WEAK CAYLEY TABLES

KENNETH W. JOHNSON, SANDRO MATTAREI
AND SURINDER K. SEHGAL

1. *Introduction*

In [**1**] Brauer puts forward a series of questions on group representation theory in order to point out areas which were not well understood. One of these, which we denote by (B1), is the following: what information in addition to the character table determines a (finite) group? In previous papers [**5**, **7–13**], the original work of Frobenius on group characters has been re-examined and has shed light on some of Brauer's questions, in particular an answer to (B1) has been given as follows.

Frobenius defined for each character χ of a group G functions $\chi^{(k)}: G^{(k)} \longrightarrow \mathbb{C}$ for $k = 1, \ldots, \deg \chi$ with $\chi^{(1)} = \chi$. These functions are called the k-characters (see [**10**] or [**11**] for their definition). The 1-, 2- and 3-characters of the irreducible representations determine a group [**7**, **8**] but the 1- and 2-characters do not [**12**]. Summaries of this work are given in [**11**] and [**13**].

We note especially that the 2-character $\chi^{(2)}$ is defined by

$$\chi^{(2)}(g, h) = \chi(g)\chi(h) - \chi(gh).$$

In the spirit of Brauer's paper, there arises the following question:

(I) What properties of a group are determined by the 1- and 2-characters of the irreducible representations?

A more specific question is the following:

(II) What properties of a group can be determined by the 1- and 2-characters which cannot be determined by the 1-characters alone?

At present we do not have satisfactory answers to questions (I) and (II). In [**13**] it was pointed out that the number of involutions in a group is determined by its irreducible 1- and 2-characters. Here we observe the following other properties determined by the irreducible 1- and 2-characters (see the discussion in Section 2).

(1) For each $z \in Z(G)$, the centre of G, the set $S(z) = \{g; g^2 = z\}$ is determined.

(2) Given any conjugacy class C of G, the class consisting of the squares of the elements of C is determined.

(3) The set of 2-power elements and their orders may be deduced.

(4) The Frobenius–Schur indicator for a character can be deduced from its 2-character.

So far no other easily described properties which are not determined by the

Received 21 March 1997; revised 22 January 1999.

2000 *Mathematics Subject Classification* 20C15.

The first author was partially supported by the Mathematical Research Institute at the Ohio State University.

J. London Math. Soc. (2) 61 (2000) 395–411

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

character table but which are determined by the extra information of the 2-characters are known.

We introduce the *weak Cayley table* of a group to be the table associating to each pair of group elements the conjugacy class of their product. It follows easily that a pair of groups have the same irreducible 1- and 2-characters if and only if they have the same weak Cayley table. This result can be deduced from [**18**], although the weak Cayley table is not explicitly introduced there. Questions may be formulated that are similar to those of Brauer, replacing 'character table' by 'weak Cayley table'. Whereas it is easy to construct pairs of groups of even order with the same character tables and different weak Cayley tables, the smallest example being the dihedral group D_8 and the quaternion group Q_8 of order 8, no such pair of groups of odd order is known.

In [**1**] Brauer asked the question which we denote by (B2): given a set of conjugacy classes of a group which form a normal subgroup, can it be determined from the character table whether the subgroup is abelian?

An example of Saksonov [**19**] showed that the answer is negative. This may also be deduced from recent work of Mattarei [**16**, **17**] in which pairs of groups are given with the same character tables and different derived lengths. We show here that the examples in [**16**] and [**17**] have the same weak Cayley tables, which answers the question obtained from (B2) by replacing 'character table' by 'weak Cayley table'. A crucial result for Mattarei's examples is that group extensions of a certain type have the same character tables. We give a direct proof that these extensions have the same weak Cayley table with the necessary restriction that the factor group is of odd order. It is somewhat remarkable that our work gives an answer to the original question (B2) using no character theory.

