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Singular infinitely divisible distributions whose characteristic functions vanish at infinity

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1. *Introduction.* In the course of discussing dynamical systems which enjoy strong mixing but have singular spectrum, E. Hewitt and the author, (2), recently constructed families $(Z_t)_{t>0}$ of symmetric random variables which satisfy *inter alia* the following properties:

(i) Z_t is purely singular and has full support,
(ii) $\chi^{(t)}(x) \to 0$ as $\pm x \to \infty$, where $\chi^{(t)}$ is the characteristic function of Z_t,
(iii)′ Z_{t+s}, $Z_t + Z_s$ have the same null events,
(iv) whenever $s \neq t$, Z_t and $Z_s + a$ are mutually singular for every (possibly zero) constant a.

(Recall that random variables X, Y are *mutually singular* (written $X \perp Y$) if there is a set E such that $P(X \in E) = 0, P(Y \in E) = 1$. Also X is *purely singular* if $P(X \in E) = 0$ whenever E has positive Lebesgue measure, i.e. $X \perp Y$ whenever Y is normally distributed.)

Property (ii), which corresponds to the strong mixing property, ensures that each Z_t is purely continuous and would, in view of the Riemann–Lebesgue lemma, be automatically true, were the Z_t absolutely continuous. Property (iii)′ can be re-expressed as the statement that $(Z_t)_{t>0}$ is a one parameter semi-group *up to equivalence* and is evidently a very weak form of infinite divisibility. Several mathematicians have subsequently asked us whether (iii)′ can be replaced by true infinite divisibility, *viz.*

$$\text{(iii)} \quad Z_{t+s} = Z_t + Z_s.$$

We show here that such variables can indeed be constructed and we do so in a particularly strong way. We prove, in fact, that it is possible to base the construction on *discrete* Lévy spectral measures of a very simple kind.

Our model is compounded coin-tossing and has parameters (A_n), (B_n) (sequences of positive real numbers). For positive integers k, n the (independent) random variables $V_{k,n}$ take the values $\pm B_n$ with probability $\frac{1}{2}$. For $t > 0$, $n \in \mathbf{N}$, $X_{t,n}$ is distributed according to a Poisson law with mean $2tA_n$. $Y_{t,n}$ is then distributed according to a compound Poisson law, being defined explicitly by

$$Y_{t,n} = \sum_{k=1}^{X_{t,n}} V_{k,n}.$$

Finally Z_t is given by

$$Z_t = \sum_{n=1}^{\infty} Y_{t,n}.$$

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(Z_t has Lévy spectral measure $\sum_{n=1}^{\infty} tA_n(\delta(B_n)+\delta(-B_n))$ and this can be regarded as the simplest possible recipe for generating continuous symmetric non-Gaussian infinitely divisible random variables.)

We obtain a suitable example by taking $A_n = 2^{4n}\log\log(4n)$, $B_n = 2^{-n^2}$; moreover, special cases of the results which enable us to specify in advance the sequence (B_n) as *any* subsequence of, say, (2^{-n}); then determine the A_n (i.e. determine the Poisson means) so that the resulting $(Z_t)_{t>0}$ satisfy (i), (ii), (iii), (iv).

We have decided to restrict attention to real-valued random variables and to keep all proofs elementary and essentially self-contained. However, the setting for (2) was locally compact abelian groups and we are able to exhibit one-parameter families satisfying (i), (ii), (iii), (iv) in a more abstract situation. (Indeed this could be tackled at the cost of heavy technical complication by combining the methods used here with some structure theory and the known techniques (see e.g. (1)) for handling analogous constructions on p-adic groups, infinite products of cyclic groups, etc.) In fact we feel that the objectives in the general case are somewhat different, as are the appropriate results and techniques. Accordingly we leave that discussion for another occasion.

It should be noted that although, to the best of our belief, this is the first time that the existence of infinitely divisible measures satisfying (i), (ii), (iii), (iv) has been demonstrated, it is actually possible to find examples using the model due to H. Rubin which was exposed in (6). One can use direct computation to modify the parameters there to ensure (i), (ii) and then one can apply the generalized purity law of W. Moran and the author, (3). This guarantees that (iv) is a consequence of (i), (iii) whenever the Lévy measure is discrete. However, the latter purity law was established on the basis of a knowledge of the cohomology of the maximal ideal space of the Banach measure algebra $M(\mathbf{R})$, so that such a treatment is not elementary. Direct verification of (iv) in the case of Rubin's model is presumably also possible but the model which we shall use seems simpler and more flexible for our current purposes.

