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Some elastostatic and potential problems for an elliptic disc

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1. *Introduction.* We consider some elastostatic and related potential problems in which an unbounded uniform medium contains a plane elliptic disc as an inhomogeneity. Various appropriate conditions on the disc and at infinity are specified. These problems have many physical applications. In the elastostatic case the disc can be a crack or rigid inclusion, for example. The potential problems include the classical one of finding the electrostatic field around an elliptic disc charged to a given potential.

Some special cases of these problems have been investigated before, in references duly mentioned below. As well as increased generality, of method and results, our treatment offers a much simpler and more direct approach. Particular results that have been previously arrived at piecemeal in a series of papers are concisely contained and generalized in the present work.

We begin by recording some relevant potential theory, in readiness for subsequent work. The suffix notation for Cartesian tensors will be employed. Suffixes preceded by a comma denote differentiation with respect to Cartesian coordinates. A repeated suffix is to be summed over 1, 2 and 3.

2. *Preliminary potential theory.* The gravitational harmonic potential ϕ, and the associated biharmonic potential ψ, of an elliptic disc can be expressed at a point $\mathbf{r}$, with Cartesian coordinates x_1, x_2 and x_3 say, as the integrals

$$\phi = \int_S \frac{\sigma\, dS}{|\mathbf{r}-\mathbf{r}'|}, \quad \psi = \int_S \sigma\, |\mathbf{r}-\mathbf{r}'|\, dS. \tag{1}$$

Here, S is the face of the disc, which we shall suppose to lie within the elliptic area $x_1^2/a^2 + x_2^2/b^2 \leqslant 1$ of the plane $x_3 = 0$. The integrations are with respect to the dashed coordinates; σ, the surface density of matter imagined to compose the disc, is some function of these coordinates. It is helpful to note the following properties of these potentials.

$$\nabla^2\psi = 2\phi, \quad \nabla^2\phi = 0, \tag{2}$$

where ∇^2 denotes the Laplacian. ϕ and its derivatives $\phi_{,11}$, $\phi_{,22}$, $\phi_{,12}$ and $\phi_{,33}$, and ψ and its first two derivatives, are all everywhere continuous. However, the normal derivatives $\phi_{,3}$ and $\psi_{,333}$ are discontinuous on the disc

$$\psi_{,333} = 2\phi_{,3} = \pm 4\pi\sigma, \tag{3}$$

according as x_3 vanishes through its negative or positive values. The discontinuities of $\phi_{,31}$ and $\phi_{,32}$ across the disc follow by differentiation of this last equality. The above

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properties of ϕ can be found established in standard works on potential theory. Those of ψ are deducible from them, on expressing ψ and its derivatives in terms of harmonic potentials as follows. Writing $|r-r'|^2 = (x_i-x_i')(x_i-x_i')$,

$$\psi = x_i x_i \int_S \frac{\sigma\, dS}{|\mathbf{r}-\mathbf{r}'|} - 2x_i \int_S \frac{x_i' \sigma\, dS}{|\mathbf{r}-\mathbf{r}'|} + \int_S \frac{x_i' x_i' \sigma\, dS}{|\mathbf{r}-\mathbf{r}'|}, \tag{4}$$

$$\psi_{,\alpha} = x_\alpha \phi - \int_S \frac{x_\alpha' \sigma\, dS}{|\mathbf{r}-\mathbf{r}'|}, \quad \psi_{,3} = x_3 \phi, \tag{5}$$

where $\alpha = 1$ or 2 only. The last equality here is particularly convenient when a derivative of ψ with respect to x_3 is required. In early literature it was known as Jellett's theorem; see, for example, Townsend (1). Another helpful case is when σ is constant or a function of $(1-x_1'^2/a^2-x_2'^2/b^2)$ only, for then

$$\psi_{,12} = \frac{a^2}{a^2-b^2} x_2 \phi_{,1} + \frac{b^2}{b^2-a^2} x_1 \phi_{,2}. \tag{6}$$

This follows on observing that $\psi_{,12}-x_1\phi_{,2}$ and $a^2(x_2\phi_{,1}-x_1\phi_{,2})/(a^2-b^2)$ are each the same unique harmonic function: each is continuous and has the same discontinuity in first derivative across the disc.

