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Classical motion in random potentials

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Abstract. We consider the motion of a classical particle under the influence of a random potential on $\mathbb{R}^d$, in particular the distribution of asymptotic velocities and the question of ergodicity of time evolution.

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1. *Introduction*

Since its introduction to physics by Einstein and its mathematical foundation by Wiener, Brownian motion has been considered one cornerstone of probability and of thermodynamics. It may thus seem natural to expect that classical motion in a spatially homogeneous force field leads in the large-time limit to deterministic diffusion. Such results can in fact be proven for a periodic Lorentz gas with finite horizon (related to the Sinai billiard) or for Coulombic periodic potentials; see [**BS81**] and [**DL91, Kna87**], respectively. However, in both cases the dynamics is non-smooth, of billiard type for

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the Lorentz gas and with orbits locally approximating Keplerian conic sections for the second case. We show here (Theorem 8.4) that the motion in a bounded smooth potential is incompatible with uniform hyperbolicity.

If periodic scatterers can lead to diffusive motion, this should be even more true for random scatterers. However, this is certainly not the case in one dimension and also wrong for more degrees of freedom and Poisson potentials (§§6 and 7). In this and other cases the Hamiltonian flow is not even ergodic (Theorem 7.1).

For the random Coulomb case, however, we show in Theorem 9.5 that the flow is typically topologically transitive for large energies. Related results on the motion in random configurations of convex scatterers have been derived by Marco Lenci and collaborators. In [**Len03**], the planar situation with a finite modification of periodic scatterers was studied. In [**CLS10**] (see also the references in that article) recurrence for particles in quenched tubes with random scatterers has been proven.

The literature on the corresponding quantum problem of Schrödinger operators with random potentials is much broader; see e.g. [**LMW03**] and [**Ves08**] with its extensive references. In the article [**RF11**] quantum diffusion in a thermal medium has been proven.

The structure of the paper is as follows. Random potentials arise in different guises. The one studied most extensively is based on a regular lattice $\mathcal{L}$ in configuration space. If J indexes the single-site potentials, one assumes a measure on the space $J^{\mathcal{L}}$, for which the $\mathcal{L}$-action is ergodic. This is studied in §2. One result is an almost deterministic distribution of asymptotic velocities (§3). One problem concerning the Liouville measure on the energy surface is discussed in §5. Depending on the exact exponential decay rate, the set of singular energy values may or may not be typically dense. Poissonian random potentials are studied in §§4 and 7. Relationships between different notions of ergodicity and some of their dynamical implications are discussed in §6. Section 8 concerns deterministic potentials, and the geometry compatible with a uniformly hyperbolic structure. Finally, we treat random Coulombic potentials in §§9 and 10, showing topological transitivity and ergodicity of the compactified flow for energy surfaces of large energy.

2. *The lattice case*

We assume the random potential to be based on *short-range* single-site potentials $W_j \in C^\eta(\mathbb{R}^d, \mathbb{R})$, indexed by $j \in J$, $|J| \in \mathbb{N}$, $|J| \geq 2$. Namely $\eta \in \mathbb{N} \cup \{\infty\}$, $\eta \geq 2$, and

$$|\partial^\alpha W_j(q)| \leq \frac{C_\alpha}{\langle q\rangle^{d+\varepsilon}} \quad (q \in \mathbb{R}^d, \alpha \in \mathbb{N}_0^d, |\alpha| \leq \eta) \tag{2.1}$$

with $\langle q\rangle := \sqrt{1 + \|q\|^2}$, for constants $C_\alpha > 0$.

The single-site potentials are placed on a regular lattice $\mathcal{L} \subseteq \mathbb{R}^d$ with basis $\ell_1, \ldots, \ell_d \in \mathbb{R}^d$ according to $\omega \in \Omega := J^{\mathcal{L}}$ to give the random potential

$$V : M := \Omega \times \mathbb{R}^d \to \mathbb{R}, \quad V(\omega, q) := \sum_{\ell\in\mathcal{L}} W_{\omega(\ell)}(q - \ell)$$

on *extended configuration space* M. We use the discrete topology on J and product topology on Ω. The probability measure β on $(\Omega, \mathcal{B}(\Omega))$ is assumed to be invariant with respect to the action

$$\vartheta : \mathcal{L} \times \Omega \to \Omega, \quad (\ell, \omega) \mapsto \vartheta_\ell(\omega) \quad \text{with } \vartheta_\ell(\omega)(\ell') := \omega(\ell' + \ell). \tag{2.2}$$

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Unless we explicitly say the contrary, β is assumed to be ϑ-ergodic (a simple example being a product measure $\beta = \bigotimes_{\ell\in\mathcal{L}} \hat{\beta}$ with a probability measure $\hat{\beta}$ on J). Ergodicity and $|J| \geq 2$ imply that β is non-atomic. The short-range conditions above imply that for all $\omega \in \Omega$ the potentials

$$V_\omega \colon \mathbb{R}^d \to \mathbb{R}, \quad V_\omega(q) := V(\omega, q),$$

are as smooth as the single-site potentials W_j. Moreover V itself is continuous and bounded, together with its partial derivatives $\partial^\alpha V$, $|\alpha| \leq \eta$.

Remark 2.1. More relevant than the *range*

$$[V_{\min}, V_{\max}] := \overline{V(M)}$$

of the potential is its *essential range* (λ^d denoting Lebesgue measure on $\mathbb{R}^d$):

$$\operatorname{supp}(V(\beta \otimes \lambda^d)) = [V_{\text{ess min}}, V_{\text{ess max}}].$$

By ϑ-ergodicity of β, this is a deterministic set:

$$\operatorname{supp}(V_\omega(\lambda^d)) = \operatorname{supp}(V(\beta \otimes \lambda^d)) \quad (\beta\text{-almost surely}).$$

As V is uniformly bounded together with its first derivatives, this is in fact a bounded interval: $\operatorname{supp}(V_\omega(\lambda^d)) = \overline{V_\omega(\mathbb{R}^d)}$. In general the essential range of V is a proper subinterval of its range.

Thus the flow $\Phi \colon \mathbb{R} \times P \to P$ generated by the Hamiltonian function

$$H \colon P \to \mathbb{R}, \quad H(\omega, p, q) := \tfrac{1}{2}\|p\|^2 + V(\omega, q),$$

on *extended phase space*

$$P := \Omega \times \mathbb{R}^d \times \mathbb{R}^d$$

uniquely exists for all times. We write

$$\Phi^t \colon P \to P, \quad \Phi^t(\omega, p_0, q_0) := \Phi(t, \omega, p_0, q_0) = (\omega, p^t(\omega, p_0, q_0), q^t(\omega, p_0, q_0)),$$

for the solution of the initial value problem at time $t \in \mathbb{R}$. Whenever a fixed $\omega \in \Omega$ is considered, we write it as a subscript (e.g. $\Phi^t_\omega \equiv (p^t_\omega, q^t_\omega) \colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$). See Figure 1 for a realization of $t \mapsto q_\omega(t, x_0)$, with lattice $\mathcal{L} = \mathbb{Z}^2$.

The space P is equipped with the locally finite Borel measure $\mu := \beta \otimes \lambda^{2d}$. The Hamiltonian flow Φ leaves μ invariant.

The lattice acts on extended phase space via the continuous group action

$$\Theta \colon \mathcal{L} \times P \to P, \quad (\ell, \omega, p, q) \mapsto \Theta_\ell(\omega, p, q) := (\vartheta_\ell(\omega), p, q - \ell),$$

and leaves μ invariant:

$$\mu \circ \Theta_\ell = \mu \quad (\ell \in \mathcal{L}). \tag{2.3}$$

Similarly for all lattice vectors $\ell \in \mathcal{L}$:

$$H \circ \Theta_\ell = H \quad \text{and thus} \quad \Theta_\ell \circ \Phi^t = \Phi^t \circ \Theta_\ell \quad (t \in \mathbb{R}). \tag{2.4}$$

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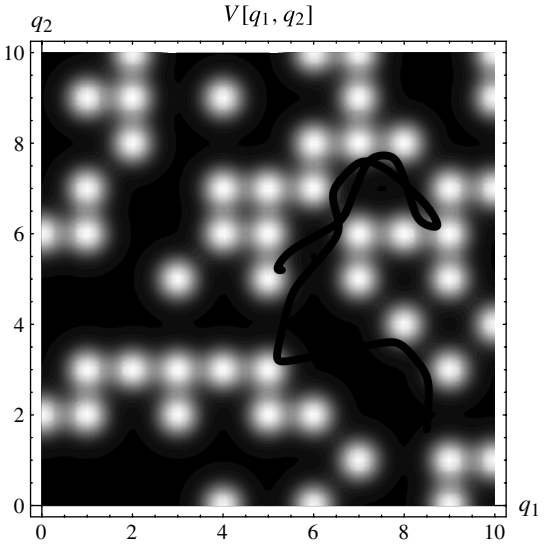


FIGURE 1. Path in configuration space.

By the Hamiltonian character of the motion, the energy is invariant under the time evolution:

$$H \circ \Phi^t = H \quad (t \in \mathbb{R}). \tag{2.5}$$

Thus we obtain a one-parameter family of Borel measures μ_E on P, given by

$$\mu_E(B) := \mu(B \cap H^{-1}(-\infty, E]) \quad (B \in \mathcal{B}(P)), \tag{2.6}$$

parametrized by the energy $E \in \mathbb{R}$. By (2.3)–(2.5) these measures μ_E are Θ- and Φ-invariant, too.

A relevant quantity is the *asymptotic* velocity

$$\overline{v}^{\pm}\colon P \to \mathbb{R}^d, \quad \overline{v}^{\pm}(\omega, x_0) := \lim_{T \to \pm\infty} \frac{q_\omega(T, x_0)}{T}. \tag{2.7}$$

Remark 2.2. (Non-existence of asymptotic velocity) For Hamiltonian motion in bounded potentials the asymptotic velocity may not exist for *any* initial condition $x_0 \in H^{-1}(E)$ for *all* energies E above some threshold. This is the case for centrally symmetric potentials ($V(q) = \tilde{V}(\|q\|)$), with $\tilde{V}$ being periodic in a slowly varying function like, for example, $\tilde{V}(r) = \cos(\log(r + 1))$.

It is also possible to construct examples of truly random potentials where for some $\omega \in \Omega$ these limits do not exist on $H_\omega^{-1}(E)$ for any $E > E_0$.

The lattice $\mathcal{L} := \mathbb{Z}^d$ with fundamental domain $\mathcal{D} = [0, 1]^d$ admits an adapted partition of unity F_ℓ ($\ell \in \mathcal{L}$)

$$\sum_{\ell \in \mathcal{L}} F_\ell(q) = 1 \quad (q \in \mathbb{R}^d),$$

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with $F_\ell(q) := F(q-\ell)$ for $F := \mathbb{1}_{\mathcal{D}} * f$ with some $f \in C_c^\infty(\mathbb{R}^d, [0,\infty))$ and $\int_{\mathbb{R}^d} f\,dx = 1$.

We now take $J := \{0, 1, \dots, d\}$ as index set for the single-site potentials $W_j := jF$. We choose $\omega \in \Omega = J^{(\mathbb{Z}^d)}$ of the form $\omega(\ell) = \sum_{k=1}^d \widetilde{\omega}_k(\ell_k)$, with $\widetilde{\omega}_k : \mathbb{Z} \to \{0, 1\}$ slowly varying like above.

Then the motion in the realization V_ω is separable in Cartesian coordinates, and thus for no $E > E_0 := d$ and initial condition $x_0 \in H_\omega^{-1}(E)$ do the asymptotic velocities $\overline{v}^{\pm}(x_0)$ exist.

By allowing for arbitrary modifications of ω over finite subsets of $\mathbb{Z}^d$, the set of these $\widehat{\omega}$ is even dense in Ω, and asymptotic velocity does not exist for any initial condition $x_0 \in H_{\widehat{\omega}}^{-1}(E)$ (possibly except for dimension $d > 1$ and a set of finite Liouville measure, for which $\overline{v}^{\pm}(\widehat{\omega}, x_0) = 0$).

In case of non-existence of the limit, we will set $\overline{v}^{\pm}(\omega, x_0) := 0$.

However, typically the above limit exists. This is shown below by invoking Birkhoff's ergodic theorem. As this deals with finite measures, we have to change our measure space. Namely, we set

$$\hat{P} := P/\Theta.$$

As $P = \bigcup_{\ell\in\mathcal{L}} \Theta_\ell(\Omega \times \mathbb{R}_p^d \times \mathcal{D})$ with *fundamental domain* of $\mathcal{L}$

$$\mathcal{D} := \left\{ \sum_{k=1}^d x_k \ell_k \;\middle|\; x_k \in [0,1] \right\} \subseteq \mathbb{R}_q^d,$$

the topological space $\hat{P}$ is homeomorphic to

$$(\Omega \times \mathbb{R}^d \times \mathcal{D})/\!\sim, \tag{2.8}$$

with $\sim$ identifying points in $\Omega \times \mathbb{R}_p^d \times \partial\mathcal{D}$ via the Θ-action. Using (2.3) and (2.4), the covering projection

$$\hat{\pi} : P \to \hat{P}$$

allows us to induce measures $\hat{\mu}$ and $\hat{\mu}_E$ on $\hat{P}$ by setting

$$\hat{\mu}(\hat{B}) := \mu(\hat{\pi}^{-1}(\hat{B}) \cap (\Omega \times \mathbb{R}_p^d \times \mathcal{D})) \quad (\hat{B} \in \mathcal{B}(\hat{P})) \tag{2.9}$$

and similarly for $\hat{\mu}_E$ (note that the Lebesgue measure satisfies $\lambda^d(\partial\mathcal{D}) = 0$).

Furthermore (2.4) allows us to define the continuous energy function $\hat{H} : \hat{P} \to \mathbb{R}$ and the flow $\hat{\Phi} : \mathbb{R} \times \hat{P} \to \hat{P}$ uniquely by $H = \hat{H} \circ \hat{\pi}$ and $\hat{\Phi}^t \circ \hat{\pi} = \hat{\pi} \circ \Phi^t$. Finally, by Θ-invariance of the momenta $p : P \to \mathbb{R}^d$, they descend to

$$\hat{p} : \hat{P} \to \mathbb{R}^d, \quad \hat{p} \circ \hat{\pi} = p. \tag{2.10}$$

PROPOSITION 2.3. *The asymptotic velocities (2.7) exist μ-almost everywhere on P, and $\overline{v}^+ = \overline{v}^-$ μ-almost everywhere. Setting $\overline{v}(x) := \overline{v}^{\pm}(x)$ in the case of equality and $\overline{v}(x) = 0$ otherwise,*

$$\overline{v} \in L^\infty_{\mathrm{loc}}(P, \mu), \quad \overline{v} \circ \Phi^t = \overline{v} \quad \textit{and} \quad \overline{v} \circ \Theta_\ell = \overline{v} \quad (t \in \mathbb{R}, \ell \in \mathcal{L}).$$

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Proof. For initial conditions $x_0 = (p_0, q_0) \in \mathbb{R}^d_p \times \mathbb{R}^d_q$ we have

$$\overline{v}^{\pm}(\omega, x_0) = \lim_{T\to\pm\infty} \frac{q_\omega(T, x_0) - q_0}{T} = \lim_{T\to\pm\infty} \frac{1}{T} \int_0^T p_\omega(t, x_0)\, dt$$
$$= \lim_{T\to\pm\infty} \frac{1}{T} \int_0^T \hat{p}(t, \hat{\pi}(\omega, x_0))\, dt.$$

But as the finite measures $\hat{\mu}_E$ on $\hat{P}$ are $\hat{\Phi}$-invariant, by Birkhoff's ergodic theorem the limits $\lim_{T\to\pm\infty}(1/T)\int_0^T \hat{p}(t, \hat{x}_0)\, dt$ exist, coincide and are $\hat{\Phi}$-invariant $\hat{\mu}_E$-almost everywhere. Moreover the limit function lies in $L^1(\hat{P}, \hat{\mu}_E)$ and is bounded by $\sqrt{2(E - V_{\min})}$ in absolute value, as $\|\hat{p}(t, \hat{x}_0)\|^2 \le 2(\hat{H}(\hat{x}_0) - V_{\min})$. Thus it is in $L^\infty(\hat{P}, \hat{\mu}_E)$.

The measure $\hat{\mu}$ is non-finite, but $\hat{\mu}\restriction_{\hat{H}^{-1}((-\infty, E])} = \hat{\mu}_E$. Thus we obtain the result. □

COROLLARY 2.4. *For β-almost every (a.e.) $\omega \in \Omega$ the asymptotic velocities $\overline{v}^{\pm}_\omega \colon \mathbb{R}^d_p \times \mathbb{R}^d_q \to \mathbb{R}^d$ exist and are equal λ^{2d}-almost everywhere.*

Proof. This follows from Proposition 2.3 and Fubini's theorem, applied to the measure $\mu = \beta \otimes \lambda^{2d}$ on extended phase space P. □

Similarly, for all regular energies E (see §6 below), the asymptotic velocities exist Liouville-almost everywhere on $H_\omega^{-1}(E)$.

3. *Distribution of asymptotic velocities*

Next we consider the joint distribution of energy and asymptotic velocity, using the measurable maps

$$\Gamma_\omega := (H_\omega, \overline{v}_\omega) \colon \mathbb{R}^d_p \times \mathbb{R}^d_q \longrightarrow \mathbb{R} \times \mathbb{R}^d \quad (\omega \in \Omega).$$

We thus consider phase space regions $\mathbb{R}^d_p \times Q_n \subseteq \mathbb{R}^d_p \times \mathbb{R}^d_q$ with

$$Q_n := \left\{ \sum_{k=1}^d x_k \ell_k \;\middle|\; \forall k \in \{1, \dots, d\} \colon -n \le x_k < n \right\} \quad (n \in \mathbb{N}),$$

and the normalized restrictions

$$\lambda_n := \frac{1}{\lambda^d(Q_n)} \lambda^{2d}\restriction_{\mathbb{R}^d_p \times Q_n} = (2n)^{-2d}\, \lambda^{2d}\restriction_{\mathbb{R}^d_p \times Q_n}$$

of Lebesgue measure to these regions.

PROPOSITION 3.1. *For β-a.e. $\omega \in \Omega$ the energy–velocity distribution*

$$\nu_\omega := \lim_{n\to\infty} \Gamma_\omega(\lambda_n)$$

exists in the sense of vague convergence and is independent of ω.