Here is the scheme of the paper: Section 2 is given over to setting up the basic model. The method to be used for proving the singularity assertions is developed (in a more general context) in section 3, and applied in section 4. The last section gives some refinements of the basic result.

2. *Notation and preliminaries – the basic model.* Given a probability space $(P, \Omega, \mathscr{E})$ and a (Borel) random variable $Z\colon \Omega \to \mathbf{R}$, the *distribution* of Z is the measure μ defined by

$$\mu(E) = P(Z^{-1}(E)) = P(Z \in E) \quad (E \text{ Borel}).$$

(There is a formal sense in which 'distribution' is the more primitive concept but we shall take the point of view that Z denotes all possible realizations of a random variable with distribution μ as $(P, \Omega, \mathscr{E})$ varies.) The *characteristic function* of Z is the function $\chi\colon \mathbf{R} \to \mathbf{C}$ defined by

$$\chi(x) = \mu^{\wedge}(x) = \int \exp(ix)\, d\mu(x).$$

Z is *infinitely divisible* if, for every positive integer n with reciprocal t, there is a random variable Z_t such that the sum of n independent repetitions of Z_t equals Z.

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Because χ determines μ hence Z uniquely, we may quote the classical characterization of the general real infinitely divisible random variable in the following way:

$$\chi(x) = \mu^{\wedge}(x) = \exp\{icx - dx^2 + \int(e^{ixy} - 1)\,d\sigma(y) + \int(e^{ixy} - 1 - ixy)\,y^{-2}\,d\tau(y)\}, \quad (1)$$

where c, d are real numbers, $d \geqslant 0$, σ is a positive bounded Borel measure on

$$\{x: x \in \mathbf{R},\ |x| \geqslant 1\}$$

and τ is a positive bounded Borel measure on $\{x: x \in \mathbf{R},\ 0 < |x| < 1\}$. It is clear from (1) that Z can be imbedded as the element Z_1 of a one parameter semi-group $(Z_t)_{t>0}$ of random variables, where the characteristic function $\chi^{(t)}$ of Z_t is given by

$$\chi^{(t)}(x) = \chi(tx) \quad (x \in \mathbf{R}). \quad (2)$$

Referring to the formula (1), we note that c corresponds to a component of Z which is simply a constant translate, d corresponds to a Gaussian variable, and σ corresponds to a simple Poisson variable. Accordingly we shall suppose that all these parameters are null and focus attention on the case where

$$\chi(x) = \mu^{\wedge}(x) = \int(e^{ixy} - 1 - ixy)\,d\nu(y), \quad (3)$$

where $\nu = y^{-2}.\tau$ is a (perhaps) unbounded measure on $\{y \in \mathbf{R}: 0 < |y| < 1\}$ which we call the *Lévy spectral measure* of Z. In general one can decompose $\nu = \sum_{n=1}^{\infty} \nu_n$, where the ν_n are mutually singular measures each with finite mean, $m_n = \int x\,d\nu_n(x)$. This leads to a convolution formula for the distribution of Z, *viz.*

$$\mu = \mathop{*}_{n=1}^{\infty} \delta(-m_n) * (\exp(\nu_n)/(\exp(\|\nu_n\|))). \quad (4)$$

The product in (4) is absolutely convergent in view of the Three Series Theorem and the condition that $\int x^2\,d\nu(x) < \infty$.

We have already declared our intention to use a symmetric Lévy measure of the form, $\nu = \sum_{n=1}^{\infty} A_n(\delta(B_n) + \delta(-B_n))$. It is now clear that we should impose the constraints $0 < B_n < 1$, $\sum_{n=1}^{\infty} A_n B_n^2 < \infty$, and that, in our case, the m_n in (4) will be zero. In fact, for reasons which will be clear in a moment, it will be convenient to change to parameters a_n, b_n where

$$2A_n = a_n b_n^2, \quad B_n = (b_1 b_2 \dots b_n)^{-1}. \quad (5)$$

We now summarize the discussion so far and establish notation for the remainder of the paper.