3. *A theorem of Dyson* (2). A valuable theorem that expresses ϕ, for a prescribed σ, is now recalled, to facilitate easy evaluation of subsequent harmonic and biharmonic potentials. Namely, if

$$\sigma = (1-x_1'^2/a^2-x_2'^2/b^2)^{\lambda-\frac{1}{2}} f(x_1', x_2'), \tag{7}$$

where λ is positive or zero, and f is any function which can be represented as a convergent series of positive integral powers of x_1' and x_2', then ϕ can be evaluated as

$$\phi = \pi ab \frac{\Gamma(\lambda+\frac{1}{2})}{\lambda\,\Gamma(\lambda)\,\Gamma(\frac{1}{2})} \int_\epsilon^\infty \frac{R^\lambda\, du}{\sqrt{\{u(a^2+u)(b^2+u)\}}} M_\lambda f\left(\frac{a^2 x_1}{a^2+u}, \frac{b^2 x_2}{b^2+u}\right), \tag{8}$$

where

$$R(u) = 1 - \frac{x_1^2}{a^2+u} - \frac{x_2^2}{b^2+u} - \frac{x_3^2}{u},$$

$$M_\lambda = 1 + \ldots + \frac{R^n u^n}{2^{2n}(\lambda+1)\ldots(\lambda+n)\,n!} D^n + \ldots,$$

with

$$D = \frac{a^2+u}{a^2}\frac{d^2}{dx_1^2} + \frac{b^2+u}{b^2}\frac{d^2}{dx_2^2},$$

and ϵ is the positive root of the equation $R(\epsilon) = 0$. For a point on the disc, where $x_3 = 0$ and $x_1^2/a^2 + x_2^2/b^2 \leqslant 1$, ϵ is zero. The theorem remains valid when λ is zero; $\lambda\,\Gamma(\lambda)$ is then put as unity.

This theorem was obtained long ago by Dyson (2), at the same time that he established a corresponding theorem for the solid ellipsoid. His results were later derived more directly and more generally by Hobson (3), and were reproduced by Routh (4) in his classical treatise. Nevertheless, the theorem seems neglected in the literature when it could often have been employed with advantage. The biharmonic potential ψ and its derivatives can be evaluated with the help of this theorem, on recalling equations (4) and (5), which express them in terms of harmonic potentials.

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If f is a polynomial of degree n, say, in x_1' and x_2', then the infinite sum in the integrand conveniently terminates. If also, λ is a positive integer, then on the disc, where ϵ vanishes, ϕ itself is a polynomial of degree $2\lambda+n$ in x_1 and x_2. We note also, for later reference in section 7, that, by differentiation, $\phi_{,11}$, $\phi_{,22}$, $\phi_{,12}$ and $\phi_{,33}$ $(=-\phi_{,11}-\phi_{,22})$ are polynomials of degree $2\lambda+n-2$ on the disc. At the same time, we observe for use in section 6 that, by (5), the second derivatives of ψ are, like ϕ, polynomials of degree $2\lambda+n$.

4. *Application to potential problems.* For the first potential problem suppose that a function ϕ is to take a prescribed value on the disc, while everywhere outside the disc it is to be continuous and satisfy Laplace's equation $\nabla^2\phi = 0$. Then, it can clearly be quite generally identified with the ϕ of equation (1), where σ is chosen to make ϕ take the required value on the disc. We consider, for simplicity, an interesting case when the problem can be matched with the theorem in section 3.

Thus, if ϕ is prescribed on the disc as a polynomial in x_1 and x_2 of degree n, say, then σ can clearly be put as

$$\sigma = (1-x_1'^2/a^2-x_2'^2/b^2)^{-\frac{1}{2}} f(x_1', x_2'),$$

where f is also a polynomial of degree n, in x_1' and x_2'. The coefficients in f can here be chosen in accordance with (8), to ensure that ϕ becomes the prescribed polynomial on the disc. On the disc $\phi_{,3}$ is proportional to σ, by (3), and is therefore singular around the edge. The result that σ takes the above form is the so-called Galin's theorem obtained otherwise, via a length excursion into ellipsoidal coordinates, by Galin (5) and likewise by Sneddon (6). However, we have shown here that the early work of Dyson (2) immediately indicates this required form for σ and provides a comparatively easy means of calculating f. As a simple example, take ϕ to be a constant ϕ_0 on the disc, as in the classical problem when an electrified elliptic disc is raised to a given constant potential. f is then immediately specifiable as $\phi_0/2\pi bK(k)$, where

$$K(k) = a\int_0^\infty \frac{du}{2\sqrt{\{u(a^2+u)\,(b^2+u)\}}}$$

is the complete elliptic integral of the first kind with modulus $k = (1-b^2/a^2)^{\frac{1}{2}}$, taking $b < a$.