Proof. In view of the Riesz representation theorem we have to check that for every function $f \in C_c(\mathbb{R}^{d+1}) = \{g \in C(\mathbb{R}^{d+1}) \mid \mathrm{supp}(g) \text{ is compact}\}$

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d+1}} f\, d\nu_{n,\omega} \quad \text{with } \nu_{n,\omega} := \Gamma_\omega(\lambda_n)$$

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exists and is independent of ω. By compactness of the support of f, there exists an $E_0 \in \mathbb{R}$ (depending on f) with $f(E, \overline{v}) = 0$ for all $\overline{v} \in \mathbb{R}^d$ and $E \geq E_0$. On the other hand we know that $\|p\| \leq \sqrt{2(E_0 - V_{\min})}$ if $(p, q) \in H^{-1}([V_{\min}, E_0])$. Thus we have the estimate

$$\int_{\mathbb{R}^{d+1}} |f| \, d\nu_{n,\omega} \leq \frac{(2(E_0 - V_{\min})\,\pi)^{d/2}}{\Gamma(\frac{1}{2}d + 1)} \cdot \|f\|_\infty,$$

which is uniform in $n \in \mathbb{N}$ and $\omega \in \Omega$. The function

$$g\colon \Omega \to \mathbb{R}, \quad g(\omega) := \int_{\mathbb{R}^{d+1}} f \, d\nu_{1,\omega},$$

is thus in $L^\infty(\Omega, \beta)$. Moreover

$$\int_{\mathbb{R}^{d+1}} f \, d\nu_{n,\omega} = \frac{1}{|\mathcal{L}_n|} \sum_{\ell \in \mathcal{L}_n} g(\vartheta_\ell(\omega)),$$

with $\mathcal{L}_n := \{\sum_{k=1}^d n_k \ell_k \mid n_k \in \{-n, -n+1, \ldots, n-1\}\} \subseteq \mathcal{L}$.

By assumption, the probability measure β is ergodic with respect to the ϑ-action on Ω. Thus a lattice version of Birkhoff's ergodic theorem (see e.g. Keller [**Kel98**, Theorem 2.1.5]) assures that

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d+1}} f \, d\nu_{n,\omega} \text{ exists and is } \omega\text{-independent}$$

β-almost surely (a.s.). □

The energy–velocity distribution is the unique non-random limit measure

$$\hat{\nu}\colon \mathcal{B}(\mathbb{R}^{d+1}) \to [0, \infty].$$

This measure always has a certain symmetry property.

PROPOSITION 3.2. *The energy–velocity distribution $\hat{\nu}$ is invariant with respect to the inversion of velocity*

$$I\colon \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}, \quad (E, \overline{v}) \mapsto (E, -\overline{v}).$$

Proof. Consider $\omega \in \Omega$ with $\nu_\omega = \hat{\nu}$ and $\overline{v}_\omega^+(p_0, q_0) = \overline{v}_\omega^-(p_0, q_0)$ for λ^{2d}-a.e. phase space point $(p_0, q_0) \in \mathbb{R}^{2d}$. (By Proposition 3.1 and Corollary 2.4 β-a.e. ω meets these conditions.)

By *reversibility* of the flow $\Phi_\omega^t = (p_\omega^t, q_\omega^t)$, that is

$$p_\omega^t(-p_0, q_0) = -p_\omega^{-t}(p_0, q_0), \quad q_\omega^t(-p_0, q_0) = q_\omega^{-t}(p_0, q_0),$$

we have $\overline{v}_\omega^+(-p_0, q_0) = -\overline{v}_\omega^-(p_0, q_0)$. Together this gives

$$\overline{v}_\omega(-p_0, q_0) = -\overline{v}_\omega(p_0, q_0) \quad (\lambda^{2d}\text{-almost everywhere}).$$

On the other hand the phase space region $\mathbb{R}^d \times Q_n$ as well as the measure λ_n on it are invariant with respect to the antisymplectic transformation $(p, q) \mapsto (-p, q)$ on phase space. Thus the image measures $\nu_{n,\omega} = \Gamma_\omega(\lambda_n)$ are I-invariant. This carries over to the vague limit $\nu_\omega = \hat{\nu}$. □

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Definition 3.3. For $\omega \in \Omega$ a phase space point $x_0 = (p_0, q_0) \in \mathbb{R}^{2d}$ is called *forward*, respectively *backward*, *bounded* if

$$q_\omega([0, \infty), x_0), \quad \text{respectively } q_\omega((-\infty, 0], x_0),$$

are bounded subsets of configuration space $\mathbb{R}^d$.

Note that λ^{2d}-a.e. $x_0 \in \mathbb{R}^{2d}$ is simultaneously bounded or unbounded in both time directions. This is a direct consequence of the flow invariance of λ^{2d}.

PROPOSITION 3.4. *For $E > V_{\text{ess max}}$ and for β-a.e. $\omega \in \Omega$ for every initial position $q_0 \in \mathbb{R}^d$ there exists an initial direction $p_0 \in \mathbb{R}^d$ with $H_\omega(p_0, q_0) = E$ and (p_0, q_0) forward unbounded, with positive minimal speed ($\inf_{t>0} \|(q(t) - q_0)/t\| > 0$).*

Proof. For β-almost all (a.a.) $\omega \in \Omega$ we have $V_\omega(q) \le V_{\text{ess max}}$ for all $q \in \mathbb{R}^d$ (see Remark 2.1 and use the continuity of V_ω). Thus the Jacobi metric $g_{E,\omega}$ on configuration space $\mathbb{R}^d$, given by

$$g_{E,\omega}(q) = (E - V_\omega(q)) g_{\text{Euclid}}, \tag{3.1}$$

is non-degenerate, and the Riemannian manifold $(\mathbb{R}^d, g_{E,\omega})$ is geodesically complete, since the conformal factor is bounded below by $E - V_{\text{ess max}} > 0$. So the Hopf–Rinow theorem (see e.g. [**GHL48**, Theorem 2.103]) implies for any $q_n \in \mathbb{R}^d$ the existence of an initial direction $v_n \in S_{q_0} := \{v \in \mathbb{R}^d \mid g_{E,\omega}(q_0)(v, v) = 1\}$ such that the forward geodesic $t \mapsto \gamma(t, q_0, v_n)$ with initial condition (q_0, v_n) meets q_n first at a time $t_n \ge 0$, and is a shortest such geodesic so that

$$\|\gamma(t, q_0, v_n) - q_0\| \ge \sqrt{2(E - V_{\text{ess max}})}\, t \quad (n \in \mathbb{N},\ t \in [0, t_n]).$$

For $\lim_{n\to\infty} \|q_n - q_0\| = \infty$, $\lim_{n\to\infty} t_n = +\infty$, and by compactness of S_{q_0} there is an accumulation point $v_\infty \in S_{q_0}$ of the v_n leading to forward unbounded geodesic motion.

Up to parametrization, the geodesic of $g_{E,\omega}$ with initial condition (q_0, v_∞) coincides with the trajectory $q_\omega(t, x_0)$ (with $x_0 := (2(E - V_\omega(q_0))\, v_\infty, q_0)$ as initial condition); see [**AM78**, Theorem 3.7.7]. The positivity of the minimal speed is preserved. □

In dimensions $d \ge 2$ bounded and unbounded motion can coexist β-a.s. for energies $E > V_{\text{ess max}}$ as well as for $E < V_{\text{ess max}}$. In one dimension this is not possible.

PROPOSITION 3.5. *For $d = 1$ and β-a.e. $\omega \in \Omega$ the motion through $x_0 = (p_0, q_0)$ is bounded if $E := H_\omega(x_0) < V_{\text{ess max}}$ and unbounded with asymptotic velocity*

$$\overline{v}_\omega(x_0) = \frac{\ell_1}{\mathbb{E}_\beta(\tau(x_0))}, \quad \tau_\omega(x_0) = \int_0^{\ell_1} \frac{\text{sign}(p_0)}{\sqrt{2(E - V_\omega(q))}}\, dq,$$

if $E > V_{\text{ess max}}$.

Remark 3.6. Here the asymptotic velocity depends on $(\omega, x_0) \in P$ only via $H_\omega(x_0)$ and $\text{sign}(p_0)$. Figure 2 shows the shape of $\text{supp}(\hat{v}) \subseteq \mathbb{R}^2$.

Proof. (i) For $E < V_{\text{ess max}}$ the connected component of x_0 in $H_\omega^{-1}(E)$ is compact β-a.s., since there exist $q^\pm \in \mathbb{R}$ with $V_\omega(q^\pm) > E$ and $q^- < q_0 < q^+$.

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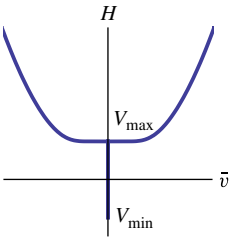


FIGURE 2. Shape of $\mathrm{supp}(\hat{\nu}) \subseteq \mathbb{R}^2$.

(ii) For $E > V_{\text{ess max}}$ the asymptotic velocity, if it exists for x_0, equals

$$\bar{v}(\omega, x_0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T p_\omega(t, x_0)\, dt = \lim_{n\to\infty} \frac{\int_0^{T_n(\omega)} p_\omega(t, x_0)\, dt}{T_n(\omega)},$$

with $T_n(\omega)$ for $n \in \mathbb{N}$ determined uniquely by $q_\omega(T_n(\omega), x_0) - q_0 = n\ell_1$. Here we assume without loss of generality that $\ell_1 > 0$ and $p_0 > 0$. The numerator equals

$$\int_0^{T_n(\omega)} \dot{q}_\omega(t, x_0)\, dt = q_\omega(T_n(\omega), x_0) - q_0 = n\ell_1,$$

whereas

$$T_n(\omega) = \sum_{k=0}^{n-1} \int_{q_0+k\ell_1}^{q_0+(k+1)\ell_1} \frac{1}{\sqrt{2(E - V_\omega(q))}}\, dq$$

for the denominator. Setting

$$g\colon \Omega \to \mathbb{R}, \quad g(\omega) := \int_{q_0}^{q_0+\ell_1} \frac{1}{\sqrt{2(E - V_\omega(q))}}\, dq,$$

we get $T_n(\omega) = \sum_{k=0}^{n-1} g(\vartheta_{k\ell_1}(\omega))$. By continuity of g, we can apply Birkhoff's theorem and get by the ergodicity assumption on β

$$\lim_{n\to\infty} \frac{1}{n} T_n(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g(\vartheta_{k\ell_1}(\omega)) = \mathbb{E}(g) \quad \beta\text{-a.s.}$$

This proves the assertion for λ^2-a.e. $x_0 \in \mathbb{R}^2$ with $H_\omega(x_0) > V_{\text{ess max}}$. The expression $(1/T) \int_0^T p_\omega(t, p_0, q_0)\, dt$ is monotonically increasing in p_0, and our formula for $\bar{v}_\omega(x_0)$ is continuous in p_0. Thus it must be valid for *all* x_0. □

For $d \geq 2$ it is an interesting question whether for large energies the asymptotic velocity distribution given by $\hat{\nu}$ is zero. As the example below shows, this is not always the case for non-trivial random potentials.

Example 3.7. (Random potential with non-zero asymptotic velocity) We use the cutoff function $F \in C_c^\infty(\mathbb{R}^d, [0, 1])$ from Remark 2.2 for the lattice $\mathcal{L} := \mathbb{Z}^d$. Given single-site potentials $\widehat{W}_j$ with $\mathrm{supp}(\widehat{W}_j) \subseteq [(1/4), (3/4)]^d$, we set

$$W_j(q) := \widehat{W}_j(q) - \sum_{k=1}^{d} \cos(2\pi q_k) F(q).$$

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Then the random potential equals

$$V_\omega(q) = \sum_{\ell\in\mathcal{L}} W_{\omega(\ell)}(q-\ell) = \hat{V}_\omega(q) - \sum_{k=1}^{d} \cos(2\pi q_k),$$

with $\hat{V}_\omega(q) := \sum_{\ell\in\mathcal{L}} \widehat{W}_{\omega(\ell)}(q-\ell)$.

Thus the Hamiltonian function $H_\omega\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ takes the form

$$H_\omega(p,q) = \hat{V}_\omega(q) + \sum_{k=1}^{d} H_\omega^{(k)}(p_k, q_k),$$

with $H_\omega^{(k)}\colon \mathbb{R}^d \to \mathbb{R},\ (p_k, q_k) \mapsto (1/2)p_k^2 - \cos(2\pi q_k)$. The phase space regions

$$P_k := \{(p,q) \in \mathbb{R}^d \times (\mathbb{R}^d \backslash S) \mid \forall\, m \in \{1,\dots,d\}\backslash\{k\} : H_\omega^{(m)}(p_m, q_m) < 0\}$$

with $S := \mathcal{L} + [(1/4), (3/4)]^d$ are invariant with respect to Φ_ω^t, since $\hat{V}_\omega(q) = 0$ for $q \in \mathbb{R}^d \backslash S$, and thus the motion separates.

For all $x = (p,q) \in P_k$ the asymptotic velocity exists. If in addition $E := H_\omega^{(k)}(p_k, q_k) > 1$, then $\overline{v}^{\pm}(x) = (0,\dots,0,\overline{v}^{(k)},0,\dots,0)$ with

$$\overline{v}^{(k)} := \frac{\operatorname{sign}(p_k)\pi\sqrt{E-1}}{\sqrt{2}\mathcal{K}(2/(1-E))},$$

$\mathcal{K}$ being the complete elliptic integral of the first kind (see [**AK98**]).

As $E \nearrow \infty$, the intersections $P_k \cap H_\omega^{-1}(E)$ have density with respect to Liouville measure on $H_\omega^{-1}(E)$ scaling like $E^{-(d-1)/2}$.

Thus for no total energy strictly above $2-d$ the distribution of asymptotic velocity is concentrated in zero.

4. *Poisson potentials*

Compared to the lattice case handled above, potentials based on marked Poisson fields have some new features like unboundedness and invariance with respect to $\mathbb{R}^d$-translations. So we discuss them in this section. Many properties should generalize to other ergodic random potentials (like Gaussian potentials).

Again we start with single-site potentials W_j indexed by $j \in J$, $|J| < \infty$, but we assume for simplicity that for some $\eta \geq 2$

$$W_j \in C_c^\eta(\mathbb{R}^d, \mathbb{R}).$$

We now consider the marked Poisson process with space

$$\tilde{\Omega} := \{\omega \mid \omega \text{ measure on } (\mathbb{R}^d \times J,\ \mathcal{B}(\mathbb{R}^d \times J)) \text{ with } \omega(K) \in \mathbb{N}_0 \text{ if } K \subseteq \mathbb{R}^d \times J \text{ is compact}\}.$$

Here $\tilde{\Omega}$ is the space of all counting measures on $\mathbb{R}^d \times J$ and carries the vague topology generated by the basis consisting of the sets

$$\mathcal{N}_\omega(\psi_0,\dots,\psi_k) := \left\{\omega' \in \tilde{\Omega} \;\middle|\; \forall i \in \{0,\dots,k\} : \left|\int \psi_i\, d\omega' - \int \psi_i\, d\omega\right| < 1\right\}, \quad (4.1)$$

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with $k \in \mathbb{N}_0$, $\omega \in \tilde{\Omega}$ and $\psi_i \in C_c(\mathbb{R}^d \times J, \mathbb{R})$ $(i \in \{0, \ldots, k\})$; see e.g. [**Rue87**]. For $j \in J$ and compact $K \in \mathcal{B}(\mathbb{R}^d)$ the random variables

$$N_{K,j} \colon \tilde{\Omega} \to \mathbb{N}_0, \quad \omega \mapsto \omega(K \times \{j\}),$$

are called *particle number functions*. We fix intensities $\rho_j \geq 0$, $j \in J$. Then β is the unique probability measure on $(\tilde{\Omega}, \mathcal{B}(\tilde{\Omega}))$ with

$$\beta(\{\omega \in \tilde{\Omega} \mid N_{K,j}(\omega) = m\}) = \frac{(\rho_j \lambda^d(K))^m}{m! \exp(\rho_j \lambda^d(K))} \tag{4.2}$$

for all $m \in \mathbb{N}_0$, $j \in J$ and $K \in \mathcal{B}(\mathbb{R}^d)$ with $\lambda^d(K) < \infty$.

Here $(\mathbb{R}^d \times J, \tilde{\Omega}, \mathcal{B}(\tilde{\Omega}), \beta)$ is called the *marked Poisson process on* $\mathbb{R}^d$ *with marks in* J *and intensities* ρ_j; see [**SKM87**, Ch. 4.2]. It induces the random potential

$$V \colon \tilde{M} := \tilde{\Omega} \times \mathbb{R}^d \longrightarrow \mathbb{R}, \quad (\omega, q) \longmapsto \int_{\mathbb{R}^d \times J} W_j(q - x)\, d\omega(x, j).$$

PROPOSITION 4.1. *The potential* $V \colon \tilde{\Omega} \to \mathbb{R}$ *is continuous, and*

$$V_\omega \in C^\eta(\mathbb{R}^d, \mathbb{R}) \quad (\omega \in \tilde{\Omega}).$$

There is a β*-measure zero subset* $N \subseteq \tilde{\Omega}$ *such that for* $\Omega := \tilde{\Omega} \backslash N$ *and extended phase space* $P := \Omega \times \mathbb{R}^d \times \mathbb{R}^d$ *the restriction* $H := \tilde{H} \restriction_P$ *of the Hamiltonian function*

$$\tilde{H} \colon \tilde{P} \to \mathbb{R}, \quad (\omega, p, q) \mapsto \tfrac{1}{2}\|p\|^2 + V(\omega, q),$$

induces a continuous Hamiltonian flow

$$\Phi \colon \mathbb{R} \times P \to P.$$

Proof. (i) To show continuity of V, we construct for $\varepsilon > 0$ a neighbourhood

$$U(\omega, q) := \mathcal{N}_\omega(\psi_1, \ldots, \psi_k) \times B_\delta(q) \subseteq \tilde{M} = \tilde{\Omega} \times \mathbb{R}^d$$

of $(\omega, q) \in \tilde{\Omega} \times \mathbb{R}^d$ such that $|V(\omega', q') - V(\omega, q)| < \varepsilon$ for all $(\omega', q') \in U(\omega, q)$. For radius $R := \sup\{\|x\| \mid x \in \bigcup_{j \in J} \operatorname{supp}(W_j)\} + 1$ only the finitely many Poisson points in

$$\mathcal{S}_{R,q} := \operatorname{supp}(\omega) \cap (B_R(q) \times J) \equiv \{(q_1, j_1), \ldots, (q_k, j_k)\}$$

can contribute to $V(\omega, q)$, and their minimal distance

$$d_{\min} := \inf\{\|q_m - q_n\| \mid 1 \leq m, n \leq k, q_m \neq q_n\}$$

is positive. We will need the loci $\mathcal{S}^j_{R,q}(\omega)$ of the support of ω with index $j \in J$, implicitly given by

$$\mathcal{S}_{R,q} = \bigcup_{j \in J} \mathcal{S}^j_{R,q}(\omega) \times \{j\},$$

and a common Lipschitz constant L for the single-site potentials $(W_j)_{j \in J}$.