LEMMA 1. *Suppose that* $a_n > 0$, $b_n > 1$, *and that* $\sum_{n=1}^{\infty} a_n (B_{n-1})^2 < \infty$, *where*

$$B_n = (b_1 b_2 \dots b_n)^{-1}.$$

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Then there is a family $(Z_t)_{t>0}$ of symmetric infinitely divisible random variables such that Z_t has distribution μ_t, characteristic function $\chi^{(t)}$, and Lévy spectral measure ν_t satisfying:

$$\chi^{(t)}(x) = \prod_{n=1}^{\infty} \chi_n^{(t)}(x), \quad \chi_n^{(t)}(x) = \exp(-ta_n b_n^2(1-\cos(xB_n))); \tag{6}$$

$$\nu_t = \sum_{n=1}^{\infty} \nu_{t,n}, \quad \nu_{t,n} = \tfrac{1}{2}ta_n b_n^2(\delta(B_n)+\delta(-B_n)); \tag{7}$$

$$\mu_t = \mathop{*}_{n=1}^{\infty} \mu_{t,n}, \quad \mu_{t,n} = \exp(-ta_n b_n^2)\exp(\nu_{t,n}). \tag{8}$$

$\chi_n^{(t)}$ is the characteristic function of the random variables $Y_{t,n}$ which has distribution $\mu_{t,n}$. For $t, s > 0$,

$$Z_{t+s} = Z_t + Z_s, \quad \mu_{t+s} = \mu_t * \mu_s, \quad \chi^{(t+s)} = \chi^{(t)}.\chi^{(s)}. \tag{9}$$

Remark. Although it is natural to introduce the coin-tossing random variables $V_{k,n}$ and the Poisson variables $X_{t,n}$ in describing Z_t from a probabilistic point of view it is much more convenient from a technical point of view to pass direct from the Lévy measure to the compound Poisson variables $Y_{t,n}$, as we have done in Lemma 1. In particular, we have no further need of $V_{k,n}$, $X_{t,n}$ and feel free to use the symbols X, X_n for general random variables in the sequel.

Now we set about ensuring that the characteristic functions $\chi^{(t)}$ vanish at infinity. Since the appropriate condition involves a_n only we have a partial justification of the parameter change from A_n, B_n to a_n, b_n. The full justification is delayed until section 4 where we use arithmetical constraints to achieve singularity.

LEMMA 2. *Suppose that the hypotheses of Lemma* 1 *are satisfied and that* $\exp(-a_n) \to 0$ *as* $n \to \infty$. *Then* $\chi^{(t)}(x) \to 0$ *as* $x \to \pm\infty$.

Proof. In view of symmetry we may restrict attention to positive x. Moreover if x is sufficiently large then it lies in the interval, $[(B_{n-1})^{-1}\pi, (B_n)^{-1}\pi]$, for some n. Because $(1-\cos t)t^{-2}$ decreases, we see that, for x in the interval just mentioned,

$$(1-\cos(xB_n)) \geqslant (xB_n)^2 2\pi^{-2} \geqslant 2(B_n/B_{n-1})^2 = 2b_n^{-2}.$$

Consequently, for such x, we have from (6),

$$\chi^{(t)}(x) \leqslant \chi_n^{(t)}(x) \leqslant \exp(-2ta_n).$$

The required result follows at once.

3. *A criterion for singularity.* In this section we establish a general criterion for the mutual singularity of two probability measures. The idea is to replace the method of comparison of orthogonal L^2-sequences by comparing sequences of square integrable random variables with low correlation. That was the mechanism of proof used by W. Moran and the author in (4), and Theorem 1 which follows, generalizes and simplifies Lemmas 1, 2, 3 of that paper. We also make use of a neat trick which we learned from J. Peyrière's note (5).

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THEOREM 1. *Let P_1, P_2 be probability measures on the measurable space $(\Omega, \mathscr{E})$ and let E_i denote expectation with respect to P_i, for $i = 1, 2$. Suppose that there exists a sequence (X_n) of complex random variables on $(\Omega, \mathscr{E})$ (square integrable with respect to P_1, P_2) such that*

$$\sum_{k=0}^{\infty} \sup_{i,n} |E_i(X_n \bar{X}_{n+k}) - E_i(X_n) E_i(\bar{X}_{n+k})| < \infty, \tag{10}$$

$$\sum_{n=1}^{\infty} |E_1(X_n) - E_2(X_n)|^2 = \infty. \tag{11}$$

Then P_1, P_2 are mutually singular.