A second kind of potential problem occurs when, instead of the function ϕ itself, the normal derivative $\phi_{,3}$ is specified on the disc. This ϕ can be represented generally as

$$\phi = \frac{\partial}{\partial x_3}\int_S \frac{\sigma\, dS}{|\mathbf{r}-\mathbf{r}'|}, \tag{9}$$

for it then satisfies Laplace's equation and its derivative

$$\phi_{,3} = -\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\int_S \frac{\sigma\, dS}{|\mathbf{r}-\mathbf{r}'|}$$

is continuous, as required. An immediate solution again follows from the theorem of section 3, if $\phi_{,3}$ is prescribed on the disc as a polynomial of degree n, say. For then σ can clearly be chosen here in the form (7), where λ is unity and where f is also a polynomial of degree n, with coefficients determined appropriately. Another solution could

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be obtained by taking λ zero and f a polynomial of degree $n+2$, with non-unique coefficients. This latter solution is, however, not relevant to physical problems where ϕ is always finite, since it makes ϕ singular on the edge of the disc. In making this remark, we recall that from (9) and (3) it follows that $\phi = \pm 2\pi\sigma$, according as x_3 vanishes through negative or positive values.

5. *Preliminary elastostatic theory.* Consider first, in preparation for the elliptic disc, a cavity of arbitrary shape, entirely surrounded by an unbounded homogeneous medium which is assumed unstressed at an infinite distance away. The displacement field u_i at a point $\mathbf{r}$ within the medium can be expressed, by means of integrals involving the displacement and traction on the bounding surface S of the cavity, as

$$u_i = -\int_S G_{ij} C_{jklm} u_{l,m} n_k dS + \int_S \frac{\partial G_{ij}}{\partial x'_k} C_{jklm} u_l n_m dS,$$

where n_i is the unit normal out of the surface, C_{ijkl} is the tensor of elastic moduli of the medium (with the symmetry $C_{ijkl} = C_{klij}$), and where $G_{ij}F_j$ is the displacement field resulting at a point $\mathbf{r}$ in an infinite homogeneous mass of the medium when a point force F_i acts at a point $\mathbf{r}'$. This expression is analogous to a well known one of potential theory and can be similarly derived. For an isotropic medium, it is essentially that due to Somigliana, see Love ((7), p. 245). If, as we shall assume for detailed analysis, the medium is isotropic, with bulk modulus κ and shear modulus μ, say, then we can write

$$C_{ijkl} = \kappa\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - 2\delta_{ij}\delta_{kl}/3),$$
$$G_{ij} = \frac{1}{4\pi\mu}\frac{\delta_{ij}}{|\mathbf{r}-\mathbf{r}'|} - \frac{\kappa+\frac{1}{3}\mu}{8\pi\mu(\kappa+\frac{4}{3}\mu)}\frac{\partial^2}{\partial x_i \partial x_j}|\mathbf{r}-\mathbf{r}'|,$$

in terms of the Kronecker delta.

The cavity is now imagined to degenerate into a plane elliptic disc. The expression for u_i, the displacement field around the disc, then becomes

$$u_i = -\int_S G_{ij} p^T_{jk} n_k dS + \int_S \frac{\partial G_{ij}}{\partial x'_k} C_{jklm} u^T_l n_m dS, \tag{10}$$

where S is now the face of the disc, and $p^T_{ij}n_j$ and u^T_i denote the increases in the traction $C_{ijkl}u_{k,l}n_j$ and the displacement u_i, respectively, measured immediately across the disc in the direction assigned to the normal n_i to S. These two integrals on the right-had side can be interpreted as the fields due to a 'layer of body force' and a 'dislocation', respectively, following Eshelby (8, 9), who considered the ellipsoidal inclusion problem. When the left-hand side is equated to the prescribed conditions on the disc, integral equations for the unknown $p^T_{ij}n_j$ and u^T_i are obtained. A concise approach to all the boundary-value problems for the disc is consequently achieved, using the Green's function G_{ij} to yield a combination of harmonic and biharmonic potentials that automatically has the desired discontinuity properties on the disc. Only the surface density $p^T_{ij}n_j$ or u^T_i remains to be determined; we shall appeal to the theorem of Dyson to accomplish this. In contrast, in the more elementary previous approaches to these problems, in particular that of Kassir and Sih (10–12), a different expression of the

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elastic field is devised for each boundary-value problem, so that each time a separate search has to be made for suitable harmonic functions. These latter harmonic functions have generally been evaluated less directly than here, for example, in ellipsoidal coordinates as Lamé triple products.