In Section 2 the weak Cayley table is defined and it is shown that the irreducible 1- and 2-characters contain the same information as the weak Cayley table. A weak Cayley table is an example of a frequency square and some necessary conditions on a frequency square for it to be a weak Cayley table are given. We also exhibit a class of 2-groups which are determined by their weak Cayley tables. In Section 3 the result on group extensions is formulated in Theorem 3.1, which is a stronger version of [**16**, Corollary 6] with the extra hypothesis that the factor group is of odd order. It follows that the examples in [**16**] and [**17**] have the same weak Cayley table and in a series of corollaries it is pointed out that several other sets of examples of pairs of groups with the same character tables fit the conditions of the theorem and consequently have the same weak Cayley tables. Three generalizations of the theorem are given in Section 4. These are then applied to show that more pairs of groups have the same weak Cayley table.

2. *The weak Cayley table of a group*

DEFINITION 2.1. Let $G = \{e, g_2, \ldots, g_n\}$ be a finite group. The *weak Cayley table* is the table whose rows and columns are indexed by the elements of G, the entry in the (g, h)th position being the conjugacy class of gh.

Note that the row which corresponds to e determines the set of conjugacy classes of G.

EXAMPLE 2.2. Let G be S_3, the symmetric group on three objects. Let $C_0 = \{0\}$,

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

TABLE 1.

	e	(123)	(132)	(12)	(23)	(13)
e	C_0	C_1	C_1	C_2	C_2	C_2
(123)	C_1	C_1	C_0	C_2	C_2	C_2
(132)	C_1	C_0	C_1	C_2	C_2	C_2
(12)	C_2	C_2	C_2	C_0	C_1	C_1
(23)	C_2	C_2	C_2	C_1	C_0	C_1
(13)	C_2	C_2	C_2	C_1	C_1	C_0

$C_1 = \{(123), (132)\}$ and $C_2 = \{(12), (23), (13)\}$ denote the conjugacy classes. The weak Cayley table of G is given in Table 1.

REMARK 2.3. In [**6**] Frobenius used the 'special group determinant' to factorize the group determinant. The determinant of the unbordered weak Cayley table is very close to the special group determinant.

PROPOSITION 2.4. *If the irreducible* 1- *and* 2-*characters of a group are given, its weak Cayley table can be constructed. Conversely, if the weak Cayley table is given, the irreducible* 1- *and* 2-*characters can be calculated.*

Proof. Assume that the irreducible 1- and 2-characters are known for a group G. The conjugacy classes of G can be determined from the 1-characters. If

$$\chi^{(2)}(g,h) = \chi(g)\chi(h) - \chi(gh)$$

is given, it follows from the knowledge of $\chi(g)\chi(h)$ that the function $(g,h) \longrightarrow \chi(gh)$ is known. If $\chi(gh)$ is known for all irreducible χ this is sufficient to determine the conjugacy class of gh.

Conversely, suppose the weak Cayley table is given. The elements in each conjugacy class can be read off from the first row. If C_i and C_j are conjugacy classes the (C_i, C_j) block of the table whose rows correspond to elements in C_i and whose columns correspond to elements in C_j determines the integers a_{ij}^k in the product

$$C_i C_j = \sum a_{ij}^k C_k.$$

Thus the class algebra, and hence the character table, can be constructed from the weak Cayley table. If χ is an irreducible character $\chi(gh)$ for any pair (g,h) may be read off from the weak Cayley table and thus $\chi^{(2)}(gh)$ can be found. □

The idea of a weak Cayley table is implicit in [**18**] and a proof of Proposition 2.4 and statements of the corollaries below can essentially be found there.

DEFINITION 2.5. The groups G_1 and G_2 *have the same* 2-*characters* if there is a bijection $\sigma: G_1 \longrightarrow G_2$ which induces a bijection between the irreducible characters $\{\chi_1, \ldots, \chi_r\}$ of G and $\{\psi_1, \ldots, \psi_r\}$ of H:

$$\chi_i(g) = \psi_i(\sigma(g))$$

and such that

$$\chi_i^{(2)}(g_1, g_2) = \psi_i^{(2)}(\sigma(g_1), \sigma(g_2))$$

for all i.

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

DEFINITION 2.6. *G_1 and G_2 have the same weak Cayley table* if there is a bijection $\alpha: G_1 \longrightarrow G_2$ such that

(1) g and h are conjugate in G_1 if and only if $\alpha(g)$ and $\alpha(h)$ are conjugate in G_2;

(2) for all $g, h \in G_1$, $\alpha(gh)$ and $\alpha(g)\,\alpha(h)$ are conjugate in G_2.

