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Nonlinear dynamics of obliquely propagating Alfvén waves

A. GAZOL, T. PASSOT and P. L. SULEM
CNRS UMR 6529, Observatoire de la Côte d'Azur, BP 4229, 06304 Nice Cedex 4, France

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The long-wave, small-amplitude dynamics of obliquely propagating Alfvén waves is shown, using a reductive perturbative expansion, to be purely linear not only in one space dimension but also in the dispersionless limit in higher dimensions. Furthermore, in the context of multidimensional wave-train modulation, all the diffraction coefficients are found to tend to zero with the dispersion, while the nonlinear terms in the envelope equation remain finite. In this 'semiclassical' limit, the envelope dynamics results in the formation of growing regions of finite-amplitude oscillations with a typical scale intermediate between the size of the wave packet and its wavelength.

1. Introduction

Special attention has been paid to the long-wavelength (or weakly dispersive) limit of small-amplitude Alfvén waves propagating in a direction parallel or quasiparallel to an ambient magnetic field. This case displays special features resulting from the equality, in the zero-dispersion limit, of the phase velocities of the Alfvén wave and of the slow ($\beta > 1$) or fast ($\beta < 1$) magnetosonic wave. In particular, a reductive perturbation expansion shows that, in one space dimension, the weakly nonlinear dynamics is governed by the derivative nonlinear Schrödinger (DNLS) equation (Rogister 1971; Mio *et al.* 1975; Mjølhus 1976). This equation, which is integrable by the inverse-scattering transform (Kaup and Newell 1978), leads to the formation of solitons. The multidimensional extension was obtained by Mjølhus and Wyller (1986, 1988) and generalized to non-localized solutions by including the coupling to magnetosonic waves in Passot and Sulem (1993, 1995). Whether the multidimensional dynamics leads to collapse or to solitonic structures – a problem already noted by Mjølhus and Wyller (1986), – is still debated (Spangler and Sheerin 1983, Pokhotelov *et al.* 1996).

The analysis in the case of oblique propagation was mostly restricted to situations where the fields vary only along the direction of propagation. This regime appears to be very specific, because the nonlinearity vanishes identically in a long-wave perturbation expansion for small-amplitude waves, leading to a dynamics governed by a linear equation with the same dispersive operator as the Korteveg–de Vries (KdV) equation (Kakutani and Ono 1969; Mio *et al.* 1976), which we refer to as the linear KdV equation. Nonlinear couplings require strong perturbations of the magnetic field component transverse to the propagation. In this context, a mKdV (for modified KdV) equation with cubic nonlinearity is obtained for the gradient of the associated precession angle (Kakutani and Ono 1969).

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In two space dimensions, long-wave, small-amplitude equations were recently derived using an approach that is not based on a systematic ordering but remains uniformly valid when the propagation direction of the Alfvén waves becomes parallel to the ambient magnetic field (Brodin 1996).

The aim of the present paper is twofold. Using a systematic reductive perturbative expansion, we first revisit the long-wave description of obliquely propagating Alfvén waves and generalize it to three-dimensional perturbations. The equations that we obtain are non-canonical and degenerate for a vanishing dispersion or for perturbations whose transverse scale is larger than the longitudinal one. We then use a modulational analysis to investigate the nonlinear phenomena resulting from cumulative effects and affecting the wave-train dynamics in the small-dispersion limit. In this context, a noticeable property is that, in contrast with magnetosonic waves, an Alfvén wave does not interact with its own harmonics, which prevents the divergence of the coefficient of the nonlinear term. We concentrate here on the regime where the wave is linearly polarized in the direction perpendicular to the plane defined by the ambient magnetic field and the direction of propagation. A main observation is that all the diffraction terms vanish while nonlinear effects remain finite in the case of a multidimensional modulation. This contrasts with the dynamics of circularly polarized Alfvén waves propagating in the direction of the ambient magnetic field (Shukla and Stenflo 1989; Passot *et al.* 1994; Champeaux *et al.* 1997*a*), where the diffraction coefficient remains finite.

The paper is organized as follows. The long-wave approximation is performed in Sec. 2, while the modulational equations are derived in Sec. 3. Section 4 discusses the envelope dynamics, stressing the difference between a purely longitudinal and a multidimensional modulation. The main conclusions are summarized in Sec. 5.

2. Long-wave dynamics

The equations for dispersive MHD are given by

$$\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \tag{2.1}$$

$$\rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}) = -\frac{\beta}{\gamma} \boldsymbol{\nabla} \rho^\gamma + (\boldsymbol{\nabla} \times \mathbf{b}) \times \mathbf{b}, \tag{2.2}$$

$$\partial_t \mathbf{b} - \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{b}) = -\frac{1}{R_i} \boldsymbol{\nabla} \times \left[\frac{1}{\rho} (\boldsymbol{\nabla} \times \mathbf{b}) \times \mathbf{b} \right], \tag{2.3}$$

$$\boldsymbol{\nabla} \cdot \mathbf{b} = 0, \tag{2.4}$$

where γ is the polytropic gas constant and R_i denotes the non-dimensional gyromagnetic frequency of the ions. The dispersion term in (2.3) arises from the Hall term in the generalized Ohm's law, where the effect of ion inertia has been retained. We assume a static homogeneous medium of density $\rho_0 = 1$ permeated by a constant magnetic field $\mathbf{b}_0$, whose direction is chosen as the x axis, and consider Alfvén waves whose phase planes propagate along a direction in the (x, y) plane making a finite angle θ with the ambient field. In order to perform a long-wave, small-amplitude reductive perturbation expansion, we define the stretched variables

$$\xi = \epsilon^{1/2}(x - t), \qquad \eta = \epsilon^{1/2} y, \qquad \zeta = \epsilon z, \qquad \tau = \epsilon^{3/2} t. \tag{2.5}$$

This choice looks unusual because the velocity of the new reference frame (ξ, η, ζ), given by the group velocity of the dispersionless Alfvén waves, cannot be identified

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in the multidimensional problem with the phase velocity of these waves. Note that the ξ and η variables are stretched similarly. We also expand the various fields

$$\rho = 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + \dots, \tag{2.6}$$

$$u_x = \epsilon u_{x1} + \epsilon^2 u_{x2} + \dots, \tag{2.7}$$

$$u_y = \epsilon u_{y1} + \epsilon^2 u_{y2} + \dots, \tag{2.8}$$

$$u_z = \epsilon^{1/2} u_{z1} + \epsilon^{3/2} u_{z2} + \dots, \tag{2.9}$$

$$b_x = 1 + \epsilon b_{x1} + \epsilon^2 b_{x2} + \dots, \tag{2.10}$$

$$b_y = \epsilon b_{y1} + \epsilon^2 b_{y2} + \dots, \tag{2.11}$$

$$b_z = \epsilon^{1/2} b_{z1} + \epsilon^{3/2} b_{z2} + \dots \tag{2.12}$$

in a way that selects to leading order a wave linearly polarized in the z direction, perpendicular to the (x, y) plane defined by the ambient field and the direction of propagation.