With $\delta := \min\{\varepsilon/(2L\omega(\mathcal{S}_{R,q})), d_{\min}/3, 1\} > 0$ we choose $A_j := B_R(q) \backslash B_\delta(\mathcal{S}^j_{R,q}(\omega))$ and, employing the Kronecker delta $\delta_{\cdot,\cdot}$ and the positive part $(\cdot)_+ := \max\{0, \cdot\}$, functions $\psi_i \in C_c(\mathbb{R}^d \times J, \mathbb{R})$ with

$$\psi_i(x, j) := \begin{cases} (1 - \operatorname{dist}(x, A_j)/\delta)_+ & (i = 0), \\ (1 - \delta_{j,j_i} \operatorname{dist}(x, B_\delta(q_i))/\delta)_+ & (i \in \{1, \ldots, k\}). \end{cases}$$

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Thus $\int_{\mathbb{R}^d} \psi_i \, d\omega = (1-\delta_{0,l})\omega\{(q_i, j_i)\}$, and by the definition of $\mathcal{N}_\omega(\psi_0, \ldots, \psi_k)$ in (4.1) all $(\omega', q') \in U(\omega, q)$ fulfil the relations

$$\|q'-q\| \le \delta, \quad \omega'(A_j \times \{j\}) = 0 \quad \text{and} \quad \omega'(B_\delta(q_i) \times \{j_i\}) = \omega\{(q_i, j_i)\},$$

$j \in J$. With this preparation we deduce

$$\begin{aligned}
&|V(\omega', q') - V(\omega, q)| \\
&\le \sum_{j\in J} \left| \Big(\sum_{\tilde q \in \mathcal{S}^j_{R,q}(\omega')} W_j(q'-\tilde q)\omega'(\tilde q, j) \Big) - \sum_{\hat q \in \mathcal{S}^j_{R,q}(\omega)} W_j(q-\hat q)\omega(\hat q, j) \right| \\
&\le \sum_{j\in J} \sum_{\hat q \in \mathcal{S}^j_{R,q}(\omega)} \left| \Big(\sum_{\tilde q \in \mathcal{S}^j_{\delta,\hat q}(\omega')} W_j(q'-\tilde q)\omega'(\tilde q, j) \Big) - W_j(q-\hat q)\omega(\hat q, j) \right| \\
&\le \sum_{j\in J} \sum_{\hat q \in \mathcal{S}^j_{R,q}(\omega)} \sum_{\tilde q \in \mathcal{S}^j_{\delta,\hat q}(\omega')} |W_j(q'-\tilde q) - W_j(q-\hat q)| \, \omega'(\tilde q, j) \\
&\le \sum_{j\in J} \sum_{\hat q \in \mathcal{S}^j_{R,q}(\omega)} 2L\delta\omega(\hat q, j) = 2L\delta\omega(\mathcal{S}_{R,q}) \le \varepsilon,
\end{aligned}$$

since $\|q'-\tilde q-(q-\hat q)\| \le \|q'-q\| + \|\tilde q - \hat q\| \le 2\delta$.

(ii) This implies that $\tilde H : \tilde P \to \mathbb{R}$, too, is continuous on $\tilde P := \tilde\Omega \times \mathbb{R}^{2d}$ and for all $\omega \in \tilde\Omega$ the Hamiltonian $\tilde H_\omega \in C^\eta(\mathbb{R}^{2d}, \mathbb{R})$. However, this only guarantees *local* unique existence of the flow. We now set

$$N := \left\{ \omega \in \tilde\Omega \;\middle|\; \liminf_{\|q\|\to\infty} V_\omega(q) + c\,\langle q\rangle^2 = -\infty \text{ for all } c > 0 \right\}. \tag{4.3}$$

For $\omega \in \tilde\Omega \backslash N$ and $E \in \mathbb{R}$ there exists a $C > 0$ with

$$\sqrt{2(E - V_\omega(q))_+} \le C\,\langle q\rangle \quad (q \in \mathbb{R}^d).$$

Thus the solution of the initial value problem with energy E exists for all times. Namely, as

$$\langle q_\omega(t)\rangle \, \frac{d}{dt} \, \langle q_\omega(t)\rangle \le \|q_\omega(t)\| \, \|p_\omega(t)\| \le C \, \langle q_\omega(t)\rangle^2 ,$$

$t \mapsto \|q_\omega(t)\|$ is at most of exponential growth.

(iii) We show that $\beta(N) = 0$. We set $W_{\max} := \max\{|W_j(q)| \mid (q, j) \in \mathbb{R}^d \times J\} > 0$, $\operatorname{diam} := \max_j \operatorname{diam}(\operatorname{supp} W_j)$ and $N_{B_r(q),J} := \sum_{j\in J} N_{B_r(q),j}$. Then $\beta(N) = 0$ follows from $\lim_{R\to\infty} \beta(N_R) = 0$ with

$$N_R := \left\{ \omega \in \tilde\Omega \;\middle|\; \min_{\|q\|\le R} V_\omega(q) \le -R^2 \right\}. \tag{4.4}$$

Now for $V_\omega(q) \le -R^2$ to occur for any point $q \in B_{\operatorname{diam}}(Q)$ we must have

$$N_{B_{2\operatorname{diam}(Q)},J}(\omega) \ge R^2 / W_{\max}. \tag{4.5}$$

As $B_R(0)$ is diam-spanned (see Walters [**Wal82**]) by a set $\mathcal{Q} \subseteq B_R(0)$ of points Q with cardinality $|\mathcal{Q}| = \mathcal{O}(R^d)$, (4.4) follows from (4.5) and (4.2).

(iv) Finally, continuity of the flow Φ follows from the theorem on continuous dependence of solutions on parameters. □

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Remark 4.2. (Comparison with quantum mechanics) We have

$$\mathbb{E}_\beta(|V_\cdot(0)|^{(d+\varepsilon)/2}) \le (W_{\max})^{(d+\varepsilon)/2}\, \mathbb{E}_\beta(N_{B_R(0),J})^{(d+\varepsilon)/2}. \tag{4.6}$$

Estimate (4.6) should be compared with a criterion for essential self-adjointness of the corresponding random Schrödinger operator $-\Delta + V$ on $L^2(\mathbb{R}^d)$. This is generally true for ϑ-ergodic V if $\mathbb{E}_\beta(|V(0)|^{\tilde{d}}) < \infty$ for $\tilde{d} := 2\lceil (d+1)/4 \rceil$; see [**PF92**, Theorem 5.1].

Unlike in the lattice case, the Poisson potential V is invariant with respect to a faithful $\mathbb{R}^d$-action on P. For $\ell \in \mathbb{R}^d$ set

$$\vartheta_\ell \colon \tilde{\Omega} \to \tilde{\Omega}, \quad \vartheta_\ell(\omega)(A) := \omega(\{(\ell + x,\, j) \mid (x,\, j) \in A\}). \tag{4.7}$$

Then N, defined in (4.3), is ϑ-invariant so that we get the $\mathbb{R}^d$-action

$$\Theta \colon \mathbb{R}^d \times P \to P, \quad (\ell,\, \omega,\, p,\, q) \mapsto (\vartheta_\ell(\omega),\, p,\, q - \ell),$$

on extended phase space, with

$$H \circ \Theta_\ell = H, \quad \mu \circ \Theta_\ell = \mu \quad \text{and} \quad \Phi^t \circ \Theta_\ell = \Theta_\ell \circ \Phi^t$$

for all $t \in \mathbb{R}^d$ and $\ell \in \mathbb{R}^d$.

In order to control the existence of the asymptotic velocities, we select an arbitrary regular lattice $\mathcal{L} \subseteq \mathbb{R}^d$, e.g. $\mathcal{L} := \mathbb{Z}^d$, and set $\Theta^{\mathcal{L}} := \Theta\restriction_{\mathcal{L} \times P}$.

Then, as in §2 above, we consider the covering projection

$$\hat{\pi} \colon P \to \hat{P} := P/\Theta^{\mathcal{L}}$$

to the factor space $\hat{P}$. Like in (2.8), $\hat{P}$ is homeomorphic to $(\Omega \times \mathbb{R}^d \times \mathcal{D})/\!\sim$.

On $\hat{P}$ we consider the measures $\hat{\mu}$ and $\hat{\mu}_E$ ($E \in \mathbb{R}$), defined like in (2.9). Again, by Θ-invariance similar to (2.10) we can define momenta on $\hat{P}$ by

$$\hat{p} \colon \hat{P} \to \mathbb{R}^d, \quad \hat{p} \circ \hat{\pi} = p.$$

But unlike in §2, the potential V_ω is β-a.s. unbounded, and thus $\hat{H}^{-1}((-\infty,\, E])$ is compact only if the W_j are non-negative. Therefore we need the following lemma.

LEMMA 4.3. *For all $E \in \mathbb{R}$ the measure $\hat{\mu}_E$ on $\hat{P}$ is finite, and*

$$\int_{\hat{P}} \|\hat{p}\|\, d\hat{\mu}_E < \infty. \tag{4.8}$$

Proof. Setting $r(q) := \sqrt{2(E - V(q))_+}$, we estimate

$$\hat{\mu}_E(\hat{P}) = \mathbb{E}_\beta\left(\int_{\mathcal{D}} \int_{\mathbb{R}^d} \mathbb{1}_{B_{r(q)}}(p)\, dp\, dq\right) = \tau_d\, \lambda^d(\mathcal{D})\, \mathbb{E}_\beta(2(E - V(0))_+^{d/2}) < \infty,$$

with $\tau_d := \lambda^d(B_1(0)) = \pi^{d/2}/\,\Gamma(1 + d/2)$, and similarly (but with exponent $(d+1)/2$) for (4.8). □

Thus we get the analogue of Proposition 2.3.

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PROPOSITION 4.4. *The asymptotic velocities* $\overline{v}^{\pm}(\omega, x_0) := \lim_{T\to\pm\infty}(q_\omega(T, x_0)/T)$ *exist and are equal* β*-a.s. on* P*. Furthermore,*

$$\overline{v} \in L^\infty_{\mathrm{loc}}(P) \quad \textit{and} \quad \overline{v} \circ \Phi^t = \overline{v} \circ \Theta_\ell = \overline{v}.$$

Proof. We invoke Birkhoff's theorem for

$$\overline{v}^{\pm}(\omega, x_0) = \lim_{T\to\pm\infty} \frac{1}{T} \int_0^T \hat{p}(t, \hat{\pi}(\omega, x_0))\, dt,$$

using Lemma 4.3. □

Remarks 4.5.

(1) The $\mathbb{R}^d$-action (4.7) is mixing on $\tilde{\Omega}$; see [**PF92**, p. 27]. Thereby the action of the lattice $\mathcal{L}$ by translations ϑ_ℓ ($\ell \in \mathcal{L}$) on $\tilde{\Omega}$ is mixing, too, and in particular β-ergodic. Thus, like in Proposition 3.1, the energy–velocity distributions Γ_ω for the Poisson potentials are β-a.s. deterministic.

(2) For $d = 1$ one gets a result similar to Proposition 3.5. Note, however, that $\mathrm{supp}(V(\beta \otimes \lambda^d))$ equals the closure of $(V_{\text{ess min}}, \infty)$, if there is a single-site potential W_j with $W_j(q) > 0$ for some $q \in \mathbb{R}$. In that case all motion is bounded β-a.s.

5. *Singular values of the Hamiltonian*

Since Hamiltonian motion enjoys conservation of energy, one has to decompose phase space into energy shells in order to find ergodic motion. This is possible for regular energy values, so we first study the set of singular values of the Hamilton function. For both the lattice and the Poisson cases we have the following result.

PROPOSITION 5.1. *The closure of the set of singular values of* V_ω *is* β*-a.s. deterministic.*

Proof. For $S_{k,m} := [m2^{-k}, (m+1)2^{-k}]$ with $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ the set

$$\Omega_{k,m} := \{\omega \in \Omega \mid \exists q \in \mathbb{R}^d : \nabla_q V(\omega, q) = 0 \text{ and } V(\omega, q) \in S_{k,m}\}$$

is $\mathcal{L}$-invariant. Thus by β-ergodicity it is of measure zero or one. The sets $S_k := \bigcup_{m\in\mathbb{Z}:\, \beta(\Omega_{k,m})=1} S_{k,m}$ are closed, and $S_{k+1} \subseteq S_k$. Thus

$$\Omega_k := \{\omega \in \Omega \mid \mathrm{CVal}_\omega \subseteq S_k\}$$

is still invariant and of measure one. The same is true for $\Omega_\infty := \bigcap_{k\in\mathbb{N}} \Omega_k$. The potentials indexed by $\omega \in \Omega_\infty$ have their critical values in the closed set $S_\infty := \bigcap_{k\in\mathbb{N}} S_k$. It is β-a.s. the closure of the set of singular values of V_ω. □

Example 5.2. (Denseness of singular values) The set of singular values may be dense in $[V_{\text{ess min}}, V_{\text{ess max}}]$. This is the case β-a.s. for the random $\mathbb{Z}$-lattice potential on $\mathbb{R}$ given by the single-site potentials

$$W_j(q) := \sum_{\ell\in\mathbb{N}} j|J|^{-\ell}\chi(q - (-1)^\ell \lfloor \ell/2 \rfloor) \quad (q \in \mathbb{R},\ j \in J := \{0, \ldots, |J| - 1\}),$$

with $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$, $\chi\restriction_{[-(1/4),(1/4)]} = 1$ and $\mathrm{supp}(\chi) \subseteq [-(1/2), (1/2)]$, if $\beta := \bigotimes_{\mathbb{Z}} \hat{\beta}$ is the product measure of $\hat{\beta} := \sum_{j\in J} \delta_j/|J|$. Then for any $\omega \in \Omega = J^{\mathbb{Z}}$ with a

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dense ϑ-orbit the set of singular values is dense in $[0, 1]$ and this is the case for β-a.e. $\omega \in \Omega$.

Note that the fall-off of the single-site potentials is not only polynomial as assumed in (2.1) but exponential, with rate $W_j(q) = \mathcal{O}(|J|^{-2|q|})$.

The exponential decay in this example is below the rate that ensures measure zero for the closure of the set $\mathrm{CVal}_\omega = V_\omega(\mathrm{CSet}_\omega)$ of singular values of V_ω.

PROPOSITION 5.3. *Let for $d = 1$ and some $\varepsilon > 0$ the single-site potentials $W_j \in C^2(\mathbb{R}, \mathbb{R})$ obey (for the lattice $\mathbb{Z}$) the decay estimate $W_j'(q) = \mathcal{O}(|J|^{-(4+\varepsilon)|q|})$. Then*

$$\lambda^1(\overline{\mathrm{CVal}_\omega}) = 0 \quad (\omega \in \Omega).$$

Proof. (i) By using the lattice translation ϑ, it suffices to show that

$$\lambda^1\left(\overline{\bigcup_{\omega\in\Omega} V_\omega(\mathrm{CSet}_\omega \cap [0, 1])}\right) = 0. \tag{5.1}$$

For this we prove that for $R \in \mathbb{N}$ large and the middle sequence $\omega^{(R)} \in J^{\{-R,\dots,R\}}$ of ω we have

$$V_\omega(\mathrm{CSet}_\omega \cap [0, 1]) \subseteq M_{\omega^{(R)}} \quad (\omega \in \Omega), \tag{5.2}$$

for suitable closed subsets $M_\tau \subseteq \mathbb{R}$ of measures

$$\lambda^1(M_\tau) \le |J|^{-(2+\varepsilon/4)R} \quad (\tau \in J^{\{-R,\dots,R\}}). \tag{5.3}$$

As this implies $\lambda^1(\overline{\bigcup_{\omega\in\Omega} V_\omega(\mathrm{CSet}_\omega\cap[0, 1])}) \le |J|^{1-\varepsilon R/4}$, (5.1) then follows as $R \to \infty$.

(ii) We have that $c^{(2)} := \sup\{\sum_{j\in\mathbb{Z}} |W_{\omega_j}''(q - j)| \mid \omega \in \Omega, q \in \mathbb{R}\} < \infty$ is a uniform upper bound for the second derivative of

$$\tilde{V}_\tau(q) := \sum_{\ell=-R}^{R} W_{\tau(\ell)}(q - \ell) \quad (q \in \mathbb{R},\ \tau \in J^{\{-R,\dots,R\}}).$$

Then for $\delta \le c^{(2)}$ the disjoint union $\{x \in [0, 1] \mid |\tilde{V}_\tau'(x)| \le \delta\}$ of closed intervals has at most $i_{\max}(\tau) \le c^{(2)}/\delta$ intervals $I_1(\tau), \dots, I_{i_{\max}(\tau)}(\tau)$ containing a point q with $\tilde{V}_\tau'(q) = 0$, since each such interval has at least length $\delta/c^{(2)}$.

We set $M_\tau := \bigcup_{i=1}^{i_{\max}(\tau)} B_{\delta^2}(\tilde{V}_\tau(I_i(\tau)))$. This union of thickened intervals has measure

$$\lambda^1(M_\tau) \le \lambda^1\left(\tilde{V}_\tau\left(\bigcup_{i=1}^{i_{\max}(\tau)} I_i(\tau)\right)\right) + 2\delta^2 i_{\max}(\tau) \le (1 + 2c^{(2)})\delta.$$

So for $\delta \equiv \delta_R := |J|^{-(2+\varepsilon/2)R}$ and R large, (5.3) is satisfied.

(iii) By the decay assumption $|W_j'(q)| \le c|J|^{-(4+\varepsilon)|q|}$ on the derivatives of the single-site potentials, for $q \in [0, 1]\backslash\bigcup_{i=1}^{i_{\max}(\tau)} I_i(\tau)$ one has

$$|V_\omega'(q)| \ge |\tilde{V}_\tau'(q)| - c \sum_{\ell\in\mathbb{Z}:|\ell|>R} |J|^{-(4+\varepsilon)|\ell|} \ge \delta_R - c\,\delta_R^2 > 0$$

for R large enough. So $\mathrm{CSet}_\omega \cap [0, 1] \subseteq \bigcup_{i=1}^{i_{\max}(\tau)} I_i(\tau)$ in (5.2).