The map α will then be called a *weak Cayley table isomorphism*. A consequence of the definition is that $\alpha(e) = e$. In fact, (1) implies that $\alpha(e)$ is central in G_2, and (2) yields that $\alpha(e)\,\alpha(e) = \alpha(e \cdot e) = \alpha(e)$, whence $\alpha(e) = e$. It also follows that α preserves inverses, in the sense that $\alpha(g^{-1}) = \alpha(g)^{-1}$ for all $g \in G_1$. Beware that $\alpha(ghk)$ and $\alpha(g)\,\alpha(h)\,\alpha(k)$ are not conjugate in general.

It is clear that α sends normal subgroups of G_1 to normal subgroups of G_2. Furthermore, α preserves cosets of normal subgroups. In fact, if N is a normal subgroup of G_1 and $gN = hN$ for some $g, h \in G_1$, then $h^{-1}g \in N$, whence $\alpha(h^{-1}g) \in N$. Since $\alpha(h^{-1}g)$ is a conjugate of $\alpha(h^{-1})\,\alpha(g) = \alpha(h)^{-1}\,\alpha(g)$ in G_2, and since $\alpha(N)$ is normal in G_2, we find that $\alpha(h)^{-1}\,\alpha(g) \in \alpha(N)$, whence $\alpha(g)\,\alpha(N) = \alpha(h)\,\alpha(N)$. It follows that, if N is a normal subgroup of G_1, the weak Cayley table isomorphism α induces a bijection $\bar{\alpha}: G_1/N \longrightarrow G_2/\alpha(N)$ via $\bar{\alpha}(gN) = \alpha(g)\,\alpha(N)$, for all $g \in G_1$.

We shall prove that $\bar{\alpha}$ is a weak Cayley table isomorphism. To prove (1), let gN and hN be conjugate cosets of N, with $g, h \in G_1$. Hence g is conjugate to hn in G_1, for some $n \in N$. As a consequence, $\alpha(g)$ is conjugate to $\alpha(hn)$ in G_2 and, in turn, is conjugate to $\alpha(h)\,\alpha(n)$. It follows that $\alpha(g)\,\alpha(N)$ and $\alpha(h)\,\alpha(n)\,\alpha(N) = \alpha(h)\,\alpha(N)$ are conjugate cosets of $\alpha(N)$. The same argument, applied to the inverse map of α in place of α, proves that (1) is satisfied. Turning to (2), let $g, h \in G_1$. Then $\bar{\alpha}(ghN) = \alpha(ghn)\,\alpha(N)$, where $n \in N$ is such that $ghn \in G_1$. Since $\alpha(ghn)$ is conjugate to $\alpha(gh)\,\alpha(n)$, the coset $\alpha(ghn)\,\alpha(N)$ of $\alpha(N)$ is conjugate to $\alpha(gh)\,\alpha(n)\,\alpha(N) = \alpha(gh)\,\alpha(N)$. Finally, since $\alpha(gh)$ and $\alpha(g)\,\alpha(h)$ are conjugate, we conclude that $\alpha(gh)\,\alpha(N)$ is conjugate to $\alpha(g)\,\alpha(h)\,\alpha(N) = \bar{\alpha}(gN)\,\bar{\alpha}(hN)$, thus proving (2). Hence α induces a weak Cayley table isomorphism $\bar{\alpha}$ between the quotient groups G_1/N and $G_2/\alpha(N)$, in a canonical way.

The 2-character table of a group is defined in [**10**], where the definition of the statement 'G_1 and G_2 have the same 2-character tables' is also given. The corollaries below follow immediately.

COROLLARY 2.7. *Let G_1 and G_2 be finite groups. The following are equivalent.*

(1) *G_1 and G_2 have the same weak Cayley table.*

(2) *G_1 and G_2 have the same 2-characters.*

(3) *G_1 and G_2 have the same 2-character tables.*

COROLLARY 2.8. *If G_1 and G_2 have the same weak Cayley table they have the same character tables.*