At the leading order of the expansion, we have

$$u_{z1} = -b_{z1}, \tag{2.13}$$

while at order $\epsilon^{3/2}$ we find

$$-\partial_\xi\rho_1 + \partial_\xi u_{x1} + \partial_\eta u_{y1} + \partial_\zeta u_{z1} = 0, \tag{2.14}$$

$$-\partial_\xi u_{x1} + \beta\partial_\xi\rho_1 + \tfrac{1}{2}\partial_\xi b_{z1}^2 = 0, \tag{2.15}$$

$$-\partial_\xi u_{y1} - \partial_\xi b_{y1} + \partial_\eta b_{x1} + \beta\partial_\eta\rho_1 + \tfrac{1}{2}\partial_\eta b_{z1}^2 = 0, \tag{2.16}$$

$$-\partial_\xi b_{x1} + \partial_\eta u_{y1} + \partial_\zeta u_{z1} = \frac{1}{R_i}\partial_{\xi\eta} u_{z1}, \tag{2.17}$$

$$-\partial_\xi b_{y1} - \partial_\xi u_{y1} = -\frac{1}{R_i}\partial_{\xi\xi} u_{z1}, \tag{2.18}$$

$$\partial_\xi b_{x1} + \partial_\eta b_{y1} + \partial_\zeta b_{z1} = 0, \tag{2.19}$$

from which we easily get

$$u_{x1} = \beta\rho_1 + \tfrac{1}{2} b_{z1}^2, \tag{2.20}$$

$$\partial_\eta b_{x1} = -\partial_\eta(\beta\rho_1 + \tfrac{1}{2} b_{z1}^2) - \frac{1}{R_i}\partial_{\xi\xi} b_{z1}, \tag{2.21}$$

$$\partial_{\eta\eta} b_{y1} = \partial_\eta(\tfrac{1}{2}\partial_\xi b_{z1}^2 + \beta\partial_\xi\rho_1 - \partial_\zeta b_{z1}) + \frac{1}{R_i}\partial_{\xi\xi\xi} b_{z1}, \tag{2.22}$$

$$\partial_\eta u_{y1} = (1 - \beta)\partial_\xi\rho_1 - \tfrac{1}{2}\partial_\xi b_{z1}^2 + \partial_\zeta b_{z1}, \tag{2.23}$$

with ρ_1 given by

$$\partial_\eta\rho_1 = -\frac{1}{R_i}\nabla_h^2 b_{z1} \tag{2.24}$$

and ∇_h^2 standing for $\partial_{\xi\xi} + \partial_{\eta\eta}$.

The solvability condition for the system arising at order ϵ^2 reads

$$\begin{aligned}-2\partial_\tau b_{z1} - (u_{x1} - \rho_1 + 2b_{x1})\partial_\xi b_{z1} - (u_{y1} + 2b_{y1})\partial_\eta b_{z1} + \partial_\zeta b_z^2 \\ -\partial_\xi(u_{x1} b_z) - \partial_\eta(u_{y1} b_{z1}) + \partial_\zeta b_{x1} + \beta\partial_\zeta\rho_1 + \frac{1}{R_i}\partial_\xi(\partial_\xi u_{y1} - \partial_\eta u_{x1}) = 0.\end{aligned} \tag{2.25}$$

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Differentiating (2.25) twice with respect to η and using (2.20)–(2.23), we finally obtain

$$2\partial_{\tau\eta\eta}b_{z1} + \left[\frac{1}{R_i^2}\partial_\xi(\partial_{\xi\xi} - \beta\nabla_h^2) + \frac{\beta}{R_i}\partial_{\eta\zeta}\right]\nabla_h^2 b_{z1} = -\frac{1}{R_i}\partial_{\xi\eta}\nabla_h^2(\tfrac{1}{2}b_{z1}^2) - \partial_{\xi\eta\eta}(\rho_1 b_{z1}). \tag{2.26}$$

In the absence of dependence on the z variable, (2.24) and (2.26) can be identified with (20) and (17) of Brodin (1996), but are here derived by means of a systematic asymptotic expansion. Note also that in the dispersionless limit, all the nonlinear terms disappear, making b_{z1} time-independent.

If the fields depend only on ζ and on the combination $\xi' = \xi\cos\theta - \eta\sin\theta$ associated with the propagation direction of the phase fronts, we obtain the linear equation

$$\partial_\tau b_{z1} + \frac{\cos\theta(\cos^2\theta - \beta)}{2R_i^2\sin^2\theta}\partial_{\xi'\xi'\xi'}b_{z1} + \frac{\beta}{2R_i\sin\theta}\partial_{\xi'\zeta}b_{z1} = 0 \tag{2.27}$$

derived by Mio *et al.* (1976) in the purely one-dimensional limit.

The same equation is still obtained when assuming a weak dependence on the coordinate $\eta' = \xi\sin\theta + \eta\cos\theta$ perpendicular to the direction of the phase-front propagation, with no contributions from terms of the form $\partial_{\xi\tau}b_z$ or $\partial_{\eta'\eta'}b_z$, present in the usual Kadomtsev–Petviashvili equation.

Note that (2.27) becomes singular in the limit $\theta \to 0$. Indeed, for parallel-propagating Alfvén waves, in contrast with the above ordering, the magnitude of u_y and b_y becomes comparable to that of u_z and b_z, and the long-wave dynamics is governed by the DNLS equation. In fact, this equation remains valid as long as the propagation angle does not exceed the magnitude of the wave amplitude (Kennel *et al.* 1989; Mjølhus and Wyller 1988; Spangler and Plapp 1992). In the limit where this angle becomes larger (although remaining small), the phase velocities of the magnetosonic and Alfvén waves become different. A reductive perturbation expansion of the DNLS equation then shows that, as in the case of a finite propagation angle (Kakutani *et al.* 1968; Mio *et al.* 1976), the long-time dynamics of the magnetosonic wave is governed by the KdV equation, while that of the Alfvén wave is governed by a linear KdV equation (Mjølhus 1989). Nonlinear effects are obtained in the form of a mKdV equation when a modulation approach is used on the DNLS equation (Kennel *et al.* 1989, Mjølhus 1989) to construct Alfvén-wave solutions with an amplitude comparable to that of the ambient magnetic field, as in Kakutani and Ono (1969). The question arises whether slow nonlinear phenomena can take place in the case of small-amplitude wave trains, as a result of cumulative effects. Indeed, while the derivation of (2.24)–(2.26) assumes localized solutions, mean fields must be included when dealing with wave packets. A usual way to analyse these effects is to perform a modulational analysis directly from the MHD equations (2.1)–(2.4).

3. The modulational analysis

In this section, the x axis of the reference frame is chosen along the direction of the wave vector $\mathbf{k}$. We thus consider magnetohydrodynamic wave trains propagating along the x axis in a static homogeneous medium of density $\rho_0 = 1$, permeated by a uniform magnetic field $\mathbf{b}_0 = (\cos\theta, \sin\theta, 0)$ making an angle θ with the propagation direction of the wave fronts.

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In the linear approximation, the solution $\mathbf{a}_1 = (\rho_1, u_{x1}, u_{y1}, u_{z1}, b_{x1}, b_{y1}, b_{z1})$ of the MHD equations is taken in the form of a monochromatic wave

$$\mathbf{a}_1 = \mathbf{A}_1 e^{i(kx-\omega t)} + \text{c.c.}\,, \tag{3.1}$$

where the components of the amplitude $\mathbf{A}_1 = (\varrho_1, U_{x1}, U_{y1}, U_{z1}, B_{x1}, B_{y1}, B_{z1})$ satisfy

$$\varrho_1 = \frac{k^2 \sin\theta}{\omega^2 - \beta k^2} B_{y_1}, \tag{3.2}$$

$$U_{x_1} = \frac{k\omega \sin\theta}{\omega^2 - \beta k^2} B_{y_1}, \tag{3.3}$$

$$U_{y_1} = -\frac{k \cos\theta}{\omega} B_{y_1}, \tag{3.4}$$

$$U_{z_1} = -\frac{ik^3 \cos^2\theta}{R_i\,(\omega^2 - k^2\cos^2\theta)} B_{y_1}, \tag{3.5}$$

$$B_{x_1} = 0, \tag{3.6}$$

$$B_{z_1} = \frac{ik^2\omega \cos\theta}{R_i\,(\omega^2 - k^2\cos^2\theta)} B_{y_1}. \tag{3.7}$$

The dispersion relation

$$\frac{\omega^2}{R_i^2} = \frac{\omega^2 - k^2\cos^2\theta}{k^4\cos^2\theta}\left[(\omega^2 - k^2\cos^2\theta) - \frac{k^2\omega^2\sin^2\theta}{\omega^2 - \beta k^2}\right], \tag{3.8}$$

defines three pairs of solutions $\omega = \pm\omega_i(k)$ with $i = 1, 2, 3$, corresponding to the contrapropagating Alfvén, slow and fast magnetosonic waves.