It will be convenient for later considerations to record an expression for the total elastic energy E in the medium surrounding the disc. Thus, integrating over the whole of space V,

$$2E = \int_V u_{k,l} C_{ijkl} u_{i,j} dV = \int_V (u_{k,l} C_{ijkl} u_i)_{,j} dV,$$

using the equations of equilibrium. Gauss's theorem now yields an integral over the face S of the disc

$$2E = -\int_S [C_{ijkl} u_{k,l} u_i n_j] dS, \tag{11}$$

where the square brackets denote the increase in their contents measured immediately across the disc in the direction of its normal n_i.

Two kinds of problems for the disc in an isotropic medium are now considered separately. First, the displacement of the disc is given, as is effectively so, for example, for a rigid disc with a given field at infinity. Secondly, the traction over the disc is prescribed, as, for example, when the disc is an empty crack with a given field at infinity. It will be supposed that the disc is situated with respect to the coordinate system as in section 2, with the direction of its normal as that of the x_3-axis.

6. *Elliptic disc with prescribed displacement.* When the displacement on the disc is prescribed, the formula (10), with u_i^T now zero, expresses the displacement field in the isotropic elastic medium surrounding the disc as

$$u_i = \frac{\kappa + \frac{1}{3}\mu}{8\pi\mu(\kappa + \frac{4}{3}\mu)} \frac{\partial^2}{\partial x_i \partial x_j} \int_S |\mathbf{r} - \mathbf{r}'| \, p_{j3}^T dS - \frac{1}{4\pi\mu} \int_S \frac{p_{i3}^T dS}{|\mathbf{r} - \mathbf{r}'|}. \tag{12}$$

The surface density p_{i3}^T of the harmonic and biharmonic potentials is to be found for the complete solution. This is readily accomplished if we assume, for simplicity, that the displacement on the disc is given as any polynomial in x_1 and x_2 of degree n, say. For then, on recalling the remarks at the end of section 3, it is apparent that the traction discontinuity is of the form

$$p_{i3}^T = (1 - x_1'^2/a^2 - x_2'^2/b^2)^{-\frac{1}{2}} P_i^n, \tag{13}$$

where each P_i^n is a polynomial of degree n in x_1' and x_2', suitably chosen in accordance with (8), to match u_i with the given value. This result is analogous to that of potential theory, Galin's theorem, considered in section 4. A weaker version of it was established otherwise by Segedin (13), who considered only a normal displacement on the disc. We notice that the individual stress components are, like the traction discontinuity, singular around the edge of the disc.

As a simple example in detail, suppose that u_i takes a uniform value u_i^0 on the disc. Then we can put

$$p_{i3}^T = \sigma A_i, \quad u_i^0 = -R_{ij} A j, \tag{14}$$

say, where $\sigma = (1 - x_1'^2/a^2 - x_2'^2/b^2)^{-\frac{1}{2}}$ and where the A_i and R_{ij} are constants. Evaluation of (12) yields the R_{ij}, and then inversion of the second equality in (14) finally determines

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A_i, and hence the required p^T_{i3}, in terms of the given u^0_i. Now, with the help of the work in section 3, the harmonic potential ϕ and the second derivatives of the biharmonic potential ψ, for the present surface density σ, can be evaluated on the disc as

$$\phi = \psi_{,33} = \pi ab \int_0^\infty \frac{du}{\sqrt{R}} = I, \quad \text{say},$$

$$\psi_{,11} - \phi = -\pi ab \int_0^\infty \frac{a^2}{a^2+u} \frac{du}{\sqrt{R}} = -I_1, \quad \text{say},$$

$$\psi_{,22} - \phi = -\pi ab \int_0^\infty \frac{b^2}{b^2+u} \frac{du}{\sqrt{R}} = -I_2, \quad \text{say},$$