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(iv) The single-site potentials themselves, and not only their derivatives, decay like $W_j(q) = \mathcal{O}(|J|^{-(4+\varepsilon)|q|})$. This means that

$$|V_\omega(q) - \tilde{V}_{\omega^{(R)}}(q)| \le c \sum_{\ell\in\mathbb{Z}:|\ell|>R} |J|^{-(4+\varepsilon)|\ell|} \le \delta^2 \quad (q \in [0,1]).$$

But δ^2 is the parameter appearing in the definition of M_τ. So (5.2) is satisfied, too. □

A similar statement, but with superexponential decay rate $\mathcal{O}(|J|^{-(c_\mathcal{L}+\varepsilon)\|q\|^d})$, should be sharp for $d \ge 1$, with a constant $c_\mathcal{L}$ depending on the lattice $\mathcal{L} \subseteq \mathbb{R}^d$.

We denote the restriction of the flow Φ_ω to the energy surface $\Sigma_{E,\omega} := H_\omega^{-1}(E) \subseteq \mathbb{R}^{2d}$ by $\Phi_{E,\omega}$. For regular values E of H_ω, $\Sigma_{E,\omega}$ carries a $\Phi_{E,\omega}$-invariant measure $\lambda_{E,\omega}$ derived from Lebesgue measure λ^{2d} on phase space (see e.g. [**AM78**, Theorem 3.4.12]).

This non-atomic *Liouville measure* thus exists for all $E > V_{\max}$, using the equality of regular values of H_ω and V_ω. For $V_\omega \in C^d(\mathbb{R}^d, \mathbb{R})$ by Sard's theorem (see e.g. [**Hir76**, Theorem 3.1.3]) it in fact exists for almost all energies $E > \inf V_\omega$.

Remark 5.4. (Denseness of singular values, Poisson case) For (non-trivial) Poisson potentials, β-a.s., the set $\overline{\mathrm{CVal}_\omega}$ equals $\mathbb{R}$, $[0,\infty)$ or $(-\infty, 0]$, depending on the signs occuring in the union of ranges of the single-site potentials. This is true, since the following hold.

(a) For any real number r between zero and an extreme value of W_j, the sum $U_a(q) := W_j(q) + W_j(q + ae_1)$ of W_j and its translate by a suitable $a = a(r) \in \mathbb{R}$ has r as a singular value. For the case $\max W_j = W_j(q_0) > 0$ we take

$$\mathrm{MM}(a) := \sup_{c\in C} \min_{t\in[0,1]} U_a(c(t)),$$

with $C := \{c \in C^\infty([0,1], \mathbb{R}^d) \mid c(0) = q_0,\ c(1) = q_0 + ae_1\}$. Then $\mathrm{MM}(0) = 2 \max W_j$, $\mathrm{MM}(a) = 0$ for $|a|$ large, and MM is continuous. Furthermore $\mathrm{MM}(a)$ is a singular value for U_a.

The case $\min W_j = W_j(q_0) < 0$ is treated by the reverse minmax problem.

(b) Integer multiples of W_j are Poisson potentials, too.

6. *Notions of ergodicity of time evolution*

In general $\lambda_{E,\omega}$ is non-finite. Nevertheless, we may ask whether it is an *ergodic* measure (in the sense of Aaronson [**Aar97**]), that is, whether the only Φ_ω-invariant measurable subsets $A \subseteq \Sigma_{E,\omega}$ have measure zero or full measure, i.e. $\lambda_{E,\omega}(\Sigma_{E,\omega}\backslash A) = 0$. Instead, we may also ask whether the restriction

$$\hat{\Phi}_E^t := \hat{\Phi}^t\restriction_{\hat{\Sigma}_E} \quad (t \in \mathbb{R}) \tag{6.1}$$

of the flow $\hat{\Phi}$ to the (extended) energy surfaces $\hat{\Sigma}_E := \hat{H}^{-1}(E) \subseteq \hat{P}$ is ergodic. Here the Liouville measure $\hat{\lambda}_E$ on $\hat{\Sigma}_E$, derived from the measure $\hat{\mu}$ on $\hat{P}$ by disintegration, is finite. Ergodicity of $\hat{\Phi}_E$ is obtained much more cheaply than ergodicity of $\Phi_{E,\omega}$, since on finite measure spaces Poincaré's recurrence theorem holds; see §10. In this section we clarify the relations between the two notions of ergodicity and give some immediate consequences. The results pertain to both lattice and Poisson potentials (using quotients by $\mathbb{Z}^d$ in the latter case).

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PROPOSITION 6.1. *If for energy $E \in \mathbb{R}$, the flow $\Phi_{E,\omega}$ on $\Sigma_{E,\omega}$ is $\lambda_{E,\omega}$-ergodic for β-a.e. $\omega \in \Omega$, then the flow $\hat{\Phi}_E$ on Σ_E is $\hat{\mu}_E$-ergodic.*

Proof. Assume that $\hat{\Phi}_E$ is not ergodic. Then there exists a measurable invariant ($\hat{\lambda}_E(\hat{A}\Delta\hat{\Phi}_E^t(\hat{A})) = 0$ for all $t \in \mathbb{R}$) subset $\hat{A} \subseteq \hat{\Sigma}_E$ with $0 < \hat{\lambda}_E(\hat{A}) < \hat{\lambda}_E(\hat{\Sigma}_E)$. Its pre-image $A := \hat{\pi}^{-1}(\hat{A}) \subseteq \Sigma_E = H^{-1}(E)$ is measurable and invariant, too, with respect to the Liouville measure λ_E on Σ_E. Finally, $\lambda_E(A) > 0$ and $\lambda_E(A^c) > 0$, as $A^c = \Sigma_E \backslash A = \hat{\pi}^{-1}(\hat{\Sigma}_E \backslash \hat{A})$.

So $A_\omega := A \cap \Sigma_{E,\omega}$ is $\Phi_{E,\omega}$-invariant β-a.s., and similarly for $A^c_\omega := A^c \cap \Sigma_{E,\omega} = \Sigma_{E,\omega} \backslash A_\omega$.

By ϑ-ergodicity of β the β-a.s. ϑ-invariant functions

$$f_A,\ f_{A^c} \colon \Omega \to \{0,\ 1\}, \qquad f_A(\omega) = \begin{cases} 1 & \lambda_{E,\omega}(A_\omega) > 0, \\ 0 & \text{otherwise} \end{cases}$$

(respectively for A^c_ω) are both β-a.s. constant with value 1. Thus $\Phi_{E,\omega}$ is not $\lambda_{E,\omega}$-ergodic β-a.s. □

We note in passing that the flow Φ_E on Σ_E is *not* λ_E-ergodic (given β is not a Dirac measure on Ω), since Ω is unchanged by that flow.

Next we ask about the dynamical consequences of ergodicity.

PROPOSITION 6.2. *Given $E \in \mathbb{R}$ and $\omega \in \Omega$ such that $\Phi_{E,\omega}$ is $\lambda_{E,\omega}$-ergodic on the regular energy surface $\Sigma_{E,\omega}$:*

(1) *then the asymptotic velocity $\overline{v}(x) = 0$ for a.e. $x \in \Sigma_{E,\omega}$; but*

(2) *the motion is unbounded almost everywhere on $\Sigma_{E,\omega}$ unless $\Sigma_{E,\omega}$ is compact.*

Remark 6.3. Note that $\Sigma_{E,\omega}$ is non-compact for $E > V_{\max}$ and β-a.s. non-compact for $E > V_{\text{ess min}}$, given ergodicity of $\Phi_{E,\omega}$.

Proof. Remember that $\overline{v}(x) = 0$ by definition if $\overline{v}^{\pm}(x)$ do not exist or are unequal. (1) Assume that $\lambda_{E,\omega}(\{x \in \Sigma_{E,\omega} \mid \overline{v}(x) \neq 0\}) > 0$. Then there exists an open half-space $H_s := \{q \in \mathbb{R}^d \mid \langle q, s\rangle > 0\}$ indexed by a unit vector $s \in S^{d-1}$ so that

$$\lambda_{E,\omega}(B_s) > 0 \quad \text{for } B_s := \{x \in \Sigma_{E,\omega} \mid \overline{v}(x) \in H_s\}.$$

By reversibility of the flow and $\overline{v}^- = \overline{v}^+$,

$$\lambda_{E,\omega}(B_{-s}) = \lambda_{E,\omega}(B_s).$$

On the other hand, $\overline{v} \circ \Phi^t = \overline{v}$ ($t \in \mathbb{R}$), so that B_s and B_{-s} are disjoint invariant subsets of positive measure. Thus $\Phi_{E,\omega}$ is not $\lambda_{E,\omega}$-ergodic.

(2) Assume that there exists a subset $B \subseteq \Sigma_{E,\omega}$ of points leading to forward bounded motion with $\lambda_{E,\omega}(B) > 0$. Then for some $r > 0$ the subset

$$B_r := \{(p, q) \in B \mid \|q\| \leq r\}$$

is of positive measure, too. Furthermore, for some $r' > r$ the same applies to

$$B_{r,r'} := \{x \in B_r \mid \|q_t(\omega, x)\| \leq r' \text{ for all } t \geq 0\}.$$

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Finally the set

$$\tilde{B} := \bigcup_{t\geq 0} \Phi^t_{E,\omega}(B_{r,r'}) \subseteq \Sigma_E$$

is $\Phi_{E,\omega}$-invariant with respect to $\lambda_{E,\omega}$, and $\lambda_{E,\omega}(\tilde{B}) > 0$. But by non-compactness of $\Sigma_{E,\omega}$, its complement $\tilde{B}^c = \Sigma_{E,\omega}\backslash\tilde{B}$ is of measure $\lambda_{E,\omega}(\tilde{B}^c) > 0$, since $\tilde{B}$ is bounded: $\|q\| \leq r'$ if $(p, q) \in \tilde{B}$. Thus $\Phi_{E,\omega}$ is not $\lambda_{E,\omega}$-ergodic. □

In one-dimensional natural mechanical systems with Hamiltonian $H\colon \mathbb{R}^2 \to \mathbb{R}$, $H(p, q) = (1/2)p^2 + V(q)$, ergodicity on regular energy surfaces $\Sigma_E = H^{-1}(E)$ is widespread and occurs for compact as well as non-compact Σ_E (examples: $V(q) = q^2$, $V(q) = q$).

In our present context, however, ergodicity in one dimension is exceptional.

PROPOSITION 6.4. *For $d = 1$ and $E > V_{\text{ess min}}$ the flow $\Phi_{t,\omega}$ on a regular energy surface $\Sigma_{E,\omega}$ is β-a.s. not $\lambda_{E,\omega}$-ergodic.*

Proof. We assume that ω is chosen so that $\overline{v}^+(x) = \overline{v}^-(x)$ for $\lambda_{E,\omega}$-a.e. $x \in \mathbb{R}$. This assumption is valid β-a.s. For $E < V_{\text{ess max}}$ the motion is bounded β-a.s., whereas for $E > V_{\text{ess max}}$ asymptotic velocity is non-zero β-a.s. according to Proposition 3.5.

Both statements contradict Proposition 6.2 (note that $\Sigma_{E,\omega}$ is non-compact β-a.s. for $E > V_{\text{ess min}}$).

But for the case left, $E = V_{\text{ess max}}$, $\Sigma_{E,\omega}$ must be connected, regular and non-compact. Thus it must be diffeomorphic to $\mathbb{R}$. This can only occur if there is a q_0 so that $V(q) > E$ for all $q \in (q_0, \infty)$ and $V(q) < E$ for all $q \in (-\infty, q_0)$ or vice versa. This, however, would contradict our initial assumption $\overline{v}^+(x) = \overline{v}^-(x)$. □

Example 3.7 together with Proposition 6.2 shows that for any dimension d there are non-trivial random lattice potentials that lead to non-ergodic motion on the regular energy surfaces $\Sigma_{E,\omega}$ for *any* E and $\omega \in \Omega$.

The statements of Proposition 6.2 can be proven under a different assumption, which by Proposition 6.1 is weaker than the $\lambda_{E,\omega}$-ergodicity assumption in Proposition 6.2 *for β-a.e. ω.*

PROPOSITION 6.5. *If $\hat{\Phi}_E$ is $\hat{\mu}_E$-ergodic on the regular energy surface $\hat{\Sigma}_E$, then:*

(1) *the asymptotic velocity satisfies $\overline{v}(x) = 0$ for a.e. $x \in \Sigma_{E,\omega}$; but*

(2) *the motion is unbounded almost everywhere on $\Sigma_{E,\omega}$ for β-a.e. ω.*

Proof. By definition, $v(x) = 0$, wherever $\overline{v}^{\pm}(x)$ do not exist or are unequal. We define $\hat{v}_E\colon \hat{\Sigma}_E \to \mathbb{R}^d$ by $\hat{v}_E \circ \hat{\pi} = \overline{v}|_{\Sigma_E}$.

(1) Assume that there exists a measurable $\Omega' \subseteq \Omega$ with $\beta(\Omega') > 0$ such that $\lambda_{E,\omega}(\{x \in \Sigma_{E,\omega} \mid \overline{v}(x) \neq 0\}) > 0$ for all $\omega \in \Omega'$. This implies that

$$\hat{\lambda}_E(\{x \in \hat{\Sigma}_E \mid \hat{v}_E(x) \neq 0\}) > 0.$$

Again, there exists an open half-space $H_s := \{q \in \mathbb{R}^d \mid \langle q, s\rangle > 0\}$ indexed by a unit vector $s \in S^{d-1}$ so that

$$\hat{\lambda}_E(\hat{B}_s) > 0 \quad \text{for } \hat{B}_s := \{x \in \hat{\Sigma}_E \mid \hat{v}_E(x) \in H_s\}.$$

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By reversibility of the flow and $\overline{v}^- = \overline{v}^+$,

$$\hat{\lambda}_{E,\omega}(\hat{B}_{-s}) = \hat{\lambda}_{E,\omega}(\hat{B}_s).$$

On the other hand, $\hat{v}_E \circ \hat{\Phi}^t = \hat{v}_E$ $(t \in \mathbb{R})$, so that $\hat{B}_s$ and $\hat{B}_{-s}$ are disjoint invariant subsets of positive measure. Thus $\hat{\Phi}_E$ is not $\hat{\lambda}_E$-ergodic.

(2) We have to show that the bounded orbits in Σ_E carry no measure. Fix $r > 0$ and let

$$B \equiv B_r := \{x \in \Sigma_E \mid \forall t \in \mathbb{R}\colon \|q_t(x)\| < r\}$$

be the set of orbits bounded by r. With the help of the projections

$$\pi_{\Omega,E}\colon \Sigma_E \to \Omega \quad \text{and} \quad \pi_\Lambda\colon \Omega \to \Omega_\Lambda := J^\Lambda$$

for finite subsets $\Lambda \subseteq \mathcal{L}$, we can partition $B = \bigcup_{\sigma\in\Omega_\Lambda} B_\sigma$, with

$$B_\sigma := B \cap (\pi_\Lambda \circ \pi_{\Omega,E})^{-1}\{\sigma\} \quad (\sigma \in \Omega_\Lambda).$$

Since the probability measure β is non-atomic, so is λ_E. We thereby conclude that

$$\sup_{\sigma\in\Omega_\Lambda} \lambda_E(B_\sigma) \xrightarrow{\Lambda\to\mathcal{L}} 0. \tag{6.2}$$

On the other hand, B_σ is Φ_E-invariant, and $\hat{B}_\sigma := \hat{\pi}(B_\sigma)$ is preserved by $\hat{\Phi}_E$. By ergodicity of $\hat{\Phi}_E$ and (6.2) we find a finite subset $\Lambda \subseteq \mathcal{L}$ such that

$$\sup_{\sigma\in\Omega_\Lambda} \hat{\lambda}_E(\hat{B}_\sigma) = 0.$$

But $\hat{\pi}$ is non-singular, i.e. preserves sets of measure 0, and we conclude that

$$\lambda_E(B) = \sum_{\sigma\in\Omega_\Lambda} \lambda_E(B_\sigma) = 0. \qquad \square$$

7. *No ergodicity in the Poisson case*

We now assume for the Poisson case that the single-site potentials are smooth, in addition to being compactly supported. Then ergodicity of the dynamics is atypical.

THEOREM 7.1. *Consider a random Poisson potential on $\mathbb{R}^d$. Then for any $E \in \mathbb{R}$ the motion on the energy surface $\Sigma_{E,\omega}$ is β-a.s. not ergodic.*

Strategy of proof and first steps

(i) The theorem is true for dimension $d = 1$. If there is a single-site potential, say W_1, and $q \in \mathbb{R}$ with $W_1(q) > 0$, then we have bounded orbits β-a.s. for all E. If, instead, all W_j are non-positive, $\Sigma_{E,\omega}$ has two connected components for $E > 0$ and one has β-a.s. bounded orbits for $E \le 0$.

(ii) So we assume $d \ge 2$. If the Poisson potential is zero, then we have free motion, which is not ergodic in any dimension. Otherwise there is a non-zero single-site potential, say W_1. The proof method depends on the sign of

$$I := \int_{\mathbb{R}^d} W \, d\lambda^d \tag{7.1}$$

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(we temporarily omit the index of W_1). Then, for given I and energy E, we construct a set $\mathcal{L}$ and thereby a *finite* sum $q \mapsto \sum_{\ell\in\mathcal{L}} W(q-\ell)$ of translated potentials, which for the given energy confines trajectories of positive measure.

This then suffices to show that β-a.s. the flow on $\Sigma_{E,\omega}$ is not ergodic.

(i) If the restriction of ω to a large ball $B_R(0) \subseteq \mathbb{R}^d$ is near to $\sum_{\ell\in\mathcal{L}} \delta_{(\ell,1)}$ with respect to the topology from (4.1), then the flow is shown to be non-ergodic, too. This event in Ω has positive probability.

(ii) By the nature of the Poisson process β, the Borel–Cantelli lemma can be applied to the ϑ_ℓ-translates of that event with $\ell \in 2R\mathbb{Z}^d$, so that the flow is even β-a.s. non-ergodic.

The $\mathrm{SO}(d)$-invariant potentials $U \in C_c^\infty(\mathbb{R}^d, [0,\infty))$ lead to integrable motion, and for suitable U to a positive measure of bounded orbits (see e.g. Arnol'd [**Arn78**, §8]).