In order to perform a long-time, large-scale modulational analysis of these waves, we define the slow variables $T = \epsilon t$ and $\mathbf{X} = (X, Y, Z) = (\epsilon x, \epsilon y, \epsilon z)$, and expand the solutions in powers of ϵ. For oblique propagation and finite dispersion, we write

$$\rho = 1 + \epsilon\rho_1 + \epsilon^2\rho_2 + +\epsilon^3\rho_3 + \ldots, \tag{3.9}$$

$$u_x = \epsilon u_{x_1} + \epsilon^2 u_{x_2} + \epsilon^3 u_{x_3} + \ldots, \tag{3.10}$$

$$b_x = \cos\theta + \epsilon^2 b_{x_2} + \epsilon^3 b_{x_3} + \ldots, \tag{3.11}$$

$$u_y = \epsilon u_{y_1} + \epsilon^2 u_{y_2} + \epsilon^3 u_{y_3} + \ldots, \tag{3.12}$$

$$u_z = \epsilon u_{z_1} + \epsilon^2 u_{z_2} + \epsilon^3 u_{z_3} + \ldots, \tag{3.13}$$

$$b_y = \sin\theta + \epsilon b_{y_1} + \epsilon^2 b_{y_2} + \epsilon^3 b_{y_3} + \ldots, \tag{3.14}$$

$$b_z = \epsilon b_{z_1} + \epsilon^2 b_{z_2} + \epsilon^3 b_{z_3} + \ldots, \tag{3.15}$$

where the perturbations arise at the same order in all the fields, except in b_x, as suggested by the above linear analysis. Substituting into the MHD equations (2.1)–(2.4), a linear system is obtained at each order of the expansion. At order ϵ, the linear approximation is recovered with an amplitude $\mathbf{A}_1$ that is now a function of the slow variables $\mathbf{X}$ and T.

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At order ϵ^2, we have (here $\mathbf{u}_1 = (u_{x_1}, u_{y_1}, u_{z_1})$)

$$\partial_t \rho_2 + \partial_x u_{x_2} = -\partial_T \rho_1 - \partial_x(\rho_1 u_{x_1}) - \boldsymbol{\nabla}\cdot\mathbf{u}_1, \tag{3.16}$$

$$\begin{aligned}\partial_t u_{x_2} + \beta\partial_x \rho_2 + \sin\theta\,\partial_x b_{y_2} = &-\partial_T u_{x_1} - \rho_1\partial_t u_{x_1} - u_{x_1}\partial_x u_{x_1}\\ &-\beta\,[\partial_X \rho_1 + (\gamma-1)\rho_1\partial_x\rho_1] - b_{z_1}\partial_x b_{z_1} - \sin\theta\,\partial_X b_{y_1} - b_{y_1}\partial_y b_{y_1},\end{aligned} \tag{3.17}$$

$$\partial_t u_{y_2} - \cos\theta\,\partial_x b_{y_2} = -\partial_T u_{y_1} - \rho_1\partial_t u_{y_1} - u_{x_1}\partial_x u_{y_1} - \beta\partial_Y\rho_1 + \cos\theta\,\partial_X b_{y_1}, \tag{3.18}$$

$$\begin{aligned}\partial_t u_{z_2} - \cos\theta\,\partial_x b_{z_2} = &-\partial_T u_{z_1} - \rho_1\partial_t u_{z_1} - u_{x_1}\partial_x u_{z_1}\\ &-\beta\partial_Z\rho_1 + \cos\theta\,\partial_X b_{z_1} + \sin\theta\,(\partial_Y b_{z_1} - \partial_Z b_{y_1}),\end{aligned} \tag{3.19}$$

$$\partial_t b_{x_2} = \sin\theta\,\partial_Y u_{x_1} - \cos\theta\,(\partial_Y u_{y_1} + \partial_Z u_{z_1}) + \frac{\cos\theta}{R_i}\partial_x(\partial_Z b_{y_1} - \partial_Y b_{z_1}), \tag{3.20}$$

$$\begin{aligned}\partial_t b_{y_2} + \partial_x(\sin\theta\,u_{x_2} - \cos\theta\,u_{y_2}) = &-\partial_T b_{y_1} + \cos\theta\,\partial_X u_{y_1}\\ &-\partial_x(u_{x_1} b_{y_1}) - \sin\theta\,(\partial_Z u_{z_1} + \partial_X u_{x_1})\\ &-\frac{1}{R_i}\partial_x[\cos\theta\,(\rho_1\partial_x b_{z_1} - \partial_x b_{z_2} - 2\partial_X b_{z_1}) - \sin\theta\,\partial_Y b_{z_1}],\end{aligned} \tag{3.21}$$

$$\begin{aligned}\partial_t b_{z_2} - \cos\theta\,\partial_x u_{z_2} = &-\partial_T b_{z_1} + \cos\theta\,\partial_X u_{z_1} - \partial_x(u_{x_1} b_{z_1}) + \sin\theta\,\partial_Y u_{z_1}\\ &+\frac{1}{R_i}\partial_x[\cos\theta\,(\rho_1\partial_x b_{y_1} - \partial_x b_{y_2} - 2\partial_X b_{y_1}) - \sin\theta\,\partial_Y b_{y_1}]\end{aligned} \tag{3.22}$$

Denoting by $\mathbf{L}$ the linear operator appearing on the left-hand side of (3.16)–(3.22), the solvability condition associated with the elimination of the resonant oscillating terms is obtained by prescribing that the right-hand side of this system be orthogonal to the oscillating elements of the null space of the adjoint operator $\mathbf{L}^\dagger$ (see Appendix A). The resulting condition reads

$$(C_1\partial_T + C_2\partial_X + C_3\partial_Y)B_{y_1} = 0, \tag{3.23}$$

where, defining

$$F = \frac{\zeta\omega k^2}{R_i\,(\,\omega^2 - k^2\cos^2\theta\,)},$$

the coefficients C_1, C_2 and C_3 are given by

$$C_1 = \frac{\omega^2 + \beta\,k^2}{(\omega^2 - \beta\,k^2)^2}k^2\sin\theta + \frac{\omega^2 + k^2\,\cos^2\theta}{\omega^2}(1 + F^2\cos^2\theta), \tag{3.24}$$

$$C_2 = \frac{2\,k\,\omega^3\sin^2\theta}{(\omega^2 - \beta\,k^2\,)^2} + 2\frac{k}{\omega}\left(1 + F^2\cos^2\theta + 2\zeta\frac{F\omega}{R_i}\right)\cos^2\theta, \tag{3.25}$$

$$C_3 = \frac{2\,k}{\omega}\left(-\frac{\beta\,k^2}{\omega^2 - \beta\,k^2} + F^2\cos^2\theta + \zeta\frac{F\omega}{R_i}\right)\cos\theta\sin\theta. \tag{3.26}$$

As usual, the vector $(C_2/C_1,\, C_3/C_1,\, 0)$ is identified with the group velocity $\mathbf{V}_g = \boldsymbol{\nabla}_k\omega$.

Under the condition (3.23), (3.16)–(3.22) are solved in the form

$$\mathbf{a}_2 = \mathbf{A}_2\,e^{i(kx-\omega t)} + \tilde{\mathbf{A}}_2\,e^{2i(kx-\omega t)} + \text{c.c.} + \bar{\mathbf{a}}_2, \tag{3.27}$$

where the components of $\mathbf{A}_2$ and $\tilde{\mathbf{A}}_2$ can easily be calculated using MAPLE software

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for symbolic calculation. The non-oscillating term $\bar{\mathbf{a}}_2$ will be determined at the next order of the expansion.