Proof. For $i = 1, 2$, let $Z_n^{(i)} = X_n - E_i(X_n)$ and note that for every non-negative integer k,

$$E_i(Z_n^{(i)} \bar{Z}_{n+k}^{(i)}) = E_i(X_n \bar{X}_{n+k}) - E_i(X_n) E_i(\bar{X}_{n+k}).$$

Accordingly we may suppose, using (10), that there exists a positive constant C such that

$$\sum_{k=0}^{\infty} \sup_{i,n} |E_i(Z_n^{(i)} \bar{Z}_{n+k}^{(i)})| \leqslant C. \tag{12}$$

Now let $\alpha = (\alpha_n)$ be an arbitrary sequence in l^2, let ϵ be an arbitrary positive number, let $\eta = (2C)^{-1} \epsilon$ and choose the positive integer N so that

$$\sum_{n=N}^{\infty} |\alpha_n|^2 < \eta. \tag{13}$$

For every positive integer K, we have, for $i = 1, 2$,

$$\begin{aligned} E_i\left\{\left|\sum_{n=N}^{N+K} \alpha_n Z_n^{(i)}\right|^2\right\} &\leqslant E_i\left\{\sum_{n=N}^{N+K} |\alpha_n|^2 |Z_n^{(i)}|^2\right\} + \sum_{k=1}^{\infty} \sum_{n=N}^{N+K} |\alpha_n| \, |\alpha_{n+k}| \, |E_i(Z_n^{(i)} \bar{Z}_{n+k}^{(i)})| \\ &\leqslant C \sum_{n=N}^{N+K} |\alpha_n|^2 + \sum_{k=1}^{\infty} \left\{\sum_{n=N}^{N+K} |\alpha_n| \, |\alpha_{n+k}|\right\} \sup_n |E_i(Z_n^{(i)} \bar{Z}_{n+k}^{(i)})| \\ &\leqslant 2C\eta, \quad \text{by Hölder's inequality and (12), (13).} \end{aligned}$$

It follows that $\sum_{n=1}^{N} \alpha_n Z_n^{(i)}$ converges in $L^2(P_i)$ for $i = 1, 2$. Consequently there is some subsequence $M(N)$ such that

$$\sum_{n=1}^{M(N)} \alpha_n Z_n^{(i)}(\omega) \quad \text{converges almost surely } P_i, \quad \text{for} \quad i = 1, 2. \tag{14}$$

Now suppose that P_1, P_2 are not mutually singular. It follows from (14) that there exists some ω in Ω such that

$$\sum_{n=1}^{M(N)} \alpha_n (Z_n^{(2)}(\omega) - Z_n^{(1)}(\omega)) = \sum_{n=1}^{M(N)} \alpha_n (E_1(X_n) - E_2(X_n))$$

converges. Since α was an arbitrary sequence in l^2, this contradicts (11) and the theorem is proved.

4. *Basic result.* It is clear that some arithmetical constraint on the points of support of the Lévy measure is a prerequisite for the type of result we seek. Our number theoretic condition is minimal, however, even in the simplest version of the result. There we demand merely that the b_n be integers. We must also find random variables

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X_n in order to apply Theorem 1. For the basic result the choice is again simple; we may take $X_n = \exp(2\pi i N_n x)$, where the integer N_n is the reciprocal of B_n. We shall see in a moment that, under a mild additional constraint, the expectation of X_n with respect to μ_t is of the same order of magnitude as (a fixed power of) $\exp(-ta_n)$. Since we intend to apply (11) to show that μ_t is singular for all $t > 0$, this means we must impose the condition that the sequence $(\exp(-ta_n))$ fails to be summable for each positive t. That said, it is reasonably easy to find tolerably neat conditions on the parameters which ensure the basic result. In fact we shall prove in this section:

THEOREM 2. *Suppose that* $a_n > 0$, $\exp(-a_n) \to 0$, *and* $\Sigma \exp(-ta_n) = \infty$ *for each positive* t. *Suppose further that* (b_n) *is a sequence of positive integers greater than one such that the sequence* $(a_{n+1}b_n^{-1})$ *is decreasing. Then there is a family* $(Z_t)_{t>0}$ *of symmetric infinitely divisible random variables such that*

(i) $Z_t + Z_s = Z_{t+s}$ $(s, t > 0)$,
(ii) $Z_t \perp L^1(\mathbf{R})$, *and* $\operatorname{supp}(Z_t) = \mathbf{R}$ $(t > 0)$,
(iii) *the characteristic function of* Z_t *vanishes at infinity*,
(iv) $Z_t + a \perp Z_s$ $(s, t > 0, a \in \mathbf{R}, s \neq t)$,
(v) *the Lévy spectral measure of* Z_t *is*

$$\sum_{n=1}^{\infty} \tfrac{1}{2} t a_n b_n^2 (\delta((b_1 b_2 \dots b_n)^{-1} + \delta(-(b_1 b_2 \dots b_n)^{-1}))).$$