where $R = u(a^2+u)(b^2+u)$, and $\psi_{,ij} = 0$ if $i \neq j$. These integrals can readily be put in standard elliptic form. Thus, taking $a > b$, $I = 2\pi b K(k)$ and

$$I_1 = 2\pi b[K(k) - E(k)]/k^2, \tag{15}$$

$$I_2 = 2\pi b[E(k) - (b^2/a^2) K(k)]/k^2, \tag{16}$$

where K and E are the complete elliptic integrals of the first and second kinds, with modulus $k = (1 - b^2/a^2)^{\frac{1}{2}}$. The desired uniform tensor R_{ij} can now be put in terms of these results as

$$R_{ij} = \frac{1}{4\pi\mu} \phi \delta_{ij} - \frac{\kappa + \frac{1}{3}\mu}{8\pi\mu(\kappa + \frac{4}{3}\mu)} \psi_{,ij},$$

and its inverse is obtainable at once, since $R_{ij} = 0$ when $i \neq j$. This tensor has in effect been evaluated, less concisely, by Kassir and Sih (11). For a circle, when $b = a$, its non-vanishing components simplify as

$$R_{11} = R_{22} = a\pi[3/\mu + 1/(\kappa + \tfrac{4}{3}\mu)]/16,$$

$$R_{33} = a\pi[1/\mu + 1/(\kappa + \tfrac{4}{3}\mu)]/8.$$

For a second example, suppose that on the disc the displacement u_i is prescribed as a linear function of x_1 and x_2. Then we can take

$$p^T_{i3} = (B_{i1} x'_1/a^2 + B_{i2} x'_2/b^2)(1 - x'^2_1/a^2 - x'^2_2/b^2)^{-\frac{1}{2}},$$

where B_{i1} and B_{i2} are constants to be determined. The resulting harmonic potential can be conveniently recast in terms of one with surface density $\sigma = (1 - x'^2_1/a^2 - x'^2_2/b^2)^{\frac{1}{2}}$:

$$\int_S \frac{p^T_{i3} dS}{|\mathbf{r} - \mathbf{r}'|} = -B_{i\alpha} \int_S \frac{\partial \sigma}{\partial x'_\alpha} \frac{dS}{|\mathbf{r} - \mathbf{r}'|} = -B_{i\alpha} \frac{\partial}{\partial x_\alpha} \int \frac{\sigma \, dS}{|\mathbf{r} - \mathbf{r}'|},$$

where Gauss's theorem and the vanishing of σ at the edge of the ellipse are used for the second equality, and where the Greek subscript α is summed over 1 and 2 only. There is a similar expression for the biharmonic potential. The displacement field everywhere can now be put as

$$u_i = -\frac{\kappa + \frac{1}{3}\mu}{8\pi\mu(\kappa + \frac{4}{3}\mu)} B_{j\alpha} \psi_{,\alpha ij} + \frac{1}{4\pi\mu} B_{i\alpha} \phi_{,\alpha},$$

where ϕ and ψ are the harmonic and biharmonic potentials of the disc for the present surface density σ. ϕ can be expressed everywhere by means of the result (8) as

$$\phi = \pi ab \int_\epsilon^\infty \frac{du}{2\sqrt{\{u(a^2+u)(b^2+u)\}}} \left(1 - \frac{x_1^2}{a^2+u} - \frac{x_2^2}{b^2+u} - \frac{x_3^2}{u}\right). \tag{17}$$

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The equations (2), (5) and (6) are now sufficient to determine all the required third derivatives of ψ in terms of ϕ. The expression for u_i takes, as intended, a linear form on the disc, which is to be equated to the prescribed displacement, $-C_{i\alpha}x_\alpha$, say. This implies that

$$C_{11} = \frac{\kappa+\frac{7}{3}\mu}{8\pi\mu(\kappa+\frac{4}{3}\mu)} B_{11} I_1/a^2 + \frac{\kappa+\frac{1}{3}\mu}{8\pi\mu(\kappa+\frac{4}{3}\mu)} (B_{11}-B_{22})(I_1-I_2)/(a^2-b^2),$$

$$C_{12} = \frac{1}{4\pi\mu} B_{12} I_2/b^2 - \frac{\kappa+\frac{1}{3}\mu}{8\pi\mu(\kappa+\frac{4}{3}\mu)} (B_{12}+B_{21})(I_1-I_2)/(a^2-b^2),$$