The condition associated with the elimination of the resonant oscillating terms at order ϵ^3 is obtained as at the previous order, and reads

$$\begin{aligned} i\ & (C_1\partial_T + C_2\partial_X + C_3\partial_Y)B_{y_2} \\ & +(C_4\partial_{TT} + C_5\partial_{TX} + C_6\partial_{TY} + C_7\partial_{XX} + C_8\partial_{XY} + C_9\partial_{YY} + C_{10}\partial_{YZ} + C_{11}\partial_{ZZ})B_{y_1} \\ & +(C_{12}\bar{\rho}_2 + C_{13}\bar{u}_{x_2} + C_{14}\bar{b}_{x_2} + C_{15}\bar{b}_{y_2} + C_{16}\bar{b}_{z_2} + C_{17}|B_{y_1}|^2)B_{y_1} = 0, \end{aligned} \tag{3.28}$$

where the coefficients $C_4, \ldots, C_{17}$ are given in Appendix B.

The equations derived up to now do not specify which type of wave is modulated, the Alfvénic or magnetosonic character being only prescribed by the choice of the solution $\omega_i(k)$ of the dispersion equation. In the following, we shall concentrate on the dispersionless limit $R_i \to \infty$, for which the present formalism remains meaningful when dealing with Alfvén waves, but not with magneto-sonic waves. Indeed, for these waves, the self-interaction coefficient C_{17} diverges at zero dispersion (see Appendix B).

In the dispersionless limit, the Alfvén wave is selected by prescribing that $\omega^2 - k^2\cos^2\theta$ vanish in (3.8), with the condition that $R_i^2(\omega^2 - k^2\cos^2\theta)$ remain finite. In this case, as can be seen from (3.2)–(3.7), the amplitude of the modulated wave can be rewritten in the form

$$\mathbf{A}_1 = \left(0, 0, 0, -\frac{k}{\omega}\cos\theta\, B_{z_1}, 0, 0, B_{z_1}\right)$$

where the only non-zero elements are the velocity and magnetic field components transverse to the plane defined by the ambient field and the direction of propagation. It is thus convenient in this limit, using (3.2)–(3.7), to rewrite the solvability condition at order ϵ^2 in terms of B_{z_1} in the form

$$(\partial_T + \cos\theta\,\partial_X + \sin\theta\,\partial_Y)B_{z_1} = 0. \tag{3.29}$$

In order to derive the solvability condition at order ϵ^3, we express the component B_y as

$$\begin{aligned} B_{y_2} = i\frac{R_i}{\omega k^2\cos\theta}(\omega^2 - k^2\cos^2\theta)B_{z_2} - \frac{R_i}{k^2\cos\theta}\Bigg[\frac{1}{\omega^2}(\omega^2 + k^2\cos^2\theta)\partial_T B_{z_1} \\ +2\frac{\omega}{k}\partial_X B_{z_1} + \frac{\sin\theta}{\omega\, k\cos\theta}(\omega^2 + k^2\cos^2\theta)\partial_Y B_{z_1} \\ +i\frac{R_i\sin\theta}{k\cos\theta}\frac{\omega^2 - k^2\cos^2\theta}{\omega^2 - \beta k^2}\partial_Z B_{z_1}\Bigg], \end{aligned} \tag{3.30}$$

where we retain terms containing $\omega^2 - k^2\cos^2\theta$ and take the limit $R_i \to \infty$ only after having substituted (3.30) into (3.28) and used (3.29). This limit can indeed be taken provided that $\theta \neq 0$, and leads to

$$i(\partial_T + \cos\theta\,\partial_X + \sin\theta\,\partial_Y)B_{z_2} - \epsilon k(\bar{u}_{x_2} + \tfrac{1}{2}\bar{b}_{x_2} - \tfrac{1}{2}\delta_2)B_{z_1} = 0, \tag{3.31}$$

where δ_i denotes the coefficient of ϵ^i in the expansion of $\delta = \bar{\rho}\cos\theta - \bar{b}_x$. This equation indicates that, to leading order, the amplitude is transported at the group velocity, which appears to be parallel to $\mathbf{b}_0$. Furthermore, the potential

$$V = k(\bar{u}_{x_2} + \tfrac{1}{2}\bar{b}_{x_2} - \tfrac{1}{2}\delta_2)$$

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can be rewritten as

$$V = -\frac{\delta\tilde{\omega}}{\delta\bar{\rho}}\bar{\rho} - \frac{\delta\tilde{\omega}}{\delta\bar{u}_x}\bar{u}_x - \frac{\delta\tilde{\omega}}{\delta\bar{b}_x}\bar{b}_x$$

in terms of the frequency Doppler shift

$$\tilde{\omega} = k\left\{\epsilon\bar{u}_x + \frac{B_{x_0} + \epsilon\bar{b}_x}{[4\pi(1+\epsilon\bar{\rho})]^{1/2}}\right\}.$$

The main point is that, in the dispersionless limit, the amplitude equation does not include diffraction terms, in contrast to the case of parallel propagation, where the diffraction remains finite in this limit (Champeaux *et al.* 1997*a*,*b*). In the absence of diffraction, two types of behaviour are then possible. If the nonlinear terms also vanish, the wave train is not affected by the modulation. In other words, the linear dynamics predicted by the long-wave asymptotics also holds at the level of a long-time, large-scale wave-train modulation. Alternatively, if the nonlinear effects remain relevant, their action is not limited by diffraction at the scale of the wave packet. As a consequence, much smaller scales are excited.

In order to estimate the efficiency of the nonlinear couplings, it is necessary to derive the equations for the mean fields. Elimination of the non-oscillating secular terms arising at order ϵ^3 leads to the system governing the mean field $\bar{\mathbf{a}}_2$ entering (3.27). In the non-dispersive limit, this system reads

$$\partial_T\bar{\rho}_2 + \partial_X\bar{u}_{x_2} + \partial_Y\bar{u}_{y_2} + \partial_Z\bar{u}_{z_2} = 0, \tag{3.32}$$

$$\partial_T\bar{u}_{x_2} + \beta\,\partial_X\bar{\rho}_2 + \sin\theta\,\partial_X\bar{b}_{y_2} - \sin\theta\,\partial_Y\bar{b}_{x_2} + \partial_X|B_{z_1}|^2 = 0, \tag{3.33}$$

$$\partial_T\bar{u}_{y_2} + \beta\,\partial_Y\bar{\rho}_2 - \cos\theta\,\partial_X\bar{b}_{y_2} + \cos\theta\,\partial_Y\bar{b}_{x_2} + \partial_Y|B_{z_1}|^2 = 0, \tag{3.34}$$

$$\partial_T\bar{u}_{z_2} + \beta\,\partial_Z\bar{\rho}_2 + \sin\theta\,\partial_Z\bar{b}_{y_2} + \cos\theta\,\partial_Z\bar{b}_{x_2} - \cos\theta\,\partial_X\bar{b}_{z_2} - \sin\theta\,\partial_Y\bar{b}_{z_2} + \partial_Z|B_z|^2 = 0, \tag{3.35}$$

$$\partial_T\bar{b}_{x_2} + \cos\theta\,\partial_Y\bar{u}_{y_2} - \sin\theta\,\partial_Y\bar{u}_{x_2} + \cos\theta\,\partial_Z\bar{u}_{z2} = 0, \tag{3.36}$$

$$\partial_T\bar{b}_{y_2} + \sin\theta\,\partial_X\bar{u}_{x_2} - \cos\theta\,\partial_X\bar{u}_{y_2} + \sin\theta\,\partial_Z\bar{u}_{z_2} = 0, \tag{3.37}$$

$$\partial_T\bar{b}_{z_2} - \cos\theta\,\partial_X\bar{u}_{z2} - \sin\theta\,\partial_Y\bar{u}_{z_2} = 0. \tag{3.38}$$

Note that (3.33), (3.34) and (3.38) degenerate for purely transverse modulation, where the fields become independent of the X and Y variables. The situation, which is analogous to that discussed in Champeaux *et al.* (1997*a*) in the case of parallel propagation of circularly polarized Alfvén waves, requires that the expansion be pushed to order ϵ^4.