COROLLARY 1. *The random variables* Z_t, *with spectral measures*

$$t \sum_{n=1}^{\infty} 2^{4n} \log\log(4n) [\delta(2^{-n^2}) + \delta(-2^{-n^2})],$$

satisfy (i), (ii), (iii), (iv) *of the statement of the theorem.*

The proof of Theorem 2 will be broken into a sequence of lemmas and we will obtain a stronger less tidy result. Although this is not strictly necessary for our present purposes it helps to clarify the mechanism of the proof. For the purpose of the lemmas we do *not* assume that N_n is the reciprocal of B_n, nor, for that matter, need we take the b_n to be integers. It is, however, a technical convenience to choose the N_n to be integers and this means we consider rational b_n. In fact let us employ the following constraints:

(M 1): $$a_n > 0, b_n > 1, B_n = (b_1 b_2 \dots b_n)^{-1}, \sum_{n=1}^{\infty} a_{n+1} B_n^2 < \infty;$$

(M 2): $$\exp(-a_n) \to 0, \sum_{n=1}^{\infty} \exp(-ta_n) = \infty, t > 0;$$

(M 3): (N_n) is an increasing sequence of positive integers such that $N_n B_k \equiv 0$ (modulo 1), $1 \leqslant k \leqslant n$;

(M 4): K_1, K_2 are finite, where

$$K_1 = \sup_n N_n B_n, K_2 = \limsup_n \sum_{m=2}^{\infty} a_{n+m} (N_n B_{n+m-1})^2;$$

(M 5): there exist positive integers k_0, n_0 such that $\sum_{k=k_0}^{\infty} K_{k,n_0} < \infty$, where

$$K_{k,n_0} = \sup_{n \geqslant n_0} \sum_{m=n+k+1}^{\infty} a_m N_n N_{n+k} (B_{m-1})^2.$$

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LEMMA 3. *Suppose that conditions* (*M* 1) *through* (*M* 4) *hold. Suppose further that* $s > t > 0$, p *is a positive integer, and* a *is a real number. Let* P_1 *denote the measure* μ_t *and let* P_2 *denote either the measure* λ_p *(which is normalized Lebesgue measure on* $[-p, p]$*) or the measure* $\delta(a) * \mu_s$. *Let* $X_n(x) = \exp(2\pi i N_n x)$. *Then*

$$\sum_{n=1}^{\infty} |\int X_n \, dP_1 - \int X_n \, dP_2|^2 = \infty. \tag{15}$$

Proof. Because (M 1) holds, we have the conclusion (and notation) of Lemma 1. We shall make particular use of (6), (9). Taking these statements in conjunction with the observation that, for any probability measure P, $\int X_n \, dP$ is the characteristic function of P evaluated at $2\pi N_n$, we deduce that

$$\int X_n \, d\mu_t = (\mu_t)^\wedge(2\pi N_n) = \prod_{k=1}^{\infty} \chi_k^{(t)}(2\pi N_n), \tag{16}$$

$$\int X_n \, d\lambda_p = (\lambda_p)^\wedge(2\pi N_n) = 0, \tag{17}$$

$$\int X_n \, d(\mu_s * \delta(a)) = \exp(2\pi i N_n a)\,(\mu_{s-t})^\wedge(2\pi N_n)\,\mu_t^\wedge(2\pi N_n). \tag{18}$$

It follows from (16), (17), (18) that, in all cases,

$$|\int X_n \, dP_1 - \int X_n \, dP_2| \geqslant (\mu_t)^\wedge(2\pi N_n)\,|1 - (\mu_{s-t})^\wedge(2\pi N_n)|. \tag{19}$$

The first part of (M 2) together with Lemma 2 shows that $(\mu_{s-t})^\wedge(2\pi N_n) \to 0$ as $n \to \infty$. This observation combined with (19) shows that we need only verify that the sequence $((\mu_t)^\wedge(2\pi N_n))$ is not square summable. Using (16) and the congruence condition (M 3), we see that the nth term of this sequence can be re-expressed in the formula

$$(\mu_t)^\wedge(2\pi N_n) = \chi_{n+1}^{(t)}(2\pi N_n) \prod_{m=2}^{\infty} \chi_{n+m}^{(t)}(2\pi N_n). \tag{20}$$

Now recall, from (6), that $\chi_k^{(t)}(x) = \exp(-ta_k b_k^2(1 - \cos(xB_k)))$, and use the fact that $1 - \cos y \leqslant y^2/2$ to see that

$$\chi_k^{(t)}(2\pi N_n) \geqslant \exp(-2\pi^2 t a_k b_k^2 B_k^2 N_n^2) = \exp(-2\pi^2 t a_k (B_{k-1} N_n)^2). \tag{21}$$

Combining (20), (21), and (M 4), we deduce that, for any $K' > K_2$,

$$(\mu_t)^\wedge(2\pi N_n) \geqslant \exp(-2\pi^2 t K_1 a_{n+1}) \exp(-2\pi^2 t K'). \tag{22}$$

Now an application of the second part of (M 2) (with $4\pi^2 t K_1$ in place of t) completes the proof.