$$C_{31} = \frac{\kappa+\frac{7}{3}\mu}{8\pi\mu(\kappa+\frac{4}{3}\mu)} B_{31} I_1/a^2,$$

together with the three further equations obtained from these by interchanging 1 and 2, and a and b. I_1 and I_2 are the integrals evaluated previously in (15) and (16). These six equations can be solved immediately for the six unknowns $B_{i\alpha}$. So far, the elastic field at infinity has been assumed to vanish. However, a uniform stress and strain field can be superimposed everywhere, with the given constants $C_{i\alpha}$ adjusted accordingly. Thus, if the disc is assumed perfectly rigid, then the symmetric part of the tensor $C_{i\alpha}$ is now determined; for the displacement on the disc must vanish, apart from the arbitrary rigid-body displacement allowed by the arbitrariness of the unsymmetric part of $C_{i\alpha}$. We can further superimpose the elastic field of the first example above, to give an additional arbitrary displacement to the disc. We have then the most general case of a rigid elliptic disc with a uniform strain field at infinity. The particular case when the only non-vanishing components of $C_{i\alpha}$ are C_{11} and C_{22} was considered by Kassir and Sih (11).

7. *Elliptic disc with prescribed traction.* When the traction over the disc is prescribed, then in the formulae (10) we put $p^T_{jk}n_k$ zero, and note that $\partial G_{ij}/\partial x'_k = -\partial G_{ij}/\partial x_k$, to obtain for the displacement field in the isotropic elastic medium surrounding the disc the expression

$$\begin{aligned} u_i = {} & \frac{\kappa+\frac{1}{3}\mu}{4\pi(\kappa+\frac{4}{3}\mu)} \frac{\partial^3}{\partial x_i \partial x_j \partial x_3} \left(\int_S u^T_j |\mathbf{r}-\mathbf{r}'|\, dS \right) \\ & - \frac{1}{4\pi} \frac{\partial}{\partial x_3} \left(\int_S \frac{u^T_i\, dS}{|\mathbf{r}-\mathbf{r}'|} \right) - \frac{1}{4\pi} \delta_{i3} \frac{\partial}{\partial x_k} \left(\int_S \frac{u^T_k\, dS}{|\mathbf{r}-\mathbf{r}'|} \right) \\ & - \frac{\kappa-\frac{2}{3}\mu}{4\pi(\kappa+\frac{4}{3}\mu)} \frac{\partial}{\partial x_i} \left(\int_S \frac{u^T_3\, dS}{|\mathbf{r}-\mathbf{r}'|} \right). \end{aligned} \tag{18}$$

It is easily verifiable, using (3), that this displacement has the discontinuity u^T_i across the disc. The biharmonic potential derivatives can conveniently be replaced by harmonic ones, with the help of the second of (5). The stress field can now be evaluated as $p_{ij} = C_{ijkl}u_{k,l}$. On the disc its traction components p_{i3} are expressible as

$$p_{\alpha 3} = \frac{\mu(\kappa-\frac{2}{3}\mu)}{4\pi(\kappa+\frac{4}{3}\mu)} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \int_S \frac{u^T_\beta\, dS}{|\mathbf{r}-\mathbf{r}'|} - \frac{\mu}{4\pi} \frac{\partial^2}{\partial x_3 \partial x_3} \int_S \frac{u^T_\alpha\, dS}{|\mathbf{r}-\mathbf{r}'|},$$

$$p_{33} = -\frac{\mu(\kappa+\frac{1}{3}\mu)}{2\pi(\kappa+\frac{4}{3}\mu)} \frac{\partial^2}{\partial x_3 \partial x_3} \int_S \frac{u^T_3\, dS}{|\mathbf{r}-\mathbf{r}'|},$$

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where the Greek subscripts take the values 1 and 2 only. We note that only the continuous second derivatives of the harmonic potentials finally appear here. The traction components are thus continuous on the disc, as intended, and u_i^T can be chosen so that they are equal to the prescribed values. Suppose, for instance, that the prescribed values are polynomials of degree n, say, in x_1 and x_2. Then it is easily observed, on recalling the remarks at the end of section 3, that the displacement discontinuity is of the form