The equations for the mean fields arising at order ϵ^4 are similar to those obtained at order ϵ^3, except for those governing the evolution of $\bar{u}_{x_3}$ and $\bar{u}_{y_3}$ which include additional terms. The equation for $\bar{u}_{y_3}$ includes a term of the form $\partial_Y(B_{z_1}\partial_Y B^*_{z_1} - B^*_{z_1}\partial_Y B_{z_1})$, while that for $\bar{u}_{x_3}$ contains a contribution proportional to $\partial_X(B_{z_1}\partial_Y B^*_{z_1} - B^*_{z_1}\partial_Y B_{z_1})$ and also a term

$$i\epsilon\frac{\omega^2\cos^2\theta}{k(\omega^2-\beta\,k^2)}(B^*_{z_1}\partial_{ZZ}B_{z_1} - B_{z_1}\partial_{ZZ}B^*_{z_1}).$$

A closed system of equations for each component of the amplitude $\epsilon\mathbf{A} = \epsilon(\mathbf{A}_1 + \epsilon\mathbf{A}_2)$ of the modulated carrying wave and of the mean field $\epsilon^2\bar{\mathbf{a}} = \epsilon^2(\bar{\mathbf{a}}_2 + \epsilon\bar{\mathbf{a}}_3)$, is obtained by combining the corresponding solvability conditions. Denoting by

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$\bar{\mathbf{u}} = (\bar{u}_x, \bar{u}_y, \bar{u}_z)$ and $\bar{\mathbf{b}} = (\bar{b}_x, \bar{b}_y, \bar{b}_z)$ the resulting mean velocity and magnetic field, we obtain to leading order

$$i(\partial_T + \mathbf{b}_0 \cdot \boldsymbol{\nabla})B_z - \epsilon k(\bar{u}_x + \tfrac{1}{2}\bar{b}_x - \tfrac{1}{2}\delta)B_z = 0, \tag{3.39}$$

$$\partial_T \bar{\rho} + \boldsymbol{\nabla} \cdot \bar{\mathbf{u}} = 0, \tag{3.40}$$

$$\partial_T \bar{\mathbf{u}} + \mathbf{b}_0 \times (\boldsymbol{\nabla} \times \bar{\mathbf{b}}) = \boldsymbol{\nabla}(|B_z|^2 - \beta\bar{\rho}) + 2\cos\theta\,(\partial_T + \mathbf{b}_0 \cdot \boldsymbol{\nabla})|B_z|^2 \mathbf{e}_x, \tag{3.41}$$

$$\partial_T \bar{\mathbf{b}} - \boldsymbol{\nabla} \times (\bar{\mathbf{u}} \times \mathbf{b}_0) = 0, \tag{3.42}$$

$$\partial_T \delta + \mathbf{b}_0 \cdot \boldsymbol{\nabla}\bar{\mathbf{u}} = 0, \tag{3.43}$$

$$\boldsymbol{\nabla} \cdot \bar{\mathbf{b}} = 0, \tag{3.44}$$

where $\mathbf{b}_0 = (\cos\theta, \sin\theta, 0)$ refers to the ambient magnetic field and $\mathbf{e}_x$ denotes the unit vector along the propagation direction of the wave fronts. In the equation for $\bar{\mathbf{u}}$, the terms arising at order ϵ^4 and including X and Y derivatives have been neglected because they are always smaller than analogous terms arising at previous order. In contrast, the contribution

$$i\epsilon \frac{\omega^2 \cos^2\theta}{k(\omega^2 - \beta k^2)}(B^*_{z_1}\partial_{ZZ}B_{z_1} - B_{z_1}\partial_{ZZ}B^*_{z_1}),$$

which remains relevant in the case of transverse modulation has been retained and, using the leading order of (3.39), rewritten in the form $-2\cos\theta\,(\partial_T + \cos\theta\,\partial_X + \sin\theta\,\partial_Y)|B_{z_1}|^2$.

4. Envelope dynamics

The nonlinear terms in (3.39) are easily calculated in three cases.

4.1. Longitudinal modulation

In the case of a purely longitudinal modulation, (3.40)–(3.44) can be integrated using the adiabatic approximation, leading to

$$\bar{\rho} = \bar{u}_x = \bar{b}_x = 0, \qquad \bar{u}_y = -\bar{b}_y = \frac{|B_z|^2}{\sin\theta}. \tag{4.1}$$

It follows that the potential vanishes in the amplitude equation, which thus describes a pure transport of the wave envelope. This shows that the results of Sec. 2 extend to the case of a purely longitudinal modulation of a wave train.

4.2. Multidimensional modulation

In the multidimensional problem, the modulation is usually viewed as either absolute or convective. The former refers to a time evolution in the whole spatial domain (Kuo *et al.* 1988), while the latter concerns the variation along the direction defined by the group velocity, at each instant of time (Kuo *et al.* 1989).

The absolute regime can be analysed from (3.39)–(3.44), but the convective one is more easily addressed in a reference frame where the direction of the group velocity (which can be identified with that of the ambient magnetic field) is used as a coordinate axis. We thus define $\xi = X\cos\theta + Y\sin\theta$ and $\eta = -X\sin\theta + Y\cos\theta$. Denoting by u_ξ, u_η and b_ξ, b_η the projections of the velocity $(\bar{u}_x, \bar{u}_y)$ and the magnetic field $(\bar{b}_x, \bar{b}_y)$ on the corresponding axes, we rewrite (3.39)–(3.44) in the forms

$$i(\partial_T B_z + \partial_\xi B_z) + \epsilon k(\bar{u}_x + \tfrac{1}{2}\bar{b}_x - \tfrac{1}{2}\delta)B_z = 0, \tag{4.2}$$

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$$\partial_T \bar{\rho} + \partial_\xi u_\xi + \partial_\eta u_\eta + \partial_Z \bar{u}_z = 0, \tag{4.3}$$

$$\partial_T u_\xi + \partial_\xi(\beta\bar{\rho} + |B_z|^2) = 2\cos^2\theta\,(\partial_T + \partial_\xi)|B_z|^2, \tag{4.4}$$

$$\partial_T u_\eta - \partial_\xi b_\eta + \partial_\eta(\beta\bar{\rho} + |B_z|^2 + b_\xi) = -2\cos\theta\sin\theta\,(\partial_T + \partial_\xi)|B_z|^2, \tag{4.5}$$

$$\partial_T \bar{u}_z - \partial_\xi b_z + \partial_Z(\beta\,\bar{\rho} + b_\xi + |B_z|^2) = 0, \tag{4.6}$$

$$\partial_{TT} b_\xi - \partial_{\xi\xi} b_\xi - (\partial_{ZZ} + \partial_{\eta\eta})(\beta\,\bar{\rho} + b_\xi + |B_z|^2) = 0, \tag{4.7}$$

$$\partial_{TT} b_\eta - \partial_{\xi\xi} b_\eta + \partial_{\eta\xi}(\beta\,\bar{\rho} + b_\xi + |B_z|^2) = 0, \tag{4.8}$$

$$\partial_T \delta + \partial_\xi(\cos\theta\, u_\xi - \sin\theta\, u_\eta) = 0, \tag{4.9}$$

where, in the potential of the nonlinear Schrödinger (NLS) equation (4.2), $\bar{u}_x = \cos\theta\, u_\xi - \sin\theta\, u_\eta$ and $\bar{b}_x = \cos\theta\, b_\xi - \sin\theta\, b_\eta$.

