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PROBLEM SECTION

Two Minor Problems

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What simple condition on a graph G will ensure that $G \succ K_t$? As usual, $G \succ K_t$ means that K_t is a minor of the graph G (in other words, G has vertex disjoint connected subgraphs $W_1, \ldots, W_t$ and at least one edge between W_i and W_j, $1 \leqslant i < j \leqslant t$).

Wagner [12] proved that the Four Colour Conjecture is equivalent to the assertion that $\chi(G) \geqslant 5$ implies $G \succ K_5$. Hadwiger [1] made the general conjecture that $\chi(G) \geqslant t$ implies $G \succ K_t$. On the face of it there is not much evidence for this conjecture; it is known only for $t \leqslant 6$, but for $t \leqslant 3$ it is trivial, and for $t = 4$ it follows from the (easily proven) fact that $\delta(G) \geqslant 3$ implies $G \succ K_4$. The case $t = 6$ was reduced to the Four Colour Theorem by Robertson, Seymour and Thomas [6].

Mader [3] showed that $G \succ K_t$ in fact follows from a much simpler condition, to wit, that the average degree of G is large. So we can write $c(t)$ for the smallest c such that $e(G) \geqslant c|G|$ implies $G \succ K_t$; Kostochka [2] (see also [8]) proved that $c(t) = \Theta(t\sqrt{\log t})$, and the asymptotic value of $c(t)$ was found in [9]. If $\chi(G) \geqslant 2c(t) + 1$ then $G \succ K_t$ and, so far, this is the best bound on $\chi(G)$ known to guarantee a K_t-minor.

The average degree needed to guarantee a K_t-minor is thus well understood, but the answer turns out to be, in a way, a bit unsatisfactory. This is because the extremal examples offered in [2, 8, 9] are disjoint unions of many graphs of fixed size, and, what is more, Myers [4] showed that all extremal examples are of this kind; they are more-or-less disjoint unions of pseudo-random graphs of fixed size.

It is natural to ask, then, whether a much smaller average degree than $c(t)$, maybe even linear in t, is enough to force a K_t-minor in a *large* and *well-connected* graph. The following question makes this precise.

Question. *Is it true that if $|G|$ is large and $\kappa(G) \geqslant t + 1$ then $G \succ K_t$?*

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Here, as usual, $\kappa(G)$ is the vertex connectivity. In [10] it was conjectured that a linear bound on $\kappa(G)$ will ensure a K_t-minor in large G, but even this, to my knowledge, remains unresolved, though Myers [5] has settled some special cases.

The join of K_{t-5} with a 5-connected planar graph shows that the question has a negative answer if the condition on $\kappa(G)$ is weakened to $\kappa(G) \geqslant t$. However, Thomas [7] has posed the following elegant version of the question combining connectivity and average degree: is it true that if $|G|$ is large, $\kappa(G) \geqslant t-1$ and $e(G) > (t-2)|G| - \binom{t-1}{2}$ then $G \succ K_t$? Maybe even $\kappa(G) \geqslant t-2$ is enough here.

As a footnote, here is a fresh airing for a slightly older question. It was posed in a different, but essentially equivalent, form in [11]. The question arises from the fact that the extremal graphs for the function $c(t)$ are random, and no good constructions are known.

Problem. *For every $C > 0$ construct, for some t, a graph G with $\delta(G) \geqslant Ct$ and $G \not\succ K_t$.*

Random graphs show that such graphs G exist. Natural candidates for G are pseudo-random graphs but most off-the-peg examples have large minors (see [11]). Part of the difficulty with this problem is that it is not easy to verify the condition $G \not\succ K_t$.

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