A concrete way of viewing in purely group-theoretic terms the information in the character table and in the weak Cayley table of a group is as follows. The knowledge of the character table of a group G is equivalent to the knowledge of the structure constants a_{ij}^k of the centre $\mathbf{Z}(\mathbf{C}G)$ of the complex group algebra $\mathbf{C}G$, with respect to its basis given by the class sums of G. In turn, knowing the structure constants amounts to knowing the probability that the product of a random conjugate of g with a random conjugate of h belongs to C, for every $g, h \in G$ and for every conjugacy class C of G. Note that such a collection of probabilities on $G \times G$ (or, better, on $\mathscr{S} \times \mathscr{S}$, where $\mathscr{S}$ is te set of conjugacy classes of G) tells us which element of G is the identity

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

(being characterized by the property $e \cdot e = e$), and therefore which subsets of G are the conjugacy classes of G. On the other hand, knowing the weak Cayley table of G means knowing for every $g, h \in G$ exactly which class C of G contains the product gh.

One usually says that two groups G_1 and G_2 have the same character table if there is a bijection α between the sets of conjugacy classes of G_1 and G_2, and a bijection β between the sets of irreducible characters of G_1 and G_2, which are compatible with evaluating characters of G_i on elements of G_i (see [**16**, Definition 2]). Thinking of a character table from the point of view illustrated above allows one to dispose of the map β, and to avoid mentioning characters at all. We shall say that groups G_1 and G_2 have the same character table if there exists a bijection $\alpha: G_1 \longrightarrow G_2$ (which will then be called a *character table isomorphism*) whose linear extension to the complex group-algebras, which we call again $\alpha: \mathbf{C}G_1 \longrightarrow \mathbf{C}G_2$, sends $\mathbf{Z}(\mathbf{C}G_1)$ onto $\mathbf{Z}(\mathbf{C}G_2)$, and such that the restriction of α to $\mathbf{Z}(\mathbf{C}G_1)$ is an algebra isomorphism (though $\alpha: \mathbf{C}G_1 \longrightarrow \mathbf{C}G_2$ itself may not be an algebra isomorphism; when this happens, it is known that G_1 and G_2 have isomorphic integral group rings). This is easily seen to be equivalent to the usual definition of having the same character table, because (keeping in mind that the class sums of G_i form a basis for $\mathbf{Z}(\mathbf{C}G_i)$) it is just a formal way of saying that G_1 and G_2 have the same structure constants a_{ij}^k, as defined above.

Now we can supplement Corollary 2.8 with the sentence 'any weak Cayley table isomorphism $\alpha: G_1 \longrightarrow G_2$ is a character table isomorphism'. Note, however, that being a character table isomorphism is not affected by rearranging the images under α inside each conjugacy class, while being a weak Cayley table is affected. This approach to a character table in terms of structure constants allows one to give a purely group-theoretic statement and proof of [**16**, Theorem 5].

We shall say that G_1 and G_2 form a *Brauer pair* via $\alpha: G_1 \longrightarrow G_2$ if α is a character table isomorphism and if $\alpha(g)^m$ is a conjugate of $\alpha(g^m)$ in G_2 for every $g \in G_1$, and for every integer m. It clearly suffices that this is satisfied for every m a prime number. Furthermore, it is enough that m ranges over the prime divisors of the order of G_1, because for m prime to $|G_1|$ the conjugacy class containing g^m can be found from algebraic conjugacy of character values.

COROLLARY 2.9. *If G_1 and G_2 are 2-groups with the same weak Cayley table, they form a Brauer pair.*

Proof. Let G_1, G_2 be 2-groups with the same weak Cayley table via the map α. We know that $\alpha(g)^2$ is conjugate to $\alpha(g^2)$ in G_2. Since odd powers of classes in 2-groups can be determined from algebraic conjugacy of character values, it follows that all power maps in conjugacy classes are preserved by α. □

We choose to state the results in the remainder of the paper in terms of the weak Cayley table, but point out that the 2-character table of a group has orthogonality conditions on rows which could repay further study.

A *frequency square* of order n and frequency vector $(\lambda_1, \ldots, \lambda_m)$ is an $n \times n$ array of m objects $a_1, \ldots, a_m$ such that a_i appears exactly λ_i times in each row and in each column (see [**4**, Chapter 12]).

The following properties follow from the above definition of the weak Cayley table.

(*W*1) The unbordered weak Cayley table of a group G is a frequency square of order $n = |G|$ in which C_i appears $|C_i|$ times in each row and column.