(i) *Transverse absolute modulation.* When the modulation is purely transverse to the plane defined by the ambient field and the direction of propagation ($\partial_\xi = \partial_\eta = 0$), the mean fields become slaved to the wave amplitude, and we have

$$\bar{\rho} = -\frac{1}{1+\beta}|B_z|^2, \tag{4.10}$$

$$u_\xi = 2\cos^2\theta|B_z|^2, \tag{4.11}$$

$$u_\eta = 0, \tag{4.12}$$

$$b_\xi = -\frac{1}{1+\beta}|B_z|^2, \tag{4.13}$$

$$b_\eta = 0. \tag{4.14}$$

After rescaling the time as $\tau = \epsilon T$, the amplitude equation reduces to

$$i\partial_\tau B_z + k\cos\theta\,\frac{3+4\beta}{2(1+\beta)} B_z|B_z|^2 = 0. \tag{4.15}$$

(ii) *Stationary convective modulation.* When a time-independent convective modulation is considered, the mean fields are given by

$$\bar{\rho} = \frac{2\cos^2\theta\, - 1}{\beta}|B_z|^2, \tag{4.16}$$

$$u_\eta = \frac{\cos\theta}{\sin\theta} u_\xi, \tag{4.17}$$

$$b_\xi = -2\cos^2\theta\,|B_z|^2, \tag{4.18}$$

$$b_\eta = 2\cos\theta\sin\theta\,|B_z|^2, \tag{4.19}$$

and, defining $\chi = \epsilon\xi$, the wave amplitude obeys

$$i\partial_\chi B_z - k\cos\theta\,\left[2 + \frac{1}{2\beta}(2\cos^2\theta - 1)\right] B_z|B_z|^2 = 0. \tag{4.20}$$

In both transverse absolute and convective cases, the nonlinear terms are thus relevant, but, because of the negligible diffraction, the multidimensional dynamics cannot be described in the framework of the usual large-scale modulation. This point is easily illustrated by considering the example of absolute modulation where the perturbation only takes place in the z direction, the fully multidimensional

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problem being still completely open. When retaining the diffraction term, whose coefficient

$$\alpha = C_{11}\frac{R_i^2(\omega^2 - k^2\cos^2\theta)^2}{k^4\omega^2\cos^2\theta}$$

vanishes like $1/R_i^2$, we have

$$i\partial_\tau B_z + \alpha\partial_{ZZ}B_z + k\cos\theta\,\frac{3+4\beta}{2(1+\beta)}|B_z|^2B_z = 0 \tag{4.21}$$

with

$$\alpha \simeq -\frac{k\cos\theta}{R_i^2\sin^4\theta}\left[\cos^4\theta + \beta(1-2\cos^2\theta)\right].$$

The limit $R_i \to \infty$ corresponds to the so-called semiclassical limit addressed by Bronski and McLaughlin (1994). It is easily seen that for $\beta < 1$ this NLS equation is of the focusing type. In this case, the most unstable mode appears at a scale of order $1/R_i$, which, if larger than the wavelength of the carrying wave, introduces an intermediate scale in the problem. Numerical simulations presented by Bronski and McLaughlin (1994) show that the solution starts to focus, leading to a finite-amplitude solitonic profile on which oscillations at the intermediate scale are superimposed. The phase gradient obeys, at large scale, an (inviscid) Burgers equation, corresponding to the geometrical-optics approximation for the linear Schrödinger equation. After a shock forms, the amplitude displays a well-delimited region of short-wavelength, finite-amplitude oscillations (referred to as a 'sea of solitons'), which extends as time elapses, while in Fourier space, the solution develops a broad energy spectrum, the precursor of a fully turbulent dynamics.

5. Conclusions

We have shown that a reductive perturbation expansion leads to non-canonical equations for the long-wave dynamics of obliquely propagating Alfvén waves. In particular, the nonlinearity disappears in the case of vanishing dispersion or when the scale of the transverse perturbation is too large. Nonlinear effects nevertheless survive in the dispersionless limit in the context of multidimensional modulation of Alfvén wave trains. The diffraction, however, vanishes in this limit, leading to dominantly unstable modes at a scale of order $1/R_i$. When this scale is intermediate between the size of the wave packet and its wavelength, the nonlinear dynamics leads, in the simplest case of purely transverse modulation, to the formation of shocks for the phase gradients at large scale, and to oscillations at the intermediate scale. This situation contrasts with the case of parallel propagation, where the diffraction retains a finite limit (Champeaux *et al.* 1997*a*,*b*), leading to wave packet collapse. It follows that for parallel propagation, the involved energies are larger than for the oblique case, and we can thus suspect that the dissipation processes will also act more efficiently.

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Appendix A

Linearization of the dispersive MHD equations (2.1)–(2.4) leads to the consideration of the eigenvalue problem

$$\mathbf{L}\mathbf{A}_1 = 0,$$

where the matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{pmatrix} -\omega & k & 0 & 0 & 0 & 0 & 0 \\ \beta k & -\omega & 0 & 0 & 0 & k\sin\theta & 0 \\ 0 & 0 & -\omega & 0 & 0 & -k\cos\theta & 0 \\ 0 & 0 & 0 & -\omega & 0 & 0 & -k\cos\theta \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & k\sin\theta & -k\cos\theta & 0 & 0 & -\omega & -i\dfrac{k^2\cos\theta}{R_i} \\ 0 & 0 & 0 & -k\cos\theta & 0 & \dfrac{i\,k^2\cos\theta}{R_i} & -\omega \end{pmatrix}. \quad \text{(A 1)}$$

Elimination of the resonant oscillating terms in the linear systems obtained at each order of the expansion requires a solvability condition involving the orthogonality between the amplitude of the first harmonics arising on the right-hand sides of these systems and the vector

$$\mathbf{A}_1^\dagger = \begin{pmatrix} \dfrac{\sin\theta\,k^2\beta}{\omega^2-\beta k^2} \\ \dfrac{\sin\theta\,k\,\omega}{\omega^2-\beta k^2} \\ -\dfrac{\cos\theta\,k}{\omega} \\ -\dfrac{i\cos^2\theta\,k^3}{R_i\,(\omega^2-k^2\cos^2\theta)} \\ 0 \\ 1 \\ \dfrac{i\cos\theta\,k^2\,\omega}{R_i\,(\omega^2-k^2\cos^2\theta)} \end{pmatrix} \quad \text{(A 2)}$$

belonging to the null space of the adjoint $\mathbf{L}^\dagger$ of $\mathbf{L}$. At order ϵ^2, this leads to the solvability condition

$$\begin{aligned} &-\frac{k^2\beta\sin\theta}{\omega^2-\beta k^2}\,(\partial_T\rho_1+\boldsymbol{\nabla}\cdot U_{x_1}) - \frac{k\omega\sin\theta}{\omega^2-\beta k^2}\,(\partial_T U_{x_1}+\beta\partial_X\rho_1+\sin\theta\,\partial_X B_{y_1}) \\ &+\frac{k\cos\theta}{\omega}\,(\partial_T U_{y_1}+\beta\partial_Y\rho_1-\cos\theta\,\partial_X B_{y_1}) \\ &-\frac{ik^3\cos^2\theta}{R_i\,(\omega^2-k^2\cos^2\theta)}\,[\partial_T U_{z_1}+\beta\partial_Z\rho_1-\cos\theta\,\partial_X B_{z_1}+\sin\theta\,(\partial_Z B_{y_1}-\partial_Y B_{z_1})] \\ &-\partial_T B_{y_1}-\sin\theta\,(\partial_Z U_{z_1}-\partial_X U_{x_1})+\cos\theta\,\partial_X U_{y_1}+\frac{ik}{R_i}(\sin\theta\,\partial_Y B_{z_1}+2\cos\theta\,\partial_X B_{z_1}) \\ &+\frac{ik^2\omega\cos\theta}{R_i\,(\omega^2-k^2\cos^2\theta)}\Big[\partial_T B_{z_1}-\sin\theta\,\partial_Y U_{z_1}-\cos\theta\,\partial_X U_{z_1} \\ &+\frac{ik}{R_i}(\sin\theta\,\partial_Y B_{y_1}+2\cos\theta\,\partial_X B_{y_1})\Big] \\ =\;&0. \end{aligned} \quad \text{(A 3)}$$

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Equation (3.23) is obtained by substituting in the above condition the explicit forms (3.2)–(3.7) of the components of $\mathbf{A}_1$.