$$u_i^T = (1 - x_1'^2/a^2 - x_2'^2/b^2)^{\frac{1}{2}} P_i^n,$$

where each P_i^n is a polynomial of degree n in x_1' and x_2' that can be suitably chosen to match p_{i3} to the given values. Weaker statements of this general result, considering the normal stress p_{33} as the only non-vanishing component of traction on the disc, have been given by Kassir and Sih (12) and Segedin (13). The general result holds also when the elastic medium is arbitrarily anisotropic, as has been shown recently by Willis (14). We remark incidentally that the corresponding result for a disc with prescribed displacement, see equation (13), is likewise valid for an anisotropic medium. The generalization from an isotropic to an anisotropic medium, together with the proof of the preceding remarks, can be carried out in full by the method of Walpole (15), which was applied to the ellipsoidal inclusion problem. We note finally that the correct traction could be obtained by taking u_i^T in the form

$$u_i^T = (1 - x_1'^2/a^2 - x_2'^2/b^2)^{-\frac{1}{2}} P_i^{n+2},$$

where each P_i^{n+2} is a (non-unique) polynomial of degree $n+2$. However, this alternative solution is disallowed for the physical reason that the displacement field would be singular around the edge of the disc. This reasoning can be compared to that in section 4 for the corresponding potential problem.

As a useful simple example, suppose that the traction components p_{i3} are to be uniform over the disc. Then we can write

$$u_i^T = \sigma B_i, \quad p_{i3} = -S_{ij} B_j, \tag{19}$$

say, where $\sigma = (1 - x_1'^2/a^2 - x_2'^2/b^2)^{\frac{1}{2}}$, and where the B_i and S_{ij} are constants. The further equalities

$$S_{ik} T_{kj} = \delta_{ij}, \quad B_i = -T_{ij} p_{j3}, \tag{20}$$

define the inverse T_{ij} of S_{ij} and express the required B_i, and hence u_i^T, in terms of the given p_{i3}. The complete elastic field is entirely expressible in terms of the harmonic potential ϕ with the present surface density σ, or, in other words, the ϕ of equation (17). Evaluation of the traction on the disc determines the S_{ij} components as follows.

$$S_{11} = \frac{\mu(\kappa + \frac{1}{3}\mu)}{2\pi(\kappa + \frac{4}{3}\mu)} I_1/a^2 + \frac{\mu}{4\pi} I_2/b^2,$$

$$S_{33} = \frac{\mu(\kappa + \frac{1}{3}\mu)}{\kappa + \frac{4}{3}\mu} E(k)/b, \quad S_{ij} = 0 \quad \text{if} \quad i \neq j,$$

and S_{22} follows from this S_{11} by interchanging 1 and 2, and a and b. I_1 and I_2 are the integrals evaluated in (15) and (16). Inversion of these components immediately

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gives T_{ij}. Results equivalent to these have been obtained before, less simply, by Green and Sneddon (16) for S_{33}, and by Kassir and Sih (10) for S_{11} and S_{22}. For a circle, when $b = a$, have simply

$$S_{11} = S_{22} = 3\pi/8a[1/\mu + 2/(3\kappa + 2\mu)],$$
$$S_{33} = \pi/2a[1/\mu + 1/(\kappa + \tfrac{1}{3}\mu)],$$

which shows that the components of $S_{ij}(T_{ij})$ are positive increasing (decreasing) functions of κ and μ, properties we refer to later. A uniform stress field p^A_{ij} can now be everywhere superimposed on the elastic field. The resulting traction $p_{i3} + p^A_{i3}$ on the disc will vanish if

$$B_i = T_{ij} p^A_{j3}. \tag{21}$$

We then have finally the elastic field around an elliptic crack with the uniform stress field at infinity.

8. *Energy considerations.* It is useful and illuminating to examine the total energy E of the elastic field in the medium surrounding the elliptic disc. It was evaluated in (11), for a medium unstressed at infinity and arbitrarily anisotropic.