Now we set about obtaining a version of the correlation estimate (10). Let us first dispense with a simple preliminary inequality.

LEMMA 4. *Suppose that* $a, b, c > 0$. *Then*

$$|\exp(-a(1 - \cos(b - c))) - \exp(-a(1 - \cos b))\exp(-a(1 - \cos c))| \leqslant 2abc.$$

Proof. We assume, without loss of generality, that $b \geqslant c$. An application of the Mean Value Theorem to $\exp(-x)$ on the appropriate positive interval shows that the quantity to be estimated is dominated by

$$|a(1 - \cos(b - c)) - [a(1 - \cos b) + a(1 - \cos c)]|.$$

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The last expression may be rewritten as

$$a\,|(\cos c - \cos(b-c)) - (1-\cos b)|.$$

Applying the Mean Value Theorem to $\cos x - \cos(b-x)$ on the interval $[0, c]$, we see that this, in turn, equals

$$ac\,|\sin\theta + \sin(b-\theta)|, \quad \text{for some} \quad 0 < \theta < c.$$

Because $\sin x < x$ for positive x, we now obtain the estimate required to complete the proof.

LEMMA 5. *Suppose that* (M 1) *holds together with the congruence condition* (M 3). *Let* $P = \mu_t * \delta(y)$, *where* y *is any real number and* $t > 0$. *Let* $X_n(x) = \exp(2\pi i N_n x)$. *Then, for* $n \geqslant q$,

$$\left|\int X_n \overline{X}_{n+k}\,dP - \int X_n\,dP \int \overline{X}_{n+k}\,dP\right| \leqslant 8\pi^2 t K_{n,q}. \tag{23}$$

Proof. As before (M 1) ensures that we have the notation and results of Lemma 1; and we are concerned with estimating

$$Q_{n,k} = |\chi(2\pi(N_n - N_{n+k})) - \chi(2\pi N_n)\,\chi(-2\pi N_{n+k})|, \tag{24}$$

where χ denotes the characteristic function of $\mu_t * \delta(y)$. It follows immediately that we can (in fact we do) suppose $y = 0$. Moreover, for sequences (z_m), (z'_m) of complex numbers of modulus not exceeding one, we have, in general,

$$|\Pi z_m - \Pi z'_m| \leqslant \Sigma\,|z_m - z'_m|;$$

therefore we use the formula $\chi = \prod_{m=1}^{\infty} \chi_m^{(t)}$ to replace (24) by

$$Q_{n,k} \leqslant \sum_{m=1}^{\infty} |\chi_m^{(t)}(2\pi(N_n - N_{n+k})) - \chi_m^{(t)}(2\pi N_n)\,\chi_m^{(t)}(2\pi N_{n+k})|. \tag{25}$$

The mth term on the right-hand side of (25) is of the same form as the quantity estimated in Lemma 4, when we take

$$a = t a_m b_m^2, \quad b = 2\pi N_{n+k} B_m, \quad c = 2\pi N_n B_m.$$

The congruence condition ensures that the first $n+k$ terms are, in fact, zero. We estimate the remainder, by means of Lemma 4, to obtain

$$Q_{n,k} \leqslant 8\pi^2 t \sum_{m=n+k+1}^{\infty} a_m N_n N_{n+k} (B_{m-1})^2.$$

The rest is clear.