Appendix B

The coefficients $C_4, \ldots, C_{17}$ appearing in (3.28) are given by

$$C_4 = \frac{k^4\omega}{{R_i}^2 D_2^3}(\omega^2 + 3k^2\cos^2\theta)\cos^2\theta + \frac{k^2}{\omega^3}\cos^2\theta + \frac{k^2\omega}{D_1^3}(\omega^2 + 3\beta k^2)\sin^2\theta,$$

$$C_5 = 2\frac{k}{\omega^2}\cos^2\theta + \frac{k^5}{{R_i}^2 D_2^3}(5\omega^2 + 3k^2\cos^2\theta)\cos^4\theta + \frac{k^3}{{R_i}^2 D_2^2}(4\omega^2 + 5k^2\cos^2\theta)\cos^2\theta$$
$$+\frac{k}{D_1^3}(\beta^2k^4 + 6\beta k^2\omega^2 + \omega^4)\sin^2\theta + \frac{k\omega^2}{D_1^2}(\omega^2 + \beta k^2)\sin^2\theta,$$

$$C_6 = \frac{k^5}{{R_i}^2 D_1^3}(5\omega^2 + 3k^2\cos^2\theta)\cos^3\theta\sin\theta + \frac{k^3}{{R_i}^2 D_1^2}(2\omega^2 + 3k^2\cos^2\theta)\cos\theta\sin\theta$$
$$- 4\frac{k^3\beta}{D_1^2}\sin\theta\cos\theta - 2\frac{k^3\beta}{\omega^2 D_1}\cos\theta\sin\theta,$$

$$C_7 = 6\frac{k^2\omega}{R_i^2 D_2}\cos^2\theta + 9\frac{k^4\omega}{R_i^2 D_2^2}\cos^4\theta + \frac{\omega}{D_1^2}(\omega^2 + \beta k^2)\sin^2\theta$$
$$+ \frac{\beta k^2\omega}{D_1^3}(3\omega^2 + \beta k^2)\sin^2\theta + 4\frac{k^6\omega}{R_i^2 D_2^3}\cos^6\theta + \frac{1}{\omega}\cos^2\theta,$$

$$C_8 = -3\frac{\beta k^2}{\omega D_1}\sin\theta\cos\theta - \frac{\beta^2 k^4}{\omega D_1^2}\sin\theta\cos\theta - 3\frac{\beta k^2\omega}{D_1^2}\sin\theta\cos\theta + 8\frac{k^6\omega}{R_i^2 D_2^3}\cos^5\theta\sin\theta$$
$$+ 14\frac{k^4\omega}{R_i^2 D_2^2}\cos^3\theta\sin\theta + 6\frac{k^2}{R_i^2 D_2}\cos\theta\sin\theta,$$

$$C_9 = \frac{\beta^2 k^4}{\omega D_1^2}\sin^2\theta + 4\frac{\omega k^6}{R_i^2 D_2^3}\cos^4\theta\sin^2\theta + 5\frac{\omega k^4}{R_i^2 D_2^2}\cos^2\theta\sin^2\theta + \frac{\beta k^2}{\omega D_1}\cos^2\theta$$
$$+ \frac{k^2\omega}{R_i^2 D_2} + \frac{\cos^2\theta}{\omega} + \frac{\omega}{D_1}\sin^2\theta,$$

$$C_{10} = \frac{k^2\sin^2\theta\cos\theta}{R_i D_1 D_2^2}[k^2\cos^2\theta(2\beta k^2 - \omega^2) - \omega^4]\sin^2\theta\cos\theta - \frac{k^4\omega^2}{R_i^3 D_2^2}\cos^3\theta + \frac{\cos\theta}{R_i}$$
$$+ 2\frac{k^4}{R_i D_2^2}\sin\theta^2\cos^3\theta,$$

$$C_{11} = \frac{k^2\omega}{R_i^2 D_2}\cos^2\theta + \frac{\beta k^6\omega}{R_i^2 D_2^2 D_1}\cos^4\theta + \frac{k^4\omega}{R_i^2 D_2^2}\cos^4\theta + \frac{\omega^3}{D_2 D_1}\sin^2\theta$$
$$+ \frac{\beta k^2\omega^3}{D_1^2 D_2}\sin^2\theta,$$

$$C_{12} = \frac{k^2}{\omega}\cos^2\theta + \frac{k^2\omega}{D_1^2}(\omega^2 - \beta\gamma k^2)\sin^2\theta + 2\frac{k^4\omega}{R_i^2 D_2}\cos^2\theta + \frac{k^6\omega}{R_i^2 D_2^2}\cos^4\theta,$$

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$$
\begin{aligned}
C_{13} &= -\frac{k^3}{D_1^2}(\omega^2+\beta\,k^2)\sin^2\theta - \frac{k^5}{R_i^2 D^2}(\omega^2+k^2\cos^2\theta)\cos^2\theta - \frac{k^3}{\omega^2}\cos^2\theta - k,\\
C_{14} &= -2\frac{k^4\omega}{R_i^2\,D_2}\cos\theta - 2\,\frac{k^6\omega}{R_i^2 D_2^2}\cos^3\theta - 2\,\frac{k^2}{\omega}\cos\theta,\\
C_{15} &= -2\frac{k^2\omega\sin\theta}{D_1}\\
C_{16} &= 2\frac{k^4\omega^2}{R_i\,D_1\,D_2}\sin\theta\cos\theta,\\
C_{17} &= 3\frac{k^6\,\omega^2}{R_i^2\,D_2\,D_3\,D_1}\sin^2\theta\cos^2\theta + \frac{15}{2}\,\frac{k^{10}\,\omega^6}{R_i^4 D_1\,D_2^4\,D_3}\sin^2\theta\cos^4\theta\\
&\quad -13\frac{k^{10}\omega^4}{R_i^4 D_1\,D_2^3\,D_3}\sin^2\theta\cos^4\theta - \frac{k^8\,\omega^2\,\beta}{R_i\,D_1^3 D_2}[\omega^2-\tfrac{1}{2}\beta\,k^2(\gamma+1)]\sin^2\theta\cos^2\theta\\
&\quad +\frac{1}{2}\,\frac{k^{10}\omega^3\beta}{R_i^2\,D_2^2\,D_1^3}(\gamma+1)\sin^2\theta\cos^2\theta + 2\frac{k^8\,\omega^2\beta}{R_i^2\,D_2\,D_3\,D_1^2}(\gamma+1)\sin^4\theta\cos^2\theta\\
&\quad -5\frac{\beta\,k^{10}\,\omega^4}{R_i^2 D_1^3\,D_3\,D_2^2}[\omega^2-\tfrac{1}{2}\beta\,k^2(\gamma+1)]\sin^4\theta\cos^2\theta + 4\frac{k^{10}\,\omega^2}{R_i^4\,D_2^2\,D_3\,D_1}\sin^2\theta\,\cos^4\theta\\
&\quad -\frac{1}{2}\frac{k^2\,\omega}{D_1} + \frac{k^{10}\,\omega^2}{R_i^2 D_1^3\,D_3\,D_2}[\omega^2-\tfrac{1}{2}\beta\,k^2(\gamma+1)]\sin^4\theta\cos^2\theta\\
&\quad -\frac{k^{10}\,\omega\,\beta}{R_i^2\,D_2\,D_1^3}(\gamma+1)\sin^2\theta\cos^2\theta - \frac{9}{2}\,\frac{k^2\omega^2}{D_3\,D_1}\sin^2\theta - 2\,\frac{k^8\,\omega^5}{R_i^2\,D_1^2 D_2^3}\sin^2\theta\cos^2\theta\\
&\quad +3\frac{k^6\,\omega^4}{R_i^2 D_1\,D_2^2\,D_3}\sin^2\theta\cos^2\theta - \frac{3}{2}\,\frac{k^8\,\omega^4\beta}{R_i^2\,D_2^2\,D_3\,D_1^2}(\gamma+1)\sin^4\theta\cos^2\theta\\
&\quad +\frac{1}{2}\,\frac{k^4\,\omega^2}{R_i\,D_1\,D_2}\cos^2\theta - 6\frac{k^8\,\omega}{R_i^2\,D_2\,D_1^2}\cos^2\theta\sin^2\theta + \frac{k^8\omega\,\beta}{D_1^4}[\tfrac{1}{2}\gamma(5-\gamma)-1]\sin^4\theta\\
&\quad -\frac{3}{2}\,\frac{k^4\beta\,\omega^2}{D_3\,D_1^2}(\gamma+1)\sin^4\theta - \frac{k^6\omega^2\beta}{D_3\,D_1^4}(\gamma+1)[\omega^2-\tfrac{1}{2}\beta\,k^2(\gamma+1)]\sin^6\theta\\
&\quad -\frac{1}{2}\,\frac{\cos^4\theta\,k^8\,\omega^4}{R_i^3\,D_2^3\,D_1} - \frac{\cos^2\theta\,k^6\,\omega}{R_i^2\,D_2\,D_1} + \frac{k^4\,\omega}{D_1^3}[\omega^2-\beta\,k^2(\gamma+1)]\sin^2\theta\\
&\quad +3\frac{k^4\,\omega^2}{D_1^3\,D_3}[\omega^2-\tfrac{1}{2}\beta\,k^2(\gamma+1)]\sin^4\theta + \frac{\cos^4\theta\,k^8\,\omega^2}{R_i^3\,D_2^2\,D_1}\\
&\quad +\frac{k^8\,\omega\,\beta}{D_1^5}(\gamma+1)\,[\omega^2-\tfrac{1}{2}\beta\,k^2(\gamma+1)]\sin^4\theta + \frac{1}{2}\,\frac{k^6\,\omega^3}{R_i^2\,D_2^2\,D_1}\cos^2\theta,
\end{aligned}
$$