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

TABLE 2.

1	2	2	3	3	3
2	1	3	2	3	3
2	3	1	3	2	3
3	2	3	2	3	1
3	3	2	3	1	2
3	3	3	1	2	2

($W2$) The weak Cayley table is symmetrical about the diagonal.

($W3$) This is the block condition. In the (C_i, C_j) block of the table each entry C_k appears $a_{ij}^k|C_k|$ times where a_{ij}^k is an integer (see above).

($W4$) $|C_i|$ divides n.

REMARK 2.10. If Q is any loop, then a weak Cayley table can be produced in a similar manner (see [**14**] for the relevant definitions). In this case ($W1$) and ($W3$) are satisfied automatically but ($W2$) and ($W4$) need not hold (but are often satisfied if the loop is constructed as an extension of a group by another group).

($W5$) Let $C(g)$ denote the class of g. For x, y in G the set of classes which contain some element of the set $C(xy)\,x^{-1} = \{gx^{-1}; g \in C(xy)\}$ is given by the weak Cayley table. This set contains $C(y)$ for all x, y in G.

REMARK 2.11. In a loop, right inverses need not be left inverses, in which case ($W5$) would have no meaning. It is possible, however, to state a condition analogous to ($W5$) which holds for the weak Cayley table of a loop.

PROBLEM 2.12. Find necessary and sufficient conditions for a frequency square to be the unbordered weak Cayley table of a group.

EXAMPLE 2.13. The frequency square in Table 2 satisfies ($W1$), ($W2$) and ($W4$), but not ($W3$) or ($W5$).

The example of Q_8 and D_8 shows that the weak Cayley table contains more information than the ordinary character table.

We rephrase question (II) of the introduction as follows.

PROBLEM 2.14. Characterize the information contained in the weak Cayley table of a group which is not contained in its character table.

It is clear that the weak Cayley table determines the set of involutions of G, whereas the ordinary character table does not (this latter is shown by the example of D_8 and Q_8). Further, as was already mentioned in the introduction, for each element z of the centre of a group the set $S(z) = \{g; g^2 = z\}$ can be determined from the weak Cayley table, and the set of 2-power elements, together with their orders, may also be read off.

We note that the Frobenius–Schur indicator of each character can be determined from the weak Cayley table (using the value of the 2-character on (g, g)). Compare this with the well-known fact that the number of involutions of a group (but not the actual set of involutions) can be computed from the character table of the group together with the Frobenius–Schur indicators.

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

Finally, observe that from the character table of a group one can read the products of central elements (by multiplying character values divided by the corresponding character degrees). In particular, one can read the isomorphism class of the centre of the group. From the weak Cayley table of a group one can read more, namely the product of any central element z by any element g (the exact product, not only up to conjugation). This follows because given $g \in G$ and $z \in Z(G)$ with product $gz = h$, $h^{(-1)}$ and therefore h may be deduced from the weak Cayley table since $gh^{(-1)} = z$. We may restate this fact by saying that a weak Cayley table isomorphism preserves multiplication *by* central elements, while a character table isomorphism only preserves multiplication *of* central elements.

In the following proposition we examine a class of 2-groups which are determined by their weak Cayley tables. In fact, they are even determined by their character tables together with the 2-power maps (all of this information is contained in the weak Cayley table of a 2-group, according to Corollary 2.9).

PROPOSITION 2.15. *Let G_1 be a 2-group of class 2 with the Frattini subgroup $\Phi(G_1) \leqslant \mathbf{Z}(G_1)$. Let G_1 and G_2 form a Brauer pair via $\alpha: G_1 \longrightarrow G_2$. Then G_1 and G_2 are isomorphic* (*although α need not be an isomorphism*).

