{2m,0}(T, F) + p_{2m,1}(T, F)$ is even. To see this, put

$$A = \{x \in X : T^{2m}x = Fx = x,\ TFx \neq x\},$$
$$B = \{x \in X : T^{2m}x = TFx = x,\ Fx \neq x\}$$

and

$$C = \{x \in X : T^{2m}x = Fx = TFx = x\}.$$

It is obvious that A, B and C are mutually disjoint, and that

$$p_{2m,0}(T, F) + p_{2m,1}(T, F) = |A| + |B| + 2|C|. \tag{4.1}$$

For $x \in A \cup B$ we denote the least period of x by $q(x)$ and define Ix by

$$Ix = \begin{cases} T^{q(x)/2}x & \text{if } q(x) \text{ is even,} \\ T^{(q(x)+1)/2}x & \text{if } q(x) \text{ is odd and } x \in A, \\ T^{(q(x)-1)/2}x & \text{if } q(x) \text{ is odd and } x \in B. \end{cases}$$

Then $I : A \cup B \to A \cup B$ is well defined and satisfies

$$I^2x = x \quad \text{and} \quad Ix \neq x \quad (x \in A \cup B).$$

Hence $|A \cup B|$ is even, and the result follows from (4.1). Now suppose that (X, σ_X, φ) is a sofic shift-flip system. Then the sets

$$\bigcup_{m=1}^{\infty} \{x_{[0,m-1]} : x \in X \text{ and } \sigma_X^{2m-1}x = \varphi x = x\},$$

$$\bigcup_{m=1}^{\infty} \{x_{[0,m]} : x \in X \text{ and } \sigma_X^{2m}x = \varphi x = x\}$$

and

$$\bigcup_{m=1}^{\infty} \{x_{[0,m-1]} : x \in X \text{ and } \sigma_X^{2m}x = \sigma_X \varphi x = x\}$$

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are recognizable [**5**, pp. 14, 55]. Hence their generating functions

$$\sum_{m=1}^{\infty} p_{2m-1,0}(\sigma_X, \varphi)t^m, \quad \sum_{m=1}^{\infty} p_{2m,0}(\sigma_X, \varphi)t^{m+1} \quad \text{and} \quad \sum_{m=1}^{\infty} p_{2m,1}(\sigma_X, \varphi)t^m$$

are $\mathbb{N}$-rational (for the definition, see [**4, 11, 12**]), and consequently, so is $2G_{\sigma_X,\varphi}$. Finally, Berstel's theorem [**4, 5, 12**] and Soittola's theorem [**4, 12**] imply that $G_{\sigma_X,\varphi}$ is $\mathbb{N}$-rational.

Let (X, σ_X, φ) and $(\mathcal{A}, \mathcal{L}, A, J)$ be as in the previous section. In the rest of this section, we will be interested in systems which are smaller than $(\mathcal{A}, \mathcal{L}, A, J)$, but satisfy conditions (1)–(5) in Proposition 2.2.

First of all, if X is irreducible, then Krieger's joint finitary state chain of X satisfies the conditions. Recall that a block $f \in \mathcal{B}(X)$ is finitary (intrinsically synchronizing) if

$$\text{for all } (\lambda, \rho) \in X^- \times X^+, \quad [\lambda f \in X^- \text{ and } f\rho \in X^+ \Rightarrow (\lambda f)\rho \in X].$$

Let $\mathcal{A}^0$ be the set of joint states $(F, a, P) \in \mathcal{A}$ such that F is the future of a left-infinite sequence in which a finitary block occurs and P is the past of a right-infinite sequence in which a finitary block occurs; and let $\mathcal{L}^0$, A^0, J^0 denote the restrictions of $\mathcal{L}$, A, J to $\mathcal{A}^0$, respectively. The topological Markov chain X_{A^0} is called *Krieger's joint finitary state chain* of X. It is clear that $(F, a, P) \in \mathcal{A}^0$ if and only if $(P^*, a^*, F^*) \in \mathcal{A}^0$ and it is known [**6**, p. 315] that $\mathcal{L}^0_\infty : \mathsf{X}_{A^0} \to X$ is a factoring. Hence the same argument as in the proof of Proposition 2.2 yields the following proposition.

PROPOSITION 4.1. *If X is irreducible, then $(\mathcal{A}^0, \mathcal{L}^0, A^0, J^0)$ satisfies conditions* (1)–(5) *in Proposition 2.2.*

If X is proper sofic, then $\mathcal{A}^0$ is considerably smaller than $\mathcal{A}$. For instance, we have $|\mathcal{A}^0| = 5$ and $|\mathcal{A}| = 13$ in the case where X is the even shift.