We first approximate a given such $U \in C_c^\infty(\mathbb{R}^d, [0,\infty))$, with $U(q) = 0$ for q near zero, by summing scaled translates of the single-site function $W \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$. We have $\int_{\mathbb{R}^d} W_a \, d\lambda^d = I$ for

$$W_a(q) := a^{-d} W(q/a) \quad (a > 0). \tag{7.2}$$

LEMMA 7.2. *For $I > 0$ in (7.1) and U as above, there exists a map $g \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ that, restricted to the interior of* $\mathrm{supp}(U)$*, is a diffeomorphism onto its (bounded) image* Im *such that the distributions*

$$D_\varepsilon := \varepsilon^d \sum_{\ell\in\varepsilon\mathbb{Z}^d\cap\mathrm{Im}} \delta_{g^{-1}(\ell)} \quad (\varepsilon > 0)$$

converge vaguely to the distribution $D_0 := U\lambda^d$ with density U with respect to Lebesgue measure λ^d for $\varepsilon \searrow 0$. Moreover, in uniform C^k norm, with $c := 1/(2(d+k+1))$,

$$I^{-1} \lim_{\varepsilon\searrow 0} D_\varepsilon * W_{\varepsilon^c} = U. \tag{7.3}$$

Proof. Write $U(q) = \tilde{U}(\|q\|)$ and set $\tilde{g}(r) := (d\int_0^r x^{d-1}\tilde{U}(x)\,dx)^{1/d}$.

Then, by the assumptions on U, the map $g\colon \mathbb{R}^d \to \mathbb{R}^d$, with $g(0) := 0$ and $g(q) := \tilde{g}(\|q\|)(q/\|q\|)$ otherwise, is zero in a neighbourhood of zero, $g \in C^k$ and bounded.

As $Dg(q) = \tilde{g}(\|q\|)(\mathbb{1} - P_q)/\|q\| + \tilde{g}'(\|q\|)P_q$ with the orthogonal projection P_q onto $\mathrm{span}(q)$, $\det(Dg)(q) = \tilde{g}'(\|q\|)\,(\tilde{g}(\|q\|)/\|q\|)^{d-1} = U(q)$. This shows that $g\restriction_{\mathrm{int}(\mathrm{supp}(U))}$ is a diffeomorphism onto Im.

The image measure of $\mathbb{1}_{\mathrm{Im}}\,\lambda^d$ under g^{-1} is $\det(Dg)\,\lambda^d$, and $\varepsilon^d \sum_{\ell\in\varepsilon\mathbb{Z}^d} \delta_\ell \to \lambda^d$. So $\lim_{\varepsilon\searrow 0} D_\varepsilon = D_0$.

To prove the statement (7.3) on the C^k norm, we notice that $\varepsilon \mapsto I^{-1}W_{\varepsilon^c}$ is an approximation to the delta distribution δ_0, and

$$I^{-1}\int_{\mathrm{Im}} W_{\varepsilon^c}(q - g^{-1}(x))\,dx = U(q) + \mathcal{O}(\varepsilon^c), \tag{7.4}$$

with corresponding formulae for the derivatives. Then (7.3) follows by a direct estimate. □

We need to scale natural Hamiltonian systems

$$H\colon P_{\mathrm{ext}} \to \mathbb{R}, \quad H(t,p,q) = \tfrac{1}{2}\|p\|^2 + V(q),$$

with flows on extended phase space $P_{\mathrm{ext}} := \mathbb{R}_t \times T^*\mathbb{R}^d_q$. We use three types of scalings, which all transform the natural Hamiltonian and its flow in a simple way.

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(1) *Motions* $M : \mathbb{R}^d_q \to \mathbb{R}^d_q$, $q \mapsto Oq + v$, with $O \in \mathrm{SO}(d)$, $v \in \mathbb{R}^d$, lift to symplectic maps M^* on $T^*\mathbb{R}^d_q$, respectively $\mathcal{M} := \mathrm{Id}_t \times M^*$ on P_{ext} and

$$\tilde{\mathcal{M}} : C^k(P_{\mathrm{ext}}, \mathbb{R}) \to C^k(P_{\mathrm{ext}}, \mathbb{R}), \quad \tilde{\mathcal{M}} H := H \circ \mathcal{M}.$$

(2) *Spatial scalings (L^∞ dilations)* $S_c : P_{\mathrm{ext}} \to P_{\mathrm{ext}}$, $S_c(t, p, q) := (t/c, p, cq)$, and

$$\tilde{S}_c : C^k(P_{\mathrm{ext}}, \mathbb{R}) \to C^k(P_{\mathrm{ext}}, \mathbb{R}), \quad \tilde{S}_c H := H \circ S_c \quad (c > 0).$$

(3) *Energy scalings* $\mathcal{E}_e : P_{\mathrm{ext}} \to P_{\mathrm{ext}}$, $\mathcal{E}_e(t, p, q) := (\sqrt{e}t, p/\sqrt{e}, q)$, and

$$\tilde{\mathcal{E}}_e : C^k(P_{\mathrm{ext}}, \mathbb{R}) \to C^k(P_{\mathrm{ext}}, \mathbb{R}), \quad \tilde{\mathcal{E}}_e H := eH \circ \mathcal{E}_e \quad (e > 0).$$

Notice that, regarding a single-site potential W as a function on P_{ext}, W_a in (7.2) is a combination of spatial and energy scalings of W.

Proof of Theorem 7.1 (The case $I > 0$). (i) Given $E \in \mathbb{R}^+$, we choose $\tilde{U} \in C^k_c([0, \infty), \mathbb{R})$ with $\tilde{U}(r) = 0$ for all r smaller than $2\,\mathrm{diam}(\mathrm{supp}(W))$ and having one maximum, with height $\max(\tilde{U}) \geq 2E$. Then $U = \tilde{U} \circ \|\cdot\|$ has the property that for all $e \in (0, E]$ the energy surface $\{(p, q) \in T^*\mathbb{R}^d \,|\, (1/2)\|p\|^2 + U(q) = e\}$ has two components.

(ii) By Lemma 7.2 for some $\varepsilon \in (0, \min\{1, I^{((1-c)d)^{-1}}\}]$ we find an approximation of U of the form $I^{-1} D_\varepsilon * W_{\varepsilon^c}$ with the same property. Then, by the combination of spatial and energy scalings, the sum

$$\sum_{\ell \in \varepsilon\mathbb{Z}^d \cap \mathrm{Im}} W\left(\cdot - \frac{g^{-1}(\ell)}{\varepsilon^c}\right) = \tilde{\mathcal{E}}_{I\varepsilon^{-(1-c)d}} \circ \tilde{S}_{\varepsilon^c}(\varepsilon^{-d} D_\varepsilon * W_1)$$

of translated single-site potentials has that property for all energies $e \in (0, I\varepsilon^{-(1-c)d}E]$, too. Since $\varepsilon^{(1-c)d} \leq I$, $E \in (0, I\varepsilon^{-(1-c)d}E]$.

(iii) For any realization $\omega \in \Omega$ that inside a ball of large radius is near enough to $\sum_{\ell \in \mathcal{L}} \delta_{(\ell, 1)}$, with $\mathcal{L}$ as above, the energy surface $\Sigma_{E,\omega}$ consists of at least two connected components. Thus the motion on $\Sigma_{E,\omega}$ is not ergodic β-a.s. □

Proof of Theorem 7.1 (The case $I < 0$). (i) We first treat the case $d = 2$ and then indicate the modifications needed for larger dimensions.

We use Lemma 7.2 in order to approximate a *non-positive* centrally symmetric function U by summing scaled translates of the single-site function W. We first assume that the energy is positive, more specifically $E = 1$. We write again $U(q) = \tilde{U}(\|q\|)$ and choose the profile $\tilde{U}$ so that there is a circular periodic orbit

$$t \mapsto q(t) = r\begin{pmatrix}\cos(\omega t)\\ \sin(\omega t)\end{pmatrix}$$

of energy E with $\omega > 0$ in the potential U.

We thus consider the effective potential $\tilde{U}_\ell(r) := \tilde{U}(r) + (\ell^2/2r^2)$ with angular momentum parameter $\ell \in \mathbb{R}$. The conditions for a circular periodic orbit of radius r are $E = \tilde{U}_\ell(r)$ and $\tilde{U}'_\ell(r) = 0$.

In order to control the stability of the orbit (make the linearized flow elliptic and let the frequency vary with the perturbation), we demand $\tilde{U}''_\ell(r) > 0$ and, say, $\tilde{U}^{(4)}_\ell(r) \neq 0$.

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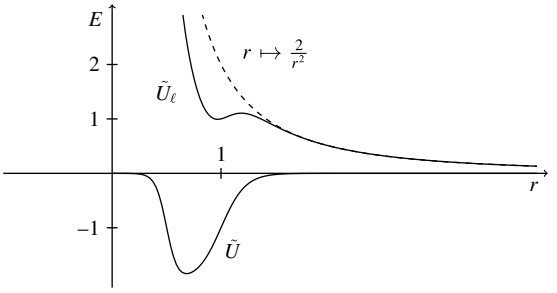


FIGURE 3. The effective radial potential $\tilde{U}_\ell$.

All these conditions can be satisfied by, for example, first choosing $r := 1$, $\ell := 2$, and then finding an appropriate $\tilde{U} \in C_c^\infty([0, \infty), [-1, 0])$ with $\tilde{U}(r) = 0$ in a neighbourhood of zero, $\tilde{U}(1) = -1$, meeting the assumptions on the derivatives and angular frequency $\omega = 2$; cf. Figure 3.

(ii) Then by Kolmogorov–Arnol'd–Moser (KAM) theory, the Hamiltonian flows with potentials C^k-near to U have, too, an elliptic orbit of energy $E = 1$, surrounded by invariant tori of positive measure; see e.g. Arnol'd [**Arn78**, Appendix 8]. According to Pöschel [**Pös82**] the differentiability condition $W \in C^k(\mathbb{R}^d, \mathbb{R})$ with $k = 3d$ suffices.

Let $\varepsilon \in (0, 1)$ be small enough so that the potential $\hat{U} := |I|^{-1} D_\varepsilon * W_{\varepsilon^c}$ (with D_ε from Lemma 7.2) meets that condition for KAM theory. Then with the scaling (7.2) of $\hat{U}$, $|I|\varepsilon^{-(1-c)d}\hat{U}_{\varepsilon^{-c}}$ is a sum of translated single-site potentials meeting that condition for energy $e := |I|\varepsilon^{-(1-c)d}$. This proves the claim for large enough energies e.

(iii) In order to solve the problem for an arbitrary smaller energy $E < e$, we first add to U a function $\Delta U \in C_c^\infty(\mathbb{R}^d, (-\infty, 0])$, which has a constant value $\Delta E := e^{-1}E - 1 = |I|^{-1}\varepsilon^{(1-c)d}(E - e) < 0$ on the support of U. Over $\mathrm{supp}(U)$, the flow at energy 1 with potential U equals the flow for energy $|I|^{-1}\varepsilon^{(1-c)d}E$ with potential $U + \Delta U$.

(iv) The case of dimension $d \geq 3$ cannot be treated in a completely analogous way, since then motion in phase space generated by rotations from $\mathrm{SO}(d)$ not only leaves the Hamiltonian invariant, but also can transform a circular orbit to *different* circular orbits. This shows that in the integrable problem periodic orbits of constant radius have degenerate transverse frequencies.

KAM theory can be applied after lifting this degeneracy. One way to do this is to construct a potential that near the circle

$$\{q \in \mathbb{R}^d \mid q_1^2 + q_2^2 = 1, q_3 = \cdots = q_d = 0\}$$

is of the form $\tilde{U}(q_1^2 + q_2^2) + \hat{U}(q_3, \ldots, q_d)$, with $\hat{U}(q_3, \ldots, q_d) = \sum_{m=3}^d (\omega_m q_m^2 + \tau_m q_m^4)$ integrable, non-resonant and non-degenerate. Since such a U locally is the sum of functions of the coordinates, U can again be locally approximated in a manner similar to Lemma 7.2. □

Proof of Theorem 7.1 (The case $I = 0$). (i) Energies $E \leq 0$ again lead to bounded motion.

(ii) By injectivity of Radon transform (see e.g. Natterer [**Nat01**, Theorem 2.1]), there exists a hyperplane $\mathcal{H} \subseteq \mathbb{R}^d$ (perpendicular to some $e_1 \in S^{d-1}$) so that $I' := \int_{\mathcal{H}} W \, d\lambda^{d-1} \neq 0$. By a translation of all the single-site potentials W_j (which does not

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change the class of Poisson Hamiltonians), we can assume that $0 \in \mathcal{H}$. By our assumption $I = \int_{\mathbb{R}^d} W \, d\lambda^d = 0$ we can even assume that $I' > 0$.

We supplement e_1 to an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$ and define the lattice $\mathcal{L} := \operatorname{span}_{\mathbb{Z}}(e_2, \ldots, e_d)$. Then for $r > 0$ large and the ball $B_r^d \subseteq \mathbb{R}^d$ of radius r, the linear combination

$$\widehat{W}_\varepsilon(q) := \varepsilon^{d-1} \sum_{\ell \in \varepsilon \mathcal{L} \cap B_{2r}^d} W(q - \ell) \quad (q \in \mathbb{R}^d) \tag{7.5}$$

converges, as $\varepsilon \to 0$, in C^k sense to $\widehat{W}_0 \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ with $\widehat{W}_0(q) = \int_{\mathcal{H}+q_1 e_1} W(x) \, dx$ for all q with $\|q - q_1 e_1\| \le r$ (setting $q_1 := \langle q, e_1 \rangle$). So inside the cylinder $\mathcal{H} \cap B_r + \operatorname{span}_{\mathbb{R}}(e_1)$, $\widehat{W}_\varepsilon$ is nearly invariant under translation perpendicular to e_1, and $\widehat{W}_\varepsilon(q) > 0$ if $q_1 = 0$.

For orbits of regular energy $E \in (0, I')$ entering supp $\widehat{W}_\varepsilon$ with velocity nearly parallel to e_1 and position q within the cylinder, the potential acts like a nearly planar mirror.

By translating two such potentials by $\pm R e_1$ with $R \gg r$, we get a system of two mirrors. Between these mirrors trajectories of appropriate energy bounce back and forth near the axis $\operatorname{span}_{\mathbb{R}}(e_1)$ for a long time.

To make the motion bounded for a set of initial conditions of positive measure, we give the mirrors inside curvatures strictly smaller than $1/R$. This can be done by changing the summand in (7.5) to $W(q - \ell - Q(\ell)e_1)$ with an appropriate quadratic form Q. Under this condition the orbit on the axes becomes linearly elliptic for a potential with perfect axial symmetry. Then by KAM theory and appropriate scaling we get the result for all positive energies. □

8. *No hyperbolicity for bounded potentials*

By far the simplest ergodic flows or maps are the uniformly hyperbolic ones. Examples include hyperbolic torus automorphisms and geodesic flows on compact manifolds of negative sectional curvature.

Although it is known that there exist severe *topological* obstructions against a flow on a manifold being Anosov, as shown in Example 8.1 below, these do not apply to the motion in a potential on configuration space $\mathbb{R}^d$. However, by Theorem 8.4 below, *geometric* obstructions exist if the potential is bounded.

There exist examples of ergodic motion in smooth bounded potentials (which are not uniformly hyperbolic); see Donnay and Liverani [**DL91**]. So our theorem does not exclude ergodicity for concrete smooth potentials on $\mathbb{R}^d$ and some energies. But it shows that it would be more difficult to prove ergodicity, and we would not expect ergodicity for open energy intervals.

For $d \ge 2$ we consider the flow $\Phi\colon \mathbb{R} \times P \to P$ on phase space $P := \mathbb{R}_p^d \times \mathbb{R}_q^d$ generated by the natural Hamiltonian function

$$H \in C^2(P, \mathbb{R}), \quad H(p, q) = \tfrac{1}{2}\|p\|^2 + V(q),$$

where V and its first and second derivatives are assumed to be bounded. This in particular ensures that $\Phi \in C^2(\mathbb{R} \times P, P)$.

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We call the flow $\Phi_E\colon \mathbb{R}\times\Sigma_E\to\Sigma_E$ restricted to an energy surface $\Sigma_E := H^{-1}(E)$ *Anosov* if there is a $d\Phi_E$-invariant splitting

$$T_x\Sigma_E = \operatorname{span}(X_H(x))\oplus E^u(x)\oplus E^s(x)$$

into a one-dimensional bundle spanned by the Hamiltonian vector field X_H, and the strong (un)stable bundles $E^{u/s}$, along which $d\Phi_E$ is exponentially contracting in backward, respectively forward, time. Even if Σ_E is not compact, this is unambiguously defined by the existence of $C\geq 1$ and $\lambda>0$ with

$$\|(d\Phi_E^t)_x(v)\|_{\Phi_E(t,x)}\leq C\exp(-\lambda t)\|v\|_x\quad (x\in\Sigma_E,\, v\in E^s(x),\, t\in[0,\infty))$$

(and analogously for E^u) if we take translation-invariant norms $\|\cdot\|_x$ on the tangent spaces T_xP in the bundle TP.

Example 8.1. For the potential $V(q)=-(1/2)\|q\|^2$ $(q\in\mathbb{R}^d)$, the flow equals

$$\begin{pmatrix}p(t)\\ q(t)\end{pmatrix}=\begin{pmatrix}\mathbb{1}\cosh(t) & \mathbb{1}\sinh(t)\\ \mathbb{1}\sinh(t) & \mathbb{1}\cosh(t)\end{pmatrix}\begin{pmatrix}p(0)\\ q(0)\end{pmatrix}\quad (t\in\mathbb{R}).$$

So we can take $\lambda=C=1$ and $E^{u/s}(x)=\left\{\begin{pmatrix}a\\ \pm a\end{pmatrix}\mid a\in\mathbb{R}^d\right\}$. We see that on P there is no *topological* obstruction against the motion generated by H to be Anosov (but we remark that here $\dim(E^{u/s}(x))=d$).

The following statement follows from specializing a theorem in [**PP94**].

THEOREM 8.2. (Paternain and Paternain) *For no value $E<\sup_q V(q)$ is the flow Φ_E on Σ_E non-wandering and Anosov.*

Proof. (i) To be Anosov, E must be a regular value of H or, equivalently, of V, since otherwise there are points on Σ_E where the Hamiltonian vector field X_H vanishes. So one assumes that E is a regular value, that Φ_E is non-wandering and Anosov, and derives a contradiction.