Proof. Our hypotheses on G_1 clearly carry on to G_2 (for this it is sufficient that α be a character table isomorphism). It follows easily that the derived subgroups G_1' and G_2' are elementary abelian. It follows from the hypothesis that $\alpha(x^2)$ is conjugate to $(\alpha(x))^2$, for x in G_1. Now x^2 lies in $\mathbf{Z}(G_1)$, whence

$$\alpha(x^2) = (\alpha(x))^2.$$

We shall prove that

$$\alpha([y, x]) = [\alpha(y), \alpha(x)].$$

The identity

$$(xy)^2 = x^2 y^2 [y, x]$$

holds in any group of class 2. For $x, y \in G_i$, the elements x^2 and y^2 are central and determined by the 2-power map. Since the character table of G_i determines G_i/G_i' it determines $xy \bmod G_i'$ and $(xy)^2$ is determined by the 2-power map. Now

$$[y, x] = (y^2)^{-1} (x^2)^{-1} (xy)^2$$

for $x, y \in G_1$, and thus

$$\alpha([y, x]) = (\alpha(y^2))^{-1} (\alpha(x^2))^{-1} \alpha((xy)^2)$$

since α is a homomorphism on $Z(G_1)$. It follows that

$$\alpha([y, x]) = (\alpha(y))^{-2} (\alpha(x))^{-2} (\alpha(xy))^2,$$

but $\alpha(xy) = \alpha(x)\,\alpha(y)\,g$ for some $g \in G_2'$ and hence $(\alpha(xy))^2 = (\alpha(x)\,\alpha(y))^2$. We conclude that

$$\alpha([y, x]) = [\alpha(y), \alpha(x)]$$

for all $x, y \in G_1$.

Now write $G_1 = \langle h_1, \ldots, h_r, \mathbf{Z}(G_1) \rangle$, with $h_1, \ldots, h_r \in G_1$ and r minimal. Then we also have $G_2 = \langle \alpha(h_1), \ldots, \alpha(h_r), \mathbf{Z}(G_2) \rangle$. Any element $g \in G_1$ can be written in a unique way as $g = \prod_i h_i^{a_i} \cdot z$, with $a_i = 0, 1$, and $z \in \mathbf{Z}(G_1)$ (where the product is naturally ordered). A similar statement holds for G_2, because α, being a character table

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

isomorphism, induces a group isomorphism from $G_1/\mathbf{Z}(G_1)$ onto $G_2/\mathbf{Z}(G_2)$. Therefore, a bijection $\beta: G_1 \longrightarrow G_2$ can be well defined by putting

$$\beta\left(\prod_i h_i^{a_i} \cdot z\right) = \prod_i \alpha(h_i)^{a_i} \cdot \alpha(z).$$

We shall prove that β is a group isomorphism.

Any product

$$\left(\prod_i h_i^{a_i} \cdot z_1\right)\left(\prod_j h_j^{b_j} \cdot z_2\right)$$

can be put in its canonical form $\prod_k h_k^{c_k} \cdot z_3$ by means of repeated operations of the following three types:

(1) replacing $h_i h_j$ with $h_j h_i [h_i, h_j]$, whenever $i > j$;
(2) replacing $h_i h_i$ with $h_i^2 \in \mathbf{Z}(G_1)$;
(3) replacing pairs of elements of $\mathbf{Z}(G_1)$ with their product.

Analogous operations can be performed on the product

$$\left(\prod_i \alpha(h_i)^{a_i} \cdot \alpha(z_1)\right)\left(\prod_j \alpha(h_j)^{b_j} \cdot \alpha(z_2)\right).$$

Since α preserves both computing squares and computing commutators, as shown in the first part of the proof, and since α obviously preserves computing products of central elements, the canonical form for the above product will be

$$\prod_k \alpha(h_k)^{c_k} \cdot \alpha(z_3) = \beta\left(\prod_k h_k^{c_k} \cdot z_3\right),$$

thus proving that β is a group isomorphism. Note that β does not necessarily equal α. In fact, the latter is not necessarily a group isomorphism: for example, any group anti-isomorphism is a character table isomorphism and preserves the power maps (it is even a weak Cayley table isomorphism). □

It is also to be remarked that it is shown in [**10**] that if a group has a faithful 2-dimensional representation, the 2-character of this representation determines the group.

PROBLEM 2.16. Obtain 'nice' invariants which in addition to the weak Cayley table determine the group.

From the examples in Section 3, it follows that the power maps between conjugacy classes are not sufficient. While the 3-characters are sufficient they are difficult to write down and it would be useful to discover more concise invariants.