where we used the notations

$$
\begin{aligned}
D_1 &= \omega^2-\beta\,k^2, \qquad D_2 = \omega^2-k^2\cos^2\theta,\\
D_3 &= \frac{\omega^3}{k^2} - \omega\cos^2\theta + 4\frac{\omega^5\,k^2\cos^2\theta}{R_i^2\,D_2^2} - 4\frac{\omega^3\,k^4\cos^4\theta}{R_i^2\,D_2^2} - \omega\,\beta\\
&\quad +\frac{k^2\,\beta\cos^2\theta}{\omega} - 4\frac{\omega^3\,k^4\,\beta\cos^2\theta}{R_i^2\,D_2^2} + 4\,\frac{\omega\,k^6\,\beta\cos^4\theta}{R_i^2\,D_2^2} - \omega\sin^2\theta,
\end{aligned}
$$

and isolated the different powers of R_i and D_2 to make the limit $R_i\to\infty$, $D_2\to 0$, with $R_i^2 D_2$ finite, more conspicuous.

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References

Brodin, G. 1996 Two-dimensional shear Alfvén-wave turbulence in a plasma with arbitrary β. *J. Plasma Phys.* **55**, 121–130.

Bronski, J. C. and McLaughlin, D. W. 1994 Semi-classical behavior in the NLS equation: optical shocks – focusing instabilities. *Singular Limits of Dispersive Waves* (ed. N. M. Ercolani, I. R. Gabitov, C. D. Levermore and D. Serre), pp. 21–38. Plenum Press, New York.

Champeaux, S., Passot, T. and Sulem, P. L. 1997*a* Alfvén-wave filamentation. *J. Plasma Phys.* **58**, 665–690.

Champeaux, S., Gazol, A., Passot, T. and Sulem, P. L. 1997*b* Plasma heating by Alfvén wave filamentation: a relevant mechanism in the solar corona and the interstellar medium. *Astrophys. J.* **486**, 477–483.

Kakutani, T. and Ono, H. 1969 Weak nonlinear hydromagnetic waves in a cold collision-free plasma. *J. Phys. Soc. Japan* **26**, 1305–1318.

Kakutani, T., Ono, H., Taniuti, T. and Wei, C. 1968 Reductive perturbation method in nonlinear wave propagation II. Application to hydrodynamic waves in cold plasma. *J. Phys. Soc. Japan*, **24**, 1159–1166.

Kaup, D. J. and Newell, A. C. 1978 An exact solution for a derivative nonlinear Schrödinger equation *J. Math. Phys.* **19**, 798–801.

Kennel, C. F., Buti, B., Hada, T. and Pellat, R. 1988 Nonlinear, dispersive, elliptically polarized Alfvén waves. *Phys. Fluids* **31**, 1949–1961.

Kuo, S. PP., Wang, M. H. and Lee, M. C. 1988 Filamentation instability of large-amplitude Alfvén waves. *J. Geophys. Res.* **93**, 9621–9627.

Kuo, S. PP., Wang, M. H. and Schmidt, G. 1989 Convective filamentation instability of circularly polarized Alfvén waves. *Phys. Fluids* **B4**, 734–740.

Mio, K., Ogino, T., Minami, K. and Takeda, S. 1975 Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas. *J. Phys. Soc. Japan* **41**, 265–271.

Mio, K., Ogino, T. and Takeda, S. 1976 A perturbation method and its application to obliquely propagating nonlinear Alfvén wave. *J. Phys. Soc. Japan*, **41**, 2114–2120.

Mjølhus, E. 1976 On the modulational instability of hydromagnetic waves parallel to the magnetic field. *J. Plasma Phys.* **16**, 321–334.

Mjølhus, E. 1989 Nonlinear Alfvén waves and the DNLS equation: oblique aspects. *Physica Scripta* **40**, 227–237.

Mjølhus, E. and Wyller, J. 1986 Alfvén solitons. *Physica Scripta* **33**, 442–451.

Mjølhus, E. and Wyller, J. 1988 Nonlinear Alfvén waves in a finite-beta plasma. *J. Plasma Phys.* **40**, 299–318.

Passot, T. and Sulem, P. L. 1993 Multidimensional modulation of Alfvén waves. *Phys. Rev.* **E48**, 2966–2974.

Passot, T. and Sulem, P. L. 1995 Nonlinear dynamics of dispersive Alfvén waves. *Small-Scale Structures in Three-Dimensional Hydrodynamic and Magnetohydrodynamic Turbulence.* (ed. M. Meneguzzi, A. Pouquet and P. L. Sulem), pp. 405–410, Lecture Notes in Physics, Vol. 462, Springer-Verlag, Berlin.

Passot, T., Sulem, C. and Sulem, P. L. 1994 Effect of longitudinal modulational on Alfvén wave filamentation. *Phys. Rev.* **E50**, 1427–1436.

Pokhotelov, O. A., Pokhotelov, D. O., Gokhberg, M. B., Feygin, F. Z., Stenflo, L. and Shukla, P. K. 1996 Alfvén solitons in the Earth's ionosphere and magnetosphere *J. Geophys. Res.* **101**, 7913–7915.

Rogister, A. 1971 Parallel propagation of nonlinear low-frequency waves in high-β plasmas. *Phys. Fluids* **14**, 2733–2739.

Shukla, P. K. and Stenflo, L. 1989 Filamentation instability of Alfvén waves. *Astrophys. Space Sci.* **155**, 145–147.

Spangler, S. R. and Plapp, B. B. 1992 Characteristics of obliquely propagating, nonlinear Alfvén waves *Phys. Fluids* **B4**, 3356–3370.

Spangler, S. R. and Sheerin, J. P. 1983 Alfvén wave collapse and the stability of a relativistic electron beam in a magnetized astrophysical plasma *Astrophys. J.* **272**, 273–278.

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