First, when the displacement on the disc is prescribed, so that u^T_i vanishes, the energy expression becomes

$$2E = -\int_S u_i p^T_{ij} n_j dS.$$

Here, only the traction jump $p^T_{ij} n_j$ obtained directly from the integral equations is required; there is no need to integrate or evaluate the complete elastic field in the medium. We now exemplify how some general properties of the solution are predictable from energy considerations alone, without the need for a full solution. Thus, if the displacement is uniform on the disc then the equalities (14) apply, where the R_{ij} and A_i are constant tensors, even for an arbitrarily anisotropic medium. E now becomes expressible as

$$E = \tfrac{1}{2} R_{ij} A_i A_j \int_S \sigma dS = \pi ab R_{ij} A_i A_j. \tag{22}$$

We observe that the quadratic form $R_{ij} A_i A_j$ must be positive for all A_i. In other words, since we show next that R_{ij} is symmetric, R_{ij} is a positive definite tensor. The symmetry $R_{ij} = R_{ji}$ follows on showing, directly by application of Betti's reciprocal theorem, that $A'_i R_{ij} A''_j = A''_i R_{ij} A'_j$, where A'_i and A''_i are two arbitrary values of A_i. A final property is that when the tensor of elastic moduli C_{ijkl} of the medium is increased (decreased) by a positive definite amount, then the elastic energy (22) decreases (increases) for a fixed arbitrary A_i, or, in other words, R_{ij} decreases (increases) by a positive definite amount. This is readily demonstrated by comparing the original and final energies, expressed as volume integrals, with the help of the same inequality that is used in deriving the well known elastic extremum principle of minimum complementary energy. We can verify that all the foregoing remarks, valid for an arbitrarily anisotropic medium, hold for the isotropic medium considered earlier: in that case the R_{11}, R_{22} and R_{33} are positive decreasing functions of κ and μ. We can compare the above

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properties of R_{ij} with those of the 'overall constraint' tensor L^* for the ellipsoidal inclusion problem; for the latter and their application see Walpole (17).

When, secondly, the traction on the disc is prescribed, so that $p^T_{ij}n_j$ vanishes, the energy expression becomes

$$2E = -\int_S p_{ij}n_j u^T_i \, dS.$$

If, for instance, the traction is uniform on the disc, the medium being arbitrarily anisotropic, then the equalities (19) and (20) can be inserted here to yield

$$E = \tfrac{1}{3}\pi ab S_{ij}B_iB_j = \tfrac{1}{3}\pi ab T_{ij}p_{i3}p_{j3}.$$

The following properties of S_{ij} and T_{ij} can be established without a complete solution, similarly to those of R_{ij}. S_{ij} and T_{ij} are symmetric positive definite tensors that increase and decrease, respectively, by a positive definite amount when the tensor of elastic moduli C_{ijkl} increases by a positive definite amount: the greater the 'strength' of the elastic medium, the smaller the energy associated with a given disc under given traction. Energy considerations thus explain why for the isotropic medium of section 7, the components of S_{ij} and T_{ij} are positive increasing and decreasing functions, respectively, of κ and μ.

So far, we have considered the energy when there is no field at infinity. Now suppose that the displacement field u^A_i is superimposed on the elastic field surrounding the disc on which the displacement or traction is given. Let $e^A_{ij} = \frac{1}{2}(u^A_{i,j} + u^A_{j,i})$ and $p^A_{ij} = C_{ijkl}e^A_{kl}$ be the accompanying strain and stress fields. Then the total elastic energy E can be readily evaluated as follows, by the method of Eshelby ((8), p. 388), who considered the interaction of a stress field with an inhomogeneity

$$E = E^A \pm \Delta E, \tag{23}$$

where E^A is the total elastic energy, $\frac{1}{2}p^A_{ij}e^A_{ij}$ per unit volume, present when the disc is removed and replaced by the homogeneous elastic medium, and ΔE is an interaction energy expressible as the following integrals over the disc.

$$\Delta E = \frac{1}{2}\int_S p^T_{ij}n_j u^A_i \, dS - \frac{1}{2}\int_S p^A_{ij}n_j u^T_i \, dS,$$

where the first or second integral vanishes according as the traction or displacement is given on the disc. The positive or negative sign is taken in (23) according as the displacement or traction is supposed to be prescribed on the infinite sphere. For an elliptic crack with a uniform stress p^A_{ij} at infinity we have, using (19) and (21), that

$$\Delta E = -\tfrac{1}{3}\pi ab p^A_{i3}T_{ij}p^A_{j3},$$

which is equivalent to the result of Eshelby ((9), p. 129) obtained via an ellipsoidal cavity. This result was in fact re-calculated by Kassir and Sih (18), by laboriously integrating over a sphere the elastic field at a distant point. We note incidentally that the fields at a large distance from the disc can easily be found from (12) and (18) if required. We simply insert the well known MacCullagh's formula for a gravitational potential at a distant point, together with the corresponding expression for the biharmonic potential.

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