Proof of Theorem 2. We start by proving the more general result that the conclusion holds under the assumptions (M 1) through (M 5). Observe first that (M 1) guarantees (v) and (i) by Lemma 1. Moreover, (M 1) shows also that Z_t has full support. To see this we use (from (8)) the formula $\mu_t = \overset{\infty}{\underset{n=1}{*}} \mu_{t,n}$, noting that the support of each $\mu_{t,n}$ is a discrete subgroup of $\mathbf{R}$ containing B_n; meanwhile B_n tends to zero as $n \to \infty$. Lemma 2 tells us that (iii) holds, so only the singularity properties remain. Obviously we intend

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to apply Theorem 1 with P_1, P_2 chosen as in Lemma 3. We relabel, if necessary, the sequence (X_n) so that the first term is $\exp(2\pi i N_{n_0} x)$, where n_0 is given by (M 5). Note also that X_n has unit modulus, so to verify (10) we are at liberty to commence the summation with $k = k_0$. It is now clear that (10) does, indeed, hold for $P_1 = \mu_t$, $P_2 = \delta(a) * \mu_s$ because of (M 1), (M 3), (M 5) and Lemma 5. The gap in the remaining case ($P_1 = \mu_t$, $P_2 = \lambda_p$) can be filled by direct appeal to orthogonality. The standing hypotheses together with Lemma 3 guarantee that (11) holds in both cases. Theorem 1 now gives (iv) and the fact that $\mu_t \perp \lambda_p$ for all $t > 0$ and all positive integers p. Letting $p \to \infty$, we deduce that $\mu_t \perp L^1(\mathbf{R})$ for all positive t.

It remains to verify that (M 1), ..., (M 5) hold under the hypotheses of Theorem 2, when we make the canonical choice of N_n as the reciprocal of B_n. Of course (M 2) was explicitly included in those hypotheses and the summability criterion in (M 1) is obviously a special case of the second statement of (M 4) (interchange m, n). (M 3) and the first assertion of (M 4) follow from the choice of N_n, so we may concentrate on (M 5) and the quantity K_2. In connexion with the latter, note that, for $n \geqslant 1$, $m \geqslant 2$,

$$a_{n+m}(N_n B_{n+m-1})^2 = a_{n+m}(b_{n+m-1})^{-2}(B_{n+m-2}/B_n)^2 \leqslant a_{n+1} b_n^{-1} 2^{1-m},$$

since $b_j \geqslant 2$ and $(a_{n+1} b_n^{-1})$ decreases. To verify (M 5) take $k \geqslant 1$, $m \geqslant n+k+1$, for the estimate

$$a_m N_n N_{n+k}(B_{m-1})^2 \leqslant a_m (b_{m-1})^{-1} 2^{3-2m+2n+k}.$$

It follows that, for $k \geqslant 1$, $n \geqslant 1$,

$$K_{k,n} \leqslant 8 a_{n+1} b_n^{-1} 2^{-k} \sum_{s=1}^{\infty} 2^{-2s},$$

and the required condition is clearly satisfied. This completes the proof of Theorem 2.

Concerning the corollary we note that the parameters are $a_n = 8 \log\log(4n)$, $b_n = 2^{2n-1}$, and these obviously satisfy the appropriate conditions.

5. *Varying the support.* When can a discrete subset E of $\mathbf{R}\backslash\{0\}$ be the support set for the spectral measure a family $(Z_t)_{t>0}$ of the type under consideration? It is clear that we must demand that E be symmetric (in the sense that $E = -E$) and that 0 is a limit point of E. Our methods give a sufficient condition in terms of divisibility properties of E. Let us say that a sequence (x_n) of real numbers is *weakly divisible* if each x_n is a non-trivial positive integer multiple of its successor x_{n+1} (i.e. x_{n+1} is an $m(n)$th root of x_n with $m(n)$ an integer greater than one).

THEOREM 3. *Let E be a countable symmetric subset of $\mathbf{R}\backslash\{0\}$ which contains a weakly divisible sequence. Then there exists a family $(Z_t)_{t>0}$ of symmetric infinitely divisible random variables satisfying* (i), (ii), (iii), (iv) *of Theorem 2 and such that the Lévy spectral measure of the family has support equal to E.*

Proof. Choose some weakly divisible sequence (x_n) of positive reals in E, and change the parameterization of $\mathbf{R}$ so that $x_1 = \frac{1}{2}$. Fix (a_n) as any sequence satisfying (M 2). (E.g. $a_n = \log\log(4n)$.) Now determine a sequence $(p(n))$ of positive integers, by induction, in the following way. Let $p(1) = 1$, $p(2) = 2$ and choose $p(n)$ sufficiently large that

$$x_{p(n+1)} < a_{n+1}(x_{p(n)})^2/(a_{n+2} x_{p(n-1)}). \tag{26}$$

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Now define (b_n) by

$$b_1 = 2 = x_1^{-1}, \quad b_n = x_{p(n-1)}/x_{p(n)} \quad (n = 2, 3, \ldots). \tag{27}$$