(ii) Regularity of E is one of the assumptions of [**PP94**, Theorem 3]. The condition of existence of a Φ_E-invariant Lagrangian subbundle E of $T\Sigma_E$ is met, too, by the centre stable bundle, with $E(x):=\operatorname{span}(X_H(x))\oplus E^s(x)\subseteq T_x\Sigma_E$.

(iii) As a conclusion [**PP94**, Theorem 3] states that E trivially intersects the vertical bundle Vert, given for $x\in P\cong T^*\mathbb{R}^d_q$ by $\operatorname{Vert}(x)=\ker(d\pi_x)$, with the projection $\pi\colon T^*\mathbb{R}^d_q\to\mathbb{R}^d_q$, $(p,q)\mapsto q$. This implies that $E\geq\sup_q V(q)$. For otherwise there is a point $x=(p,q)\in\Sigma_E$ with $V(q)=E$. By regularity of the value E then $0\neq X_H(x)\in\operatorname{Vert}(x)$. □

Remark 8.3. (The non-wandering condition)

(1) If one wants to show ergodicity using the Anosov property, then the non-wandering condition is somewhat natural.

The Liouville measure μ_E on Σ_E is invariant under the flow Φ_E. For a regular value E of H, μ_E is absolutely continuous with respect to the Riemannian measure. The flow is called *ergodic* (in the sense of Aaronson [**Aar97**]) if every Φ_E-invariant

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measurable subset A of Σ_E is of measure zero or the complement of a measure zero set.

Under the assumption of ergodicity, the non-wandering set of Φ_E equals Σ_E. Assume that $x \in \Sigma_E$ is wandering. Then by definition there is an open neighbourhood $U \subseteq \Sigma_E$ of x and $T > 0$, so that $\Phi_E(t, U) \cap U = \emptyset$ if $|t| \geq T$. Since $\dim(\Sigma_E) > 1$, there is a neighbourhood $W \subseteq U$ of x so that $\mu_E(\bigcup_{t\in[-T,T]} \Phi_E(t, W)) < \mu_E(U)$. So both the Φ_E-invariant set $\bigcup_{t\in\mathbb{R}} \Phi_E(t, W)$ and its complement have positive measures, contradicting ergodicity.

(2) Besides that, there are many alternatives to the non-wandering condition in the above theorem.

One choice is to assume that the boundary $V^{-1}(E) \subseteq \mathbb{R}^d_q$ of Hill's region contains a compact component that is not diffeomorphic to S^{d-1}, or more than one compact component. Then there exists a closed (*brake*) orbit with positive Maslov class, contradicting the existence of a section of the Lagrangian bundle (see [**Kna90**, Theorem 2] and [**KK08**, §6]).

Another such alternative is the assumption that there exists an $e < E$ such that for every $r > 0$ there exists a ball $B_r(Q) \subseteq \mathbb{R}^d_q$ with $V\restriction_{B_r(Q)} \leq e$. Then the proof of Theorem 8.4 below can be adapted.

For large energies we do not need the non-wandering condition.

THEOREM 8.4. *For no $E > \sup_q V(q)$ the flow Φ_E on Σ_E is Anosov.*

Proof. (i) For $E > \sup_q V(q)$ we use Riemannian geometry. The metric g of a Riemannian manifold (M^d, g) defines a connection and thus a canonical decomposition of TTM into a horizontal and a vertical subspace:

$$T_x TM = T_{x,h} TM \oplus T_{x,v} TM \quad (x \in TM).$$

Both $T_{x,h}TM$ and $T_{x,v}TM$ are canonically isomorphic to the d-dimensional vector space T_qM (for $x \in T_qM$).

For a Lagrangian subspace $\lambda \subseteq T_xTM$ that is transversal to the vertical subspace, there exists a symmetric operator

$$S\colon T_{x,h}TM \to T_{x,v}TM \tag{8.1}$$

such that the vertical and horizontal components of any vector $w = w_h + w_v \in \lambda$ obey the relation (see e.g. Klingenberg [**Kli95**, Proposition 3.2.16])

$$w_v = Sw_h.$$

The covariant derivative $\nabla Y(t)$ of a stable Jacobi field $Y(t)$ along a geodesic trajectory equals $S(t)Y(t)$. Hence the operator S satisfies the Riccati equation

$$S^2 = -\nabla S - R_X \tag{8.2}$$

along the geodesic. In our case we use on $M := \mathbb{R}^d_q$ the Jacobi–Maupertuis metric g, with $g(q) := (E - V(q)) \cdot g_{\text{Euclid}}(q)$. Up to a reparametrization of time t, the geodesics in this metric coincide with the projection of the solutions $t \mapsto \Phi_E(t, x)$ of our Hamiltonian equation to configuration space M. Since $E - V(q)$ is bounded from below and above

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by positive constants, the derivative of the time reparametrization is bounded below and above, too. We denote the geodesic flow by

$$\Psi : \mathbb{R} \times T_1 M \to T_1 M.$$

By [**PP94**, Theorem 3] we can write the Lagrangian subbundle E as the graph of a symmetric operator-valued function of the form (8.1).

We integrate the trace of (8.2) over the unit tangent bundle $T_1 B_r$ of the ball $B_r = B_r(0) \subseteq \mathbb{R}^d$ of radius r.

(ii) The integral of the covariant derivative is of order

$$\int_{T_1 B_r} \operatorname{trace}(\nabla S)\, dm\, do = \mathcal{O}(r^{d-1}), \tag{8.3}$$

where we denote by $dm(q) = \sqrt{\det g(q)}\, dq_1 \wedge \cdots \wedge dq_d$ the measure on M and by do the measure on the unit sphere ($\int_{S^{d-1}} do = \operatorname{vol}(S^{d-1})$). We show (8.3) by reducing it to a term scaling with the volume of the boundary ∂B_r of the ball. To this end we decompose the region $T_1 B_r$ of the energy surface into

$$T_1 B_r = \mathcal{S} \,\dot\cup\, \mathcal{B} \,\dot\cup\, \mathcal{T}$$

using the maximal time interval $I(x)$ containing 0 for which the geodesic flow line through x stays within $T_1 B_r$:

- the *scattering set*† $\mathcal{S} := \{x \in T_1 B_r \mid I(x) = [T^-(x), T^+(x)]\}$;
- the *bounded set* $\mathcal{B} := \{x \in T_1 B_r \mid I(x) = \mathbb{R}\}$; and
- the *trapped set* $\mathcal{T} := T_1 B_r \setminus (\mathcal{B} \cup \mathcal{S})$.

All three sets are measurable.

- The trapped set $\mathcal{T}$ consists of wandering points and thus is of measure zero.
- The bounded set $\mathcal{B}$ is Ψ-invariant. So we can use the relation

$$\int_0^T \operatorname{trace}(\nabla S \circ \Psi(t, x))\, dt = \operatorname{trace}(S \circ \Psi(T, x) - S(x)) \tag{8.4}$$

  to show that

$$\begin{aligned} T \int_{\mathcal{B}} \operatorname{trace}(\nabla S)\, dm\, do &= \int_{\mathcal{B}} \int_0^T \operatorname{trace}(\nabla S \circ \Psi(t, x))\, dt\, dm\, do \\ &= \int_{\mathcal{B}} \operatorname{trace}(S \circ \Psi(T, x) - S(x))\, dm\, do = 0. \end{aligned}$$

- So the only contribution to (8.3) comes from the scattering set $\mathcal{S}$. Every $x \in \mathcal{S}$ can be uniquely written as $x = \Psi(t, y)$ with $t \in [0, T^+(y)]$ and $T^-(y) = 0$. Conversely, for the points in $\mathcal{V} := \{y \in \mathcal{S} \mid T^-(y) = 0\}$ all $\Psi(t, y)$ with $t \in [0, T^+(y)]$ are in $\mathcal{S}$. So we rewrite the integral:

$$\int_{\mathcal{S}} \operatorname{trace}(\nabla S)\, dm\, do = \int_{\mathcal{V}} \int_0^{T^+(y)} \operatorname{trace}(\nabla S \circ \Psi(t, y)) J(y)\, dt\, dy.$$

  Since the Jacobian $J : \mathcal{V} \to \mathbb{R}^+$ is bounded above by 1, and $\operatorname{trace}(S)$ is bounded on $T_1 M$, we obtain (8.3), reusing (8.4).

† The names should not be taken too seriously, since, for example, the intersection of a Ψ-orbit with $\mathcal{S}$ can consist of several components.

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(iii) For the second term on the right-hand side of (8.2),

$$\int_{T_1 B_r} \operatorname{trace}(R_X)\, dm\, do = \frac{\operatorname{vol}(S^{d-1})}{d} \int_{B_r} \mathcal{R}(q)\, dm,$$

where $\mathcal{R}(q)$ denotes the scalar curvature. If the motion takes place on a two-dimensional plane $M = \mathbb{R}^2_q$, then $\int_{B_r} \mathcal{R}(q)\, dm = \mathcal{O}(r)$ as a consequence of the Gauss–Bonnet formula. For dimension $d \geq 3$, that equality is wrong in general. But in our case the Jacobi metric is conformally flat. Defining the positive function $u\colon M \to \mathbb{R}^+$ by $u(q) := (E - V(q))^{(d-2)/4}$, the measure dm on M equals $dm = u^{2d/(d-2)} dq_1 \wedge \cdots \wedge dq_d$. The scalar curvature equals

$$\mathcal{R} = \frac{1-d}{(E-V)^3}\left[(V-E)\Delta V + \frac{d-6}{4}(\nabla V)^2\right] = 4\frac{1-d}{d-2} u^{-(d+2)/(d-2)} \Delta u$$

(with the Euclidean Laplacian $\Delta = \sum_{k=1}^d (\partial^2/(\partial q_k^2))$). Therefore

$$\begin{aligned}\int_{B_r} \mathcal{R}\, dm &= -4\frac{d-1}{d-2} \int_{B_r} u^{-(d+2)/(d-2)} (\Delta u) u^{2d/(d-2)} dq_1 \wedge \cdots \wedge dq_d \\ &= -4\frac{d-1}{d-2} \int_{B_r} u(\Delta u)\, dq_1 \wedge \cdots \wedge dq_d \\ &= +4\frac{d-1}{d-2} \int_{B_r} (\nabla u)(\nabla u)\, dq_1 \wedge \cdots \wedge dq_d + f(r) \geq f(r). \end{aligned} \tag{8.5}$$

The surface integral is of order $f(r) = \mathcal{O}(r^{d-1})$. Concerning the left-hand side of (8.2), the integral of $\operatorname{trace}(S^2)$ is positive. For uniform hyperbolicity, one would need $\limsup_{r\to\infty} r^{-d} \int_{B_r} \operatorname{trace}(S^2)(q)\, dm > 0$, which is impossible, since the corresponding lim sup of the right-hand side is non-positive. So (8.5) is compatible with (8.2) only if the flow is not Anosov. □

9. *Random Coulombic potentials*

As it does not seem to be so simple to find smooth random potentials that lead to ergodic motion, we now study the example of Coulombic potentials; see Figure 4 for a numerical realization. We restrict ourselves to dimension $d = 2$. Here for $j \in J$ the single-site potentials

$$\tilde{W}_j \in C^\eta(\mathbb{R}^2 \backslash \{s_j\}, \mathbb{R})$$

(with $\eta \in \mathbb{N} \cup \{\infty\}$, $\eta \geq 2$) diverge at the position $s_j \in \mathcal{D}$ in the fundamental domain $\mathcal{D} := \{x_1\ell_1 + x_2\ell_2 \mid 0 \leq x_i < 1\}$ of the lattice $\mathcal{L} = \operatorname{span}_{\mathbb{Z}}(\ell_1, \ell_2)$.

Our assumptions are as follows:

(1) the *decay at infinity* is short-range, that is, for $\alpha \in \mathbb{N}_0^2$, $|\alpha| \leq \eta$,

$$\partial^\alpha \tilde{W}_j(q) = \mathcal{O}(\|q\|^{-2-\varepsilon}) \quad (\|q\| \to \infty);$$

(2) the *local singularity* at $s_j \in \mathbb{R}^2 \cong \mathbb{C}$ is controlled by

$$f_j\colon \mathbb{C}^* := \mathbb{C}\backslash\{0\} \to \mathbb{R}, \qquad f_j(z) := |z|^2 \tilde{W}_j(z^2 + s_j)$$

and we assume that for all multi-indices $\alpha \in \mathbb{N}_0^2$, $|\alpha| \leq \eta$, $\partial^\alpha f_j$ can be continuously extended to zero, with $f_j(0) < 0$. We allow for an additional single-site potential $\tilde{W}_0 = 0$.

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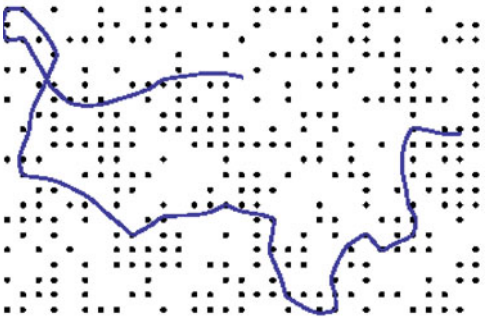


FIGURE 4. Motion in the configuration space of a random Coulombic potential.

Example 9.1. The *Yukawa potential* with parameters $c_j,\ \mu_j > 0$ is defined via

$$\tilde{W}_j(q) = -c_j \frac{\exp(-\mu_j \|q - s_j\|)}{\|q - s_j\|}.$$

Example 9.2. Finite-range potentials are given by

$$\tilde{W}_j(q) := -\frac{g_j(\|q - s_j\|)}{\|q - s_j\|},$$

with $g_j \in C_c^\eta(\mathbb{R}, \mathbb{R}),\ g_j(0) > 0$.

The random potential is determined by the probability space $(\Omega, \mathcal{B}(\Omega), \beta)$ with $\Omega = \mathcal{L}^J$. The probability measure β is assumed to be ϑ-invariant (see equation (2.2)) and to give probability $\beta\{\omega_0\} = 0$ to the configuration $\omega_0 \in \Omega$ with $\omega_0(\ell) = 0$, $\ell \in \mathcal{L}$. No $\mathcal{L}$-ergodicity of β is assumed here.

For what follows we fix $\omega \in \Omega$. The *punctured configuration space* $\tilde{M}_\omega := \mathbb{C}\backslash\mathcal{S}_\omega$ now depends on the *singularity set* $\mathcal{S}_\omega := \{s_{\omega(\ell)} + \ell \mid \ell \in \mathcal{L},\ \omega(\ell) \neq 0\}$. The former supports the random potential

$$\tilde{V}_\omega \colon \tilde{M}_\omega \to \mathbb{R}, \quad \tilde{V}_\omega(q) := \sum_{\ell \in \mathcal{L}} \tilde{W}_{\omega(\ell)}(q - \ell).$$

Assumption (1) guarantees the convergence of $\tilde{V}_\omega$ and its derivatives, and assumption (2) implies that the Coulombic singularities are attractive and

$$\tilde{V}_{\omega,\max} := \sup \tilde{V}_\omega(\tilde{M}_\omega) < \infty.$$

The Hamiltonian flow $\tilde{\Phi}_\omega \colon \tilde{U}_\omega \to \tilde{P}_\omega$ on $\tilde{P}_\omega := T^*\tilde{M}_\omega = \mathbb{R}^2 \times \tilde{M}_\omega$, generated by the Hamiltonian function $\tilde{H}_\omega \colon \tilde{P} \to \mathbb{R},\ (p, q) = (1/2)\|p\|^2 + \tilde{V}_\omega(q)$, is now, due to the Coulombic singularities, incomplete and only defined on a maximal open subset $\tilde{U}_\omega \subseteq \mathbb{R} \times \tilde{P}_\omega$.

It is known that the flow can be continuously regularized by reflecting collision orbits at their singularity. This is possible (see [**KK92**, Proposition 2.3] or [**Kna11**, Theorem 11.23]) by smoothly extending the incomplete Hamiltonian system

$$(\tilde{P}_\omega, dq \wedge dp, \tilde{H}_\omega) \quad \text{to a Hamiltonian system} \quad (P_\omega, \sigma_\omega, H_\omega),$$

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with $H_\omega\colon P_\omega \to \mathbb{R}$ generating a complete smooth Hamiltonian flow on the symplectic manifold $(P_\omega, \sigma_\omega)$. In fact, similarly to the construction in §2, these data extend continuously to a triple (P, σ, H), with extended phase space $P := \bigcup_{\omega\in\Omega} \{\omega\} \times P_\omega$.

For $d = 2$ and energies $E > V_{\max}$ regularization can also be performed, as in [**KK92, Kna87**], with the help of the twofold covering

$$\pi_\omega\colon \mathbf{M}_\omega := \left\{(q, Q) \in \mathbb{C}^2 \mid f_\omega(q) = Q^2\right\} \longrightarrow \mathbb{C} \tag{9.1}$$

with branch points in the singularity set $\mathcal{S}_\omega$, where $f_\omega\colon \mathbb{C} \to \mathbb{C}$ is an entire function with simple zeroes in and only in $\mathcal{S}_\omega$: $f_\omega^{-1}\{0\} = \mathcal{S}_\omega$. By a Weierstrass product construction we can choose $(\omega, q) \mapsto f_\omega(q)$ as a continuous function. The lift $\tilde{\mathbf{g}}_{\omega,E} := (\pi_\omega)^* \tilde{g}_{\omega,E}$ of the Jacobi–Maupertuis metric

$$\tilde{g}_{\omega,E}(q) := (1 - E^{-1}\tilde{V}_\omega(q)) g_{\text{Euclid}}(q) \quad (q \in \widehat{M}_\omega)$$

can be continued to all of $\mathbf{M}_\omega$ by taking limits:

$$\mathbf{g}_{\omega,E}(q, 0) := \lim_{\mathbf{M}_\omega \ni (q', Q) \to (q,0)} \hat{\mathbf{g}}_{\omega,E}(q', Q) \quad (\pi_\omega(q) \in \mathcal{S}_\omega)$$

(see [**Kna87, KK92**]). This gives a smooth and complete Riemannian metric on $\mathbf{M}_\omega$, and its geodesics are, up to a reparametrization of time and modulo π_ω, trajectories of the Hamiltonian flow.