It is clear that there is a smallest system $(\mathcal{A}^m, \mathcal{L}^m, A^m, J^m)$ satisfying the conditions in Proposition 2.2 in the sense that $|\mathcal{A}^m|$ is minimal. It is also clear that there are only a finite number of such smallest systems if we regard isomorphic ones as the same. The authors, however, do not know how to obtain such smallest systems and whether there are non-isomorphic ones in the general case.

We conclude this paper with simple examples in which the smallest systems are completely determined.

Example 4.2. Let X be the even shift and φ be given by

$$\varphi(x)_i = x_{-i}. \tag{4.2}$$

Then $(\mathcal{A}^0, \mathcal{L}^0, A^0, J^0)$ is isomorphic to $(\mathcal{A}', \mathcal{L}', A', J')$, where $\mathcal{A}' = \{1, 2, 3, 4, 5\}$, $\mathcal{L}'(1) = 1$, $\mathcal{L}'(2) = \cdots = \mathcal{L}'(5) = 0$,

$$A' = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad J' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

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moreover, any smallest system is isomorphic to $(\mathcal{A}^m, \mathcal{L}^m, A^m, J^m)$, where $\mathcal{A}^m = \{1, 2, 3, 4\}$, $\mathcal{L}^m(1) = 1$, $\mathcal{L}^m(2) = \cdots = \mathcal{L}^m(4) = 0$,

$$A^m = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad J^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We observe that $(\mathcal{A}^m, \mathcal{L}^m, A^m, J^m)$ is isomorphic to a subsystem of $(\mathcal{A}, \mathcal{L}, A, J)$ and that the topological Markov chains $\mathsf{X}_{A'}$ and X_{A^m} are not conjugate. We also observe that $\mathsf{X}_{A'}$ and X_{A^m} are reducible. In fact, it is easy to see that for any system $(\tilde{\mathcal{A}}, \tilde{\mathcal{L}}, \tilde{A}, \tilde{J})$ satisfying the conditions in Proposition 2.2 the topological Markov chain $\mathsf{X}_{\tilde{A}}$ is reducible. A straightforward calculation using this minimal system and Theorem B gives

$$G_{\sigma_X,\varphi}(t) = \frac{2t + 2t^2 - t^3 - t^4 - 2t^5 - t^6}{(1 - t^2)(1 - t^2 - t^4)}.$$

Example 4.3. Let $S = \{3, 6, 9, \ldots, 3n, \ldots\}$, X be the S-gap shift and $\varphi : X \to X$ be given by (4.2). For the definition of the S-gap shift, see [**9**, p. 7]. We have $|\mathcal{A}| = 29$, $|\mathcal{A}^0| = 17$ and (as before) X_{A^0} is reducible. However, $(\mathcal{A}^0, \mathcal{L}^0, A^0, J^0)$ has a subsystem which satisfies the conditions in Proposition 2.2 and the corresponding topological Markov chain is irreducible: if we define $(\mathcal{A}', \mathcal{L}', A', J')$ by $\mathcal{A}' = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{L}'(1) = 1$, $\mathcal{L}'(2) = \cdots = \mathcal{L}'(6) = 0$,

$$A' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad J' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

then $(\mathcal{A}', \mathcal{L}', A', J')$ satisfies the conditions, is isomorphic to a subsystem of $(\mathcal{A}^0, \mathcal{L}^0, A^0, J^0)$ and $\mathsf{X}_{A'}$ is irreducible. On the other hand, any smallest system is isomorphic to $(\mathcal{A}^m, \mathcal{L}^m, A^m, J^m)$, where $\mathcal{A}^m = \{1, 2, 3, 4\}$, $\mathcal{L}^m(1) = 1$, $\mathcal{L}^m(2) = \cdots = \mathcal{L}^m(4) = 0$,

$$A^m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad J^m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We observe that $(\mathcal{A}^m, \mathcal{L}^m, A^m, J^m)$ is not isomorphic to any subsystems of $(\mathcal{A}, \mathcal{L}, A, J)$, the topological Markov chains $\mathsf{X}_{A'}$ and X_{A^m} are conjugate, but the shift-flip systems $(\mathsf{X}_{A'}, \sigma_{A'}, \varphi_{J',A'})$ and $(\mathsf{X}_{A^m}, \sigma_{A^m}, \varphi_{J^m,A^m})$ are not conjugate. In this case,

$$G_{\sigma_X,\varphi}(t) = \frac{t + t^2 + t^3 + t^4 - t^7 - 2t^9 - 2t^{10} - t^{11}}{(1 - t^3)(1 - t^6 - t^8)}.$$

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