It follows from (26), (27), that (b_n) is a sequence of integers, each greater than one, such that $(a_{n+1}b_n^{-1})$ is a decreasing sequence. Moreover, for each positive integer n, we may write

$$B_n = (b_1 b_2 \ldots b_n)^{-1} = x_{p(n)}. \tag{28}$$

It follows from Theorem 2, that there is a family $(Z_t')_{t>0}$, say, of infinitely divisible random variables with the properties specified for $(Z_t)_{t>0}$ in (i) ... (iv) of the statement of that theorem, and such that the Lévy spectral measure of Z_t' is given by

$$\nu_{t,n}' = \sum_{n=1}^{\infty} \tfrac{1}{2} t a_n b_n^2 (\delta(x_{p(n)}) + \delta(-x_{p(n)})). \tag{29}$$

Now we consider the complement in E of the set $\{\pm x_{p(n)}: n = 1, 2, \ldots\}$. The set in question could be empty or finite – but there is clearly no loss of generality in restricting discussion to the case where there are infinitely many points of E outside the support of $\nu_{t,n}'$. Let us enumerate these points as $\{\pm y_n: n = 1, 2, \ldots\}$ and fix some strictly positive sequence (c_n) with $\sum_{n=1}^{\infty} c_n < \infty$. We define a family $(Z_t'')_{t>0}$ of infinitely divisible random variables by specifying the Lévy measures

$$\nu_{t,n}'' = \sum_{n=1}^{\infty} \tfrac{1}{2} t c_n (\delta(y_n) + \delta(-y_n)). \tag{30}$$

Of course each Z_t'' is discrete. This follows from well known results, but is also immediately evident from (8) since we may write the distribution μ_t'' of Z_t'' as an infinite convolution

$$\mu_t'' = \mathop{*}_{n=1}^{\infty} \exp(-tc_n) \exp(\nu_{t,n}'') = \mathop{*}_{n=1}^{\infty} (\alpha_n \delta(0) + (1-\alpha_n)\tau_n),$$

where $\alpha_n = \exp(-tc_n)$ and τ_n is a discrete probability measure. Now we can define the required family $(Z_t)_{t>0}$ by specifying the spectral measures,

$$\nu_{t,n} = \nu_{t,n}' + \nu_{t,n}''. \tag{31}$$

From (29), (30), (31) we see that $\nu_{t,n}$ has support E and that the distribution μ_t of Z_t satisfies $\mu_t = \mu_t' * \mu_t''$. This shows, in particular, that μ_t is a sum of translates of μ_t' and, since the zero translate is present, we see that (i), (ii), (iv) hold. Finally (iii) is true because the characteristic function of μ_t' is a factor of the characteristic function of μ_t. This completes the proof.

It follows from the theorem that there exist suitable families $(Z_t)_{t>0}$ with the rationals as the support set of the Lévy measures but it is clear from the proof that the 'interesting' case lies rather in the direction of very thin support – e.g. $E = \{\pm 2^{-k(n)}\}$ where $k(n)$ increases arbitrarily fast. We have been unable to determine whether weak divisibility is necessary. It would, for example, be interesting to know whether the set $\{\pm p_n^{-1}\}$, where (p_n) is an enumeration of the prime numbers, supports Lévy spectral measures of the required kind.

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Returning to the case where we can handle the support set E, we feel it worth commenting on the rôle of the parameter sequence (a_n). This is essentially at our disposal within the limits prescribed by (M 2). In consequence we may vary (a_n) for fixed (b_n) to obtain collections of one parameter semi-groups with all the properties discussed so far. Moreover, it is possible to apply Theorem 1 as a tool for comparing different families. Lemma 5 ensures that there are no difficulties in proving mutual singularity provided we can obtain an estimate for the sum of the squares of the mean differences. The latter problem corresponds, of course, to Lemma 3, a variant of which reduces the essential problem to a consideration of

$$\sum_{n=1}^{\infty} |\exp(-ta_n) - \exp(-t'a'_n)|^2 \quad \text{(where we should consider also } t = t'\text{)}.$$

Details are tedious so we simply invite the reader to convince himself that one can in this way keep the parameters (b_n) fixed and vary (a_n) to obtain uncountably many families $(Z_t)_{t>0}$ such that for any distinct pair $(Z_t)_{t>0}$, $(Z'_t)_{t>0}$ we have $Z_t + a \perp Z'_s + b$, for all $a, b \in \mathbf{R}$, all $s, t > 0$ (including $a = b$, $s = t$). This species of behaviour was established for the distributions considered in (2).

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