This geometric regularization allows us to take full advantage of Riemannian geometry. For two-dimensional compact Riemannian manifolds of negative curvature, ergodicity of the geodesic flow was established by [**Hop41**]. The Gaussian curvature of $(\widehat{M}_\omega, \hat{g}_{\omega,E})$ at $q \in \widehat{M}_\omega$ is given by

$$\begin{aligned} \tilde{K}_{\omega,E}(q) &= \frac{(E - \tilde{V}_\omega(q))\Delta\tilde{V}_\omega(q) + (\nabla\tilde{V}_\omega(q))^2}{2(E - \tilde{V}_\omega(q))^3} \\ &= \frac{(1 - E^{-1}\tilde{V}_\omega(q))E^{-1}\Delta\tilde{V}_\omega(q) + (E^{-1}\nabla\tilde{V}_\omega(q))^2}{2E(1 - E^{-1}\tilde{V}_\omega(q))^3}. \end{aligned} \tag{9.2}$$

Since $\pi_\omega^{-1}(\tilde{M}_\omega)$ is a local isometry, the curvature of all non-branch points of $\mathbf{M}_\omega$ is determined by equation (9.2). In Example 9.1 it turns out that for high enough energy $E > E_{\text{th}} > \tilde{V}_{\omega,\max}$ the curvature is non-positive. This motivates the following definition.

Definition 9.3. The pair $(\tilde{V}_\omega, E)$ is of *non-positive (strictly negative) curvature*, if $(\mathbf{M}_\omega, \mathbf{g}_{\omega,E})$ has non-positive (strictly negative) curvature.

A direct consequence of equation (9.2) is that, given a pair $(\tilde{V}_\omega, E)$ of non-positive curvature, all $(\tilde{V}_\omega, E')$ with $E' \geq E$ are of non-positive curvature, too. As $E \nearrow \infty$ the curvature concentrates in the singularities. In Example 9.2 non-positive curvature can be achieved for all energies above a threshold with suitable choices for g_j, e.g.

$$g_j(r) := \begin{cases} -c_j \cos^{\eta+1}(\lambda_j r), & r < \dfrac{\pi}{2\lambda}, \\ 0, & r \geq \dfrac{\pi}{2\lambda}, \end{cases}$$

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with $c_j, \lambda_j > 0$; see [**Kna87**]. From now on we will assume non-positive curvature. This assumption and the following lemma explain why we allow only $W_0 = 0$ as smooth single-site potential.

LEMMA 9.4. *If a single-site potential $\tilde{W}\colon \mathbb{C} \to \mathbb{R}$ without singularity and an energy E are of non-positive curvature, then $\tilde{W} = \tilde{W}_0 \equiv 0$.*

Proof. From (2.1) it follows that the curvature of one single-site potential is integrable. By the Gauss–Bonnet theorem the integral of the curvature over $\mathbb{C}$ vanishes. Since the curvature is non-positive, it has to vanish, too, and so does the single-site potential. □

Our goal is expressed in the following theorem.

THEOREM 9.5. *If $(\tilde{V}_\omega, E)$ is of non-positive curvature, then the geodesic flow $\Phi_\omega\colon \mathbb{R} \times T_1\mathbf{M}_\omega \to T_1\mathbf{M}_\omega$ is* topologically transitive *β-a.s.*

The strategy for the proof will be as follows. In Proposition 9.8 we show that the periodic orbits of Φ_ω are dense in $T_1\mathbf{M}_\omega$. Note that the same is true for Anosov diffeomorphisms on compact manifolds, cf. [**Bow75**, 3.8: Anosov's closing lemma]. Then, in the proof of Theorem 9.5, in order to connect two open subsets of $T_1\mathbf{M}_\omega$, we will connect two periodic orbits with an intertwining orbit.

We start with a useful lemma, which singles out the set of full measure on which we establish topological transitivity. We denote the Euclidean cone intersected with the lattice $\mathcal{L}$

$$\triangle_q^{(\varphi)}(x) := \{\ell \in \mathcal{L} \mid \sphericalangle(x, \ell - q) < \varphi\},$$

with $q \in \mathbb{C}$, $\varphi \in [0, \pi]$ and $x \in T_{1,q}\mathbb{C} = S^1$.

LEMMA 9.6. *The ϑ-invariance of β and $\beta\{\omega_0\} = 0$ imply that there are β-a.s. no Euclidean cones without non-vanishing single-site potential. More precisely,*

$$\beta\{\omega \in \Omega \mid \exists (q, \varphi, x) \in \mathbb{R}^2 \times (0, \pi] \times S^1\colon \omega(\triangle_q^{(\varphi)}(x)) = \{0\}\} = 0.$$

Proof. We study the sets

$$A_{q,\varphi,x} := \{\omega \in \Omega \mid \omega(\triangle_q^{(\varphi)}(x)) = \{0\}\},$$

$(q, \varphi, x) \in \mathbb{R}^2 \times (0, \pi] \times S^1$, first. For $(q, \varphi, \ell) \in \mathbb{R}^2 \times (0, \pi] \times \mathcal{L}$ we have

$$\beta(A_{q,\varphi,\ell/\|\ell\|}) = \lim_{n\to\infty} \beta(A_{q-n\ell,\varphi,\ell/\|\ell\|}) = \beta\left(\bigcap_{n\in\mathbb{N}} A_{q-n\ell,\varphi,\ell/\|\ell\|}\right) = \beta\{\omega_0\} = 0.$$

Note now that the set in question can be written as the denumerable union

$$\bigcup_{(q,\varphi,\ell)\in\mathbb{Q}^2\times\mathbb{Q}_+\times\mathcal{L}} A_{q,\varphi,\ell/\|\ell\|}.$$ □

We denote the tangent bundle of $\mathbf{M}_\omega$ with $\tau_{\mathbf{M}_\omega}\colon T\mathbf{M}_\omega \to \mathbf{M}_\omega$ and the natural length metric on $(\mathbf{M}_\omega, \mathbf{g}_{\omega,E})$ with $d_{\mathbf{M}_\omega}$.

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LEMMA 9.7. *For β-almost all $\omega \in \Omega$ the following holds. Given a point $q \in \mathbf{M}_\omega$ and a non-empty open subset $U \subseteq T_{1,q}\mathbf{M}_\omega := T_1\mathbf{M}_\omega \cap T_q\mathbf{M}_\omega$ of the sphere bundle, there exists an initial direction $v \in U$ such that the corresponding geodesic hits a branch point at $t > 0$,* i.e.

$$\exp_{\mathbf{M}_\omega}(tv) \in \pi_\omega^{-1}(\mathcal{S}_\omega).$$

Proof. Without loss of generality, we assume $U \neq T_{1,q}\mathbf{M}_\omega$. For every $\omega \in \Omega$ such that there is an open subset $U \subseteq T_{1,q}\mathbf{M}_\omega$ with no branch points in

$$\blacktriangle := \exp_{\mathbf{M}_\omega}(\mathbb{R}_+ U) \subseteq \mathbf{M}_\omega,$$

we will establish the existence of a Euclidean cone $\triangle \subseteq \mathcal{L}$ with $\omega(\triangle) = \{0\}$. Then Lemma 9.6 applies and gives the desired result.

Fix $v \in U$. Since the curvature of $(\mathbf{M}_\omega, \mathbf{g}_{\omega,E})$ is non-positive, we find a constant $\rho_0 > 0$ such that

$$d_{\mathbf{M}_\omega}(\exp_{\mathbf{M}_\omega}(tv), \mathbf{M}_\omega \backslash \blacktriangle) \geq 2\rho_0 t \tag{9.3}$$

for all $t \geq 0$; see [**BBI01**]. Note that, due to the absence of branch points in $\blacktriangle$, $\pi_\omega|_\blacktriangle$ is a homeomorphism onto its image, and since the conformal factor is bounded by $h := 1 - E^{-1}\tilde{V}_{\omega,\max}$, we know for all $q, q' \in \mathbf{M}_\omega$ that

$$d_{\mathbf{M}_\omega}(q, q') \leq h\, |\pi_\omega(q) - \pi_\omega(q')|.$$

This implies that

$$\blacktriangle' := \bigcup_{t \geq 0} B_{h^{-1}\rho_0 t}(\pi_\omega \circ \exp_{\mathbf{M}_\omega}(tv)) \subseteq \pi_\omega(\blacktriangle) \subseteq \mathbb{C},$$

and in $\blacktriangle'$ we search for our Euclidean cone $\triangle$. To this end all we have to show is that the 'axis' of $\blacktriangle'$,

$$\mathbb{R}_+ \ni t \longmapsto (q_\omega(t),\, p_\omega(t)) := T\pi_\omega \circ \Phi_\omega(t, v),$$

converges to a definite direction: $\lim_{t_0 \to \infty} \sup_{t \geq t_0} |p_\omega(t + t_0, v) - p_\omega(t_0, v)| = 0$. But this is clear from assumption (1), i.e. that all single-site potentials are short-range, and equation (9.3), which together imply that the curvature along the axis vanishes. □

With these tools at hand it is easy to prove the following proposition.

PROPOSITION 9.8. *The set of periodic orbits of $\Phi_\omega \colon \mathbb{R} \times T_1\mathbf{M}_\omega \to T_1\mathbf{M}_\omega$ is dense in $T_1\mathbf{M}_\omega$ for β-a.e. $\omega \in \Omega$.*

Proof. To any non-empty open set $U \subseteq T_1\mathbf{M}_\omega$, we construct a periodic orbit whose trajectory intersects U. Lemma 9.7 guarantees that we hit a branch point at some time $t > 0$. We choose a branch point

$$x \in \exp_{\mathbf{M}_\omega}(tU) \cap \pi_\omega^{-1}(\mathcal{S}_\omega) \subseteq \mathbf{M}_\omega.$$

Now Φ_ω is continuous, which shows that

$$U' := -\Phi_\omega(t_1, U) \cap T_{1,x}\mathbf{M}_\omega$$

is open in $T_{1,x}\mathbf{M}_\omega$. Again by Lemma 9.7 we find $v \in U'$ and $t' > 0$ such that $\Phi_\omega(t', v)$ again hits a branch point. By construction, the trajectory of v intersects U and is periodic. □

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To gain more insight, we introduce the universal covering $\pi_\omega^*\colon \mathbf{M}_\omega^* \to \mathbf{M}_\omega$ of $\mathbf{M}_\omega$ and equip it with the Riemannian metric $\mathbf{g}_{\omega,E}^* := (\pi_\omega^*)^*\mathbf{g}_{\omega,E}$. This makes $\mathbf{M}_\omega^*$ a Hadamard manifold. We denote the natural length metric with $d_{\mathbf{M}_\omega^*}$. The fact that there is a certain amount of negative curvature in $\mathbf{M}_\omega^*$ is expressed in the following proposition.

PROPOSITION 9.9. *For β-almost all $\omega \in \Omega$ the Riemannian surface $(\mathbf{M}_\omega^*, \mathbf{g}_\omega^*)$ is a* visibility manifold*, i.e. for all $p \in \mathbf{M}_\omega^*$ and $\varepsilon > 0$ there exists $r = r(p, \varepsilon) > 0$ such that all geodesic segments $\sigma\colon [a, b] \to \mathbf{M}_\omega^*$ with distance at least r from p are seen from p under an angle less than ε (cf.* [**EO73**]*):*

$$\text{for all } p \in \mathbf{M}_\omega^*,\ \varepsilon > 0, \text{ there exists } r > 0\colon d_{\mathbf{M}_\omega^*}(p, \sigma[a, b]) \geq r \implies \sphericalangle_p(\sigma(a), \sigma(b)) < \varepsilon.$$

Proof. We need to show that the set of all $\omega \in \Omega$ which allow a positive ε such that we find for all $r > 0$ a geodesic segment $\sigma_r\colon [a_r, b_r] \to \mathbf{M}_\omega^*$ with (1) $d_{\mathbf{M}_\omega^*}(p, \sigma_r) > r$ and (2) $\sphericalangle_p(\sigma_r(a_r), \sigma_r(b_r)) \geq \varepsilon$ is of β-measure 0. To do this, we find a cone without singularities and invoke Lemma 9.7.

By compactness of $T_{1,p}\mathbf{M}_\omega^*$ we get $x, y \in T_{1,p}\mathbf{M}_\omega^*$ with $\sphericalangle_p(x, y) > \varepsilon$ and

$$d_{\mathbf{M}_\omega^*}(p, \gamma_t) \xrightarrow{t\to\infty} \infty,$$

where $\gamma_t\colon [a_t, b_t] \to \mathbf{M}_\omega^*$ is the unique geodesic segment connecting $\gamma_t(a_t) = \exp_{\mathbf{M}_\omega^*}(tx)$ and $\gamma_t(b_t) = \exp_{\mathbf{M}_\omega^*}(ty)$.

The Gauss–Bonnet theorem implies that the curvature integrated over the cone

$$\blacktriangle := \bigcup_{t>0} \gamma_t[a_t, b_t]$$

between x and y is bounded by $-\pi$ from below. This means that $\blacktriangle$ cannot cover two singularities, since then $\blacktriangle$, being geodesically convex, would cover the periodic orbit connecting these two, and this would contradict Gauss–Bonnet. Thereby $\blacktriangle$ contains a cone without singularity and Lemma 9.7 applies. □

Remark 9.10. Two geodesic rays $\gamma, \gamma'\colon \mathbb{R}_+ \to \mathbf{M}_\omega^*$ are called *asymptotic* if their distance is bounded. Being asymptotic is an equivalence relation, and the set of equivalence classes (equipped with a suitable topology, see [**EO73**]) is the *ideal boundary* $\partial\mathbf{M}_\omega^* \equiv \mathbf{M}_\omega^*(\infty)$ of $\mathbf{M}_\omega^*$. A point $\gamma(\infty) := [\gamma] \in \mathbf{M}_\omega^*(\infty)$ is a *zero point* if for all $\gamma' \in \gamma(\infty)$ the distance between γ and γ' vanishes. Eberlein and O'Neill [**EO73**] explain that a visibility manifold satisfies *Axiom 1*, i.e. any two distinct boundary points can be connected with a (not necessarily unique) geodesic.

LEMMA 9.11. *For all $\omega \in \Omega$ the following holds. Every lift of a periodic geodesic in $\mathbf{M}_\omega$ to $\mathbf{M}_\omega^*$ connects two zero points of $\mathbf{M}_\omega^*(\infty)$.*

Proof. Given a periodic geodesic $\gamma\colon S^1 \to \mathbf{M}_\omega$, its lift $\gamma^*\colon \mathbb{R} \to \mathbf{M}_\omega^*$, and another geodesic $\gamma'\colon \mathbb{R} \to \mathbf{M}_\omega^*$ with $\gamma^*(\infty) = \gamma'(\infty)$, we integrate curvature over the area bounded by $\gamma^*(\mathbb{R}_+)$, $\gamma'(\mathbb{R}_+)$ and the geodesic segment connecting $\gamma^*(0)$ and $\gamma'(0)$. Gauss–Bonnet assures us that this quantity is bounded, and this is only possible if γ^* and γ' have distance 0. Otherwise, since γ^* covers a periodic orbit, the curvature integral is unbounded. □

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Proof of Theorem 9.5. Proposition 9.8 tells us that for almost all $\omega \in \Omega$ periodic orbits of Φ_ω are dense in $T_1\mathbf{M}_\omega$. Therefore for any two open and non-empty sets $U, V \subseteq T_1\mathbf{M}_\omega$ we find two periodic geodesics $\gamma_U, \gamma_V : \mathbb{R} \to \mathbf{M}_\omega$ which intersect U and V, respectively, i.e. $\dot\gamma_U(\mathbb{R}) \cap U \neq \emptyset$ and $\dot\gamma_U(\mathbb{R}) \cap V \neq \emptyset$.

There are lifts $\gamma_U^*, \gamma_V^* : \mathbb{R} \to \mathbf{M}_\omega^*$ of γ_U and γ_V to the universal covering $\pi_\omega^* : \mathbf{M}_\omega^* \to M_\omega$ of $\mathbf{M}_\omega$. By Proposition 9.9 and Remark 9.10 the endpoints $\gamma_U^*(-\infty)$ and $\gamma_V^*(\infty)$ can be joined with a geodesic $\gamma^* : \mathbb{R} \to \mathbf{M}_\omega^*$, i.e. $\gamma^*(-\infty) = \gamma_U^*(-\infty)$ and $\gamma^*(\infty) = \gamma_V^*(\infty)$. Lemma 9.11 makes sure that the distance between $\gamma^*(t)$ and $\gamma_U^*((0, \infty))$, respectively $\gamma_V^*((-\infty, 0))$, vanishes as $t \to \pm\infty$, respectively

$$\lim_{t\to\infty} d_{\mathbf{M}_\omega^*}(\gamma^*(t), \gamma_U^*((0,\infty))) = \lim_{t\to-\infty} d_{\mathbf{M}_\omega^*}(\gamma^*(t), \gamma_V^*((-\infty,0))) = 0.$$

This implies that $T\pi_\omega^* \circ \dot\gamma^*(\mathbb{R})$ intersects U and V. □

10. *Ergodicity of the finite factor*

Similar to the lattice and the Poissonian case, the Coulombic system does have a finite factor. Assuming (ϑ, β) to be ergodic again and the single-site potentials to be $\eta \geq 3$ times continuously differentiable, we show ergodicity of this factor. This is analogous to a result in [**Len03**] in the setting of billiards. Thanks to the smoothness of our system, our proof can rely directly on [**Hop41**] without the need for technical generalizations to cope with singularities like, for example, in [**LW95**].

The following construction of the finite factor is carried out in detail in [**Sch04, Sch10**]. As above the geodesic motion on the energy surface is regularized with the twofold covering (9.1). There is one non-trivial deck transformation

$$\mathbf{G}_\omega : \mathbf{M}_\omega \to \mathbf{M}_\omega, \quad \mathbf{G}_\omega(q, Q) := (q, -Q).$$

For each $\omega \in \Omega$ and every $\ell \in \mathcal{L}$ there exists an isometry

$$\phi_{\omega,\ell} : \mathbf{M}_\omega \to \mathbf{M}_{\vartheta_\ell\omega}$$

that shifts q to $q - \ell$:

$$\pi_{\vartheta_\ell\omega} \circ \phi_{\omega,\ell}(q, Q) = q - \ell \quad ((q, Q) \in \mathbf{M}_\omega).$$

The deck transformation gives another such map $\phi_{\omega,\ell} \circ \mathbf{G}_\omega = \mathbf{G}_{\vartheta_\ell\omega} \circ \phi_{\omega,\ell}$. Before we can bundle the maps $((\phi_{\omega,\ell})_{\omega\in\Omega})_{\ell\in\mathcal{L}}$ into a group action of $\mathcal{L}$ on $\mathbf{M} := \bigcup_{\omega\in\Omega}\{\omega\} \times \mathbf{M}_\omega$, we have to divide through the group of deck transformations.

But π_ω is a branched covering, so to keep the differentiable structure we first follow [**Kna87, KK92**] and restrict the geodesic flow to the energy surface

$$\Sigma_{\omega,E} := \{(x, v) \in T\mathbb{C}^2 \mid \mathbf{g}_{\omega,E}(v, v) = 1\}.$$

We divide $\Sigma_{\omega,E}$ by the group $\mathbb{Z}_2 \cong \{\mathrm{Id}_{\Sigma_{\omega,E}}, D\mathbf{G}_\omega\restriction_{\Sigma_{\omega,E}}\}$ and get the smooth manifold

$$\Sigma_{\omega,E} := \Sigma_{\omega,E}/\mathbb{Z}_2.$$

The disjoint union

$$\Sigma_E := \bigcup_{\omega\in\Omega} \{\omega\} \times \Sigma_{\omega,E}$$

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$$\begin{array}{ccccccc} \Sigma^*_{\omega,E} & \xrightarrow{\Pi^*_\omega} & \Sigma_{\omega,E} & \longrightarrow & \Sigma_E & \longrightarrow & \Omega \times T\mathbb{C}^2 \\ & & \downarrow{\scriptstyle \pi_{\omega,\mathbb{Z}_2}} & & \downarrow{\scriptstyle \pi_{\mathbb{Z}_2}} & & \\ & & \Sigma_{\omega,E} & \xrightarrow{\iota_{\omega,E}} & \Sigma_E & \xrightarrow{\pi_{\mathcal{L}}} & \hat{\Sigma}_E \end{array}$$

FIGURE 5. Overview of the relations between the different phase spaces.

inherits its topology from the embedding into $\Omega \times T\mathbb{C}^2$. The quotient

$$\Sigma_E := \Sigma_E/\mathbb{Z}_2 = \bigcup_{\omega\in\Omega} \{\omega\} \times \Sigma_{\omega,E}$$

is the phase space for which we can construct the desired group action

$$\Theta^E \colon \mathcal{L} \times \Sigma_E \to \Sigma_E, \quad \Theta^E_\ell(\omega, [x]_{\mathbb{Z}_2}) := (\vartheta_\ell \omega, [D\phi_{\omega,\ell}(x)]_{\mathbb{Z}_2}).$$

This group action is continuous. Analogous to §2 the quotient

$$\pi_{\mathcal{L}} \colon \Sigma_E \to \hat{\Sigma}_E := \Sigma_E/\mathcal{L}$$

is compact and metrizable, inherits the geodesic flow

$$\hat{\Phi}_E \colon \mathbb{R} \times \hat{\Sigma}_E \to \hat{\Sigma}_E$$

and carries the finite and $\hat{\Phi}_E$-invariant Liouville measure $\hat{\mu}_E$; see [**Sch10**]. An overview of the different phase spaces including the embedding into $\Omega \times T\mathbb{C}^2$ and the unit tangent bundle $\Sigma^*_{\omega,E}$ of the universal cover $\pi^*_\omega \colon \mathbf{M}^*_\omega \to \mathbf{M}_\omega$ is given in Figure 5.

THEOREM 10.1. *Let the shift action $\vartheta \colon \mathcal{L} \times \Omega \to \Omega$ be ergodic with respect to β, the potential $\tilde{V}_\omega$ three times continuously differentiable and $(\tilde{V}_\omega, E)$ of strictly negative curvature for β-almost all $\omega \in \Omega$.*

Then the flow $\hat{\Phi}_E$ is ergodic with respect to $\hat{\mu}_E$.

Proof. In [**Hop41**] the ergodicity of a recurrent and hyperbolic geodesic flow on a two-dimensional Riemannian surface is proven. We use that work here.

We need to show that for every continuous function $\hat{f} \colon \hat{\Sigma}_E \to \mathbb{R}$ the limits

$$\bar{f}^{\pm} := \lim_{T\to\pm\infty} \frac{1}{T} \int_0^T \hat{f} \circ \hat{\Phi}_E(t, \cdot)\, dt \tag{10.1}$$

exist and are almost everywhere constant. The existence almost everywhere of $\bar{f}^{\pm}$ is guaranteed by Birkhoff's ergodic theorem, and so is $\bar{f}^+ = \bar{f}^- =: \bar{f}$ $\hat{\mu}_E$-almost everywhere.

Via the quotients and embeddings (see Figure 5) we introduce

$$f^*_\omega \colon \Sigma^*_{\omega,E} \to \mathbb{R}, \quad f^*_\omega := \hat{f} \circ \pi_{\mathcal{L}} \circ \iota_{\omega,E} \circ \pi_{\omega,\mathbb{Z}_2} \circ \Pi^*_\omega \quad (\omega \in \Omega).$$

By Poincaré's recurrence theorem the finite measure-preserving dynamical system $(\hat{\Sigma}_E, \hat{\Phi}_E, \hat{\mu}_E)$ is recurrent, so that $\liminf_{t\to\infty} |f^*_\omega \circ (\Phi^*_{\omega,E})_t - f^*_\omega| = 0$ almost everywhere; see e.g. [**Aar97**, pp. 14–16]. Hyperbolicity is guaranteed by the curvature assumption. Using the almost everywhere constancy of $\bar{f}^*_\omega$ along stable and unstable manifolds like in [**Hop41**] we see that for each $\omega \in \Omega$ the limit $\bar{f}^*_\omega$ is a.s. constant.

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Finally we use the ergodicity of (ϑ, β) to conclude that $\bar{f}$ is constant on $\hat{\Sigma}_E$. By denoting this almost everywhere value by $F(\omega)$ for β-a.a. $\omega \in \Omega$ we define β-almost everywhere a function $F\colon \Omega \to \mathbb{R}$. Since the whole construction depends measurably on $\omega \in \Omega$, F itself is measurable. To see that F is in fact ϑ-invariant, we introduce

$$f_{\Sigma_E} := \bar{f} \circ \pi_{\mathcal{L}}\colon \Sigma_E \to \mathbb{R} \quad (\omega \in \Omega) \tag{10.2}$$

and write for a.e. $\omega \in \Omega$, a.e. $x \in \Sigma_{\vartheta_\ell \omega}$ and a.e. $y \in \Sigma_\omega$,

$$F(\vartheta_\ell \omega) = f_{\Sigma_E} \circ \iota_{\vartheta_\ell \omega}(x) = f_{\Sigma_E} \circ \Theta^E \circ \iota_\omega(y) = f_{\Sigma_E} \circ \iota_\omega(y) = F(\omega),$$

using that f_{Σ_E} by definition is Θ^E-invariant; see (10.1) and (10.2). Therefore, by ergodicity of (ϑ, β), F is β-a.s. constant, which then implies that $\bar{f}$ is constant modulo $\hat{\mu}_E$. □

The above theorem relates to a geodesic flow on an (extended and compactified) unit tangent bundle. This construction has been made in order to make use of the well-known consequences of negative curvature on the dynamics.

Like in the case of smooth potentials treated in §6, we are finally interested in the (analogue of) the Hamiltonian flow (6.1) generated by $\hat{H}$ (based on the Hamiltonian $H\colon P \to \mathbb{R}$ on the extended phase space P of the first regularization in §9). As this flow is related to the geodesic flow of Theorem 10.1 by a continuous time change, we get the following corollary.

COROLLARY 10.2. *Under the conditions of Theorem 10.1 the Coulombic Hamiltonian flow (6.1) on the compactified energy surface $\hat{\Sigma}_E$ is ergodic with respect to Liouville measure $\hat{\lambda}_E$.*

Remark 10.3. (Markov partition) In a forthcoming article, one of us (CS) shows for all large energies, under some additional conditions, the existence of Markov partitions for the system; cf. [**Sch10**].

In [**CLS10**] Cristadoro *et al* show recurrence for random Lorenz tubes. Their configuration spaces are contained in a connected union of translates of a fundamental polygon. We make a similar statement for the Coulomb case.

Again with $H\colon P \to \mathbb{R}$ from §9, for $\ell \in \mathcal{L}\backslash\{0\}$ we define the factors

$$\check{\Sigma}_E := H^{-1}(E)/\mathrm{span}_{\mathbb{Z}}(\ell) \quad \text{and} \quad \check{\Sigma}_{E,\omega} := H_\omega^{-1}(E)/\mathrm{span}_{\mathbb{Z}}(\ell),$$

for ℓ-periodic $\omega \in \Omega$. Again we assume (V_ω, E) to be of strictly negative curvature β-a.s. As we divided by the free Θ-action of a subgroup of $\mathcal{L}$, by the analogue of (2.4) we obtain smooth flows on the non-compact 3-manifolds $\check{\Sigma}_{E,\omega}$.

We assume for simplicity that $\ell \in \mathcal{L}$ is primitive. Then $\mathcal{L}/\mathrm{span}_{\mathbb{Z}}(\ell) \cong \mathbb{Z}$ (and via the element ℓ' of a positively oriented basis (ℓ, ℓ') of $\mathcal{L}$, this isomorphism is unique). We may consider random potentials indexed by $\Omega_\ell := J^{\mathbb{Z}}$. The $\mathbb{Z}$-ergodic probability measures β_ℓ on Ω_ℓ give rise to $\mathcal{L}$-ergodic image measures β on Ω, via the injection $\mathbf{i}\colon \Omega_\ell \to \Omega$, $\mathbf{i}(\omega)_{a\ell+b\ell'} = \omega_b$. These are analogues of the measures considered in [**CLS10**]. For non-trivial β_ℓ the flow on $\check{\Sigma}_E$ is not ergodic with respect to the measure induced by $\check{\mu}_E$ via the projection $\check{\Sigma}_E \to \hat{\Sigma}_E$.

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PROPOSITION 10.4. (Ergodicity and recurrence of Coulomb tubes) *Under the conditions of Theorem 10.1 the following hold:*

(1) *β-a.s. the motion on $\check{\Sigma}_E$ is recurrent; and*

(2) *for $\mathbf{i}(\beta_\ell)$-almost all $\omega \in \Omega$ the motion on $\check{\Sigma}_{E,\omega}$ is recurrent and ergodic.*

Proof. By Corollary 10.2 the flow $\hat{\Phi}_E$ is ergodic with respect to $\hat{\lambda}_E$. Thus by the analogue of Proposition 6.5 for Coulomb potentials, for β-a.e. ω the asymptotic velocity satisfies $\overline{v}(x) = 0$ almost everywhere on $\Sigma_{E,\omega}$, and thus on $\check{\Sigma}_E$. If β is of the form $\mathbf{i}(\beta_\ell)$, then for β-a.e. ω the asymptotic velocity satisfies $\overline{v}(x) = 0$ almost everywhere on $\check{\Sigma}_{E,\omega}$.

We compare with Schmidt [**Sch98**] (and the references cited therein). In the case of an ergodic transformation T on a standard probability space (X, μ), a Borel map $f : X \to \mathbb{R}$ and the induced cocycle (orbit sum) $\hat{f} : \mathbb{Z} \times X \to \mathbb{R}$, he defines $\sigma_k(A) := \mu(\{x \in X \mid \hat{f}(k, x)/k \in A\})$ $(k \in \mathbb{N})$, for Borel sets $A \subseteq \mathbb{R}$.

Under the assumption that vaguely $\lim_{k\to\infty} \sigma_k = \delta_0$, he notes that f is *recurrent*, meaning that $\liminf_{k\to\infty} |\hat{f}(k, x)| = 0$, μ-a.s.

Here we use a Poincaré map discretization (X, μ, T) of the flow $\hat{\Phi}_E$ on $\hat{\Sigma}_E$ (by Remark 10.3 such a discretization exists and is ergodic). For $f(x_0)$, with $x_0 = [(\omega, p_0, q_0)] \in X$, we use the difference $\langle q(T(x_0)) - q_0, \ell^\perp \rangle$ in position along the direction $\ell^\perp := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \ell \in \mathbb{R}^2$ perpendicular to ℓ. The map f is well-defined on X.

The assumptions of [**Sch98**] apply and f is recurrent. This means that for a.e. x_0 the motion in the Coulomb tube returns infinitely often to every neighbourhood of the circle in the configuration torus given by $\langle q_0, \ell^\perp \rangle = 0$. Then one obtains, by Poincaré's recurrence theorem for the induced map, recurrence in the usual sense.

Ergodicity of the flow on $\check{\Sigma}_{E,\omega}$ follows from local ergodicity, using Hopf's argument [**Hop39**]. □

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REFERENCES

[Aar97] J. Aaronson. *An Introduction to Infinite Ergodic Theory (Mathematical Surveys and Monographs, 50)*. American Mathematical Society, Providence, RI, 1997.

[AK98] J. Asch and A. Knauf. Motion in periodic potentials. *Nonlinearity* **11**(1) (1998), 175–200.

[AM78] R. Abraham and J. E. Marsden. *Foundations of Mechanics*, 2nd edn. Benjamin/Cummings, Reading, MA, 1978.

[Arn78] V. I. Arnol'd. *Mathematical Methods of Classical Mechanics*. Springer, Berlin, 1978.

[BBI01] D. Burago, Y. Burago and S. Ivanov. *A Course in Metric Geometry (Graduate Studies in Mathematics, 33)*. American Mathematical Society, Providence, RI, 2001.

[Bow75] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. Springer, Berlin, 1975.

[BS81] L. A. Bunimovich and Y. G. Sinai. Statistical properties of Lorentz gas with periodic configuration of scatterers. *Comm. Math. Phys.* **78** (1981), 479–497.

[CLS10] G. Cristadoro, M. Lenci and M. Seri. Recurrence for quenched random Lorentz tubes. *Chaos* **20** (2010), 023115.

[DL91] V. Donnay and C. Liverani. Potentials on the two-torus for which the Hamiltonian flow is ergodic. *Comm. Math. Phys.* **135** (1991), 267–302.

https://doi.org/10.1017/etds.2012.141 Published online by Cambridge University Press

[EO73] P. Eberlein and B. O'Neill. Visibility manifolds. *Pacific J. Math.* **46** (1973), 45–109.
[GHL48] S. Gallot, D. Hulin and J. Lafontaine. *Riemannian Geometry*. Springer, Berlin, 1948.
[Hir76] M. W. Hirsch. *Differential Topology (Graduate Texts in Mathematics, 33)*. Springer, Berlin, 1976.
[Hop39] E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. *Ber. Verh. Sächs. Akad. Wiss. Leipzig* **91** (1939), 261–304.
[Hop41] E. Hopf. Statistik der Lösungen geodätischer Probleme vom unstabilen Typus II. *Math. Ann.* **117** (1941), 590–608.
[Kel98] G. Keller. *Equilibrium States in Ergodic Theory (London Mathematical Society Student Texts, 42)*. Cambridge University Press, Cambridge, 1998.
[KK92] M. Klein and A. Knauf. *Classical Planar Scattering by Coulombic Potentials (Lecture Notes in Physics, 13)*. Springer, Berlin, 1992.
[KK08] A. Knauf and M. Krapf. The non-trapping degree of scattering. *Nonlinearity* **21** (2008), 2023–2041.
[Kli95] W. P. A. Klingenberg. *Riemannian Geometry*, 2nd edn *(de Gruyter Studies in Mathematics, 1)*. de Gruyter, Berlin, 1995.
[Kna87] A. Knauf. Ergodic and topological properties of Coulombic periodic potentials. *Comm. Math. Phys.* **110** (1987), 89–112.
[Kna90] A. Knauf. Closed orbits and converse KAM theory. *Nonlinearity* **3** (1990), 961–973.
[Kna11] A. Knauf. *Mathematische Physik: Klassische Mechanik (Master Class)*. Springer, Berlin, 2011.
[Len03] M. Lenci. Aperiodic Lorentz gas: recurrence and ergodicity. *Ergod. Th. & Dynam. Sys.* **23** (2003), 869–883.
[LMW03] H. Leschke, P. Müller and S. Warzel. A survey of rigorous results on random Schrödinger operators for amorphous solids. *Markov Process. Related Fields* **9** (2003), 729–760.
[LW95] C. Liverani and M. P. Wojtkowski. Ergodicity in Hamiltonian systems. *Dynamics Reported (Dynam. Report. Expositions Dynam. Systems (N.S.), 4)*. Springer, Berlin, 1995, pp. 130–202.
[Nat01] F. Natterer. *The Mathematics of Computerized Tomography (Classics in Applied Mathematics, 32)*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2001.
[PF92] L. A. Pastur and A. Figotin. *Spectra of Random and Almost-Periodic Operators (Grundlehren der mathematischen Wissenschaften, 297)*. Springer, Berlin, 1992.
[Pös82] J. Pöschel. Integrability of Hamiltonian systems on cantor sets. *Commun. Pure Appl. Math.* **35** (1982), 653–695.
[PP94] G. P. Paternain and M. Paternain. On Anosov energy levels of convex Hamiltonian systems. *Math. Z.* **217**(1) (1994), 367–376.
[RF11] W. D. Roeck and J. Fröhlich. Diffusion of a massive quantum particle coupled to a quasi-free thermal medium. *Comm. Math. Phys.* **303** (2011), 613–707.
[Rue87] D. Ruelle. A mathematical reformulation of Derrida's REM and GREM. *Comm. Math. Phys.* **108** (1987), 225–239.
[Sch98] K. Schmidt. On joint recurrence. *C. R. Acad. Sci. Paris* **327** (1998), 837–842.
[Sch04] C. Schumacher. Klassische Bewegung in zufälligen Potenzialen. *Diplomarbeit*, University Erlangen–Nuremberg, 2004.
[Sch10] C. Schumacher. Klassische Bewegung in zufälligen Potenzialen mit Coulomb–Singularitäten. *PhD Thesis*, University Erlangen–Nuremberg, 2010.
[SKM87] D. Stoyan, W. S. Kendall and J. Mecke. *Stochastic Geometry and its Applications*. John Wiley & Sons, Chichester, 1987.
[Ves08] I. Veselić. *Existence and Regularity Properties of the Integrated Density of States of Random Schrödinger Operators (Lecture Notes in Mathematics, 1917)*. Springer, Berlin, 2008.
[Wal82] P. Walters. *An Introduction to Ergodic Theory (Graduate Texts in Mathematics, 79)*. Springer, Berlin, 1982.

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