3. *Groups with the same weak Cayley table*

The following theorem is based on [**16**, Corollary 6], where the conclusion is that the ordinary character tables are the same, but the factor group is not restricted. The pair (H, N), with N a normal subgroup of H, is a *Camina pair* if $h^H \cdot N = h^H$ for all $h \in H - N$.

THEOREM 3.1. *Suppose that G is a group of odd order and that N is an abelian group which is a G-module. Suppose further that G_1 and G_2 are nonisomorphic groups*

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

which are extensions of N by G such that (G_1, N) and (G_2, N) are Camina pairs. Then G_1 and G_2 have the same weak Cayley tables.

Proof. As is well known from extension theory we may write G_1 and G_2 as structures on $N \times G$ with multiplication $\circ_i$, for $i = 1, 2$, where

$$(n_1, g_1) \circ_i (n_2, g_2) = (n_1 n_2^{g_1^{-1}} f_i(g_1, g_2), g_1 g_2). \tag{3.1}$$

It will be convenient to assume, as we may, that $f_i(g, e) = f_i(e, g) = e$ for all $g \in G_i$.

We define a bijection ϕ from $N \times G$ to itself as follows. Divide $G - \{e\}$ arbitrarily into subsets S_1 and S_2 such that $S_1 \cup S_2 = G$ and $S_2 = \{g^{-1}; g \in S_1\}$. Then define

$$\begin{aligned} \phi(n, e) &= (n, e) && \text{for } n \in N, \\ \phi(n, g) &= (n, g) && \text{for } g \in S_1, \\ \phi((n, g)^{\mathrm{inv}(1)}) &= (n, g)^{\mathrm{inv}(2)} && \text{for } g \in S_1, \end{aligned}$$

where $(n, g)^{\mathrm{inv}(i)}$ denotes the inverse of (n, g) in G_i, for $i = 1, 2$.

We claim that the conjugacy classes of G_1 and G_2 regarded as subsets of $N \times G$ coincide and that ϕ is a bijection which induces an identification of the weak Cayley tables of G_1 and G_2.

Since the inverse of (n, g) in either group is of the form (m, g^{-1}) for some $m \in N$ it is easy to check that ϕ is a bijection. Using the assumption of the theorem, and by direct calculation, the conjugacy classes of each G_i consist of

(a) subsets of the form $\{(n^g, e); g \in G\}$;

(b) for a given conjugacy class $C \neq \{e\}$ of G, the set $\{(n, g); n \in N, g \in C\}$.

It follows directly that ϕ preserves conjugacy classes. It will now be shown that $P_1 = (n_1, g_1) \circ_1 (n_2, g_2)$ and $P_2 = \phi(n_1, g_1) \circ_2 \phi(n_2, g_2)$ lie in the same conjugacy class. We discuss three cases:

(i) $g_1 = g_2 = e$. Here we have $P_1 = P_2 = (n_1 n_2, e)$.

(ii) $g_1 \neq g_2^{-1}$. From (3.1), $P_1 = (m_1, g_1 g_2)$ and $P_2 = (m_2, g_1 g_2)$ where $m_1, m_2 \in N$ so that P_1 and P_2 lie in the same class of type (b).

(iii) $g_2 = g_1^{-1} \neq e$. Without loss of generality we may assume that $g_1 \in S_1$, since xy and yx necessarily lie in the same conjugacy class. We note that in both extensions $(n, g) = (n, e) \circ (e, g)$ for all $n \in N$, $g \in G$. Consequently,

$$(n, g) \circ (m, g)^{-1} = (n, e) \circ (e, g) \circ (e, g)^{-1} \circ (m, e)^{-1} = (nm^{-1}, e),$$

independently of the specific extension in which it is computed. Now (n_2, g_2) can be written as $(m, g_1)^{\mathrm{inv}(1)}$ for some $m \in N$. Hence

$$P_1 = (n_1, g_1) \circ_1 (m, g_1)^{\mathrm{inv}(1)} = (n_1 m^{-1}, e)$$

and

$$P_2 = \phi(n, g_1) \circ_2 \phi((m, g_1)^{\mathrm{inv}(1)}) = (n_1, g_1) \circ_2 (m, g_1)^{\mathrm{inv}(2)} = (n_1 m^{-1}, e).$$

We conclude that G_1 and G_2 have the same weak Cayley table. □

Since D_8 and Q_8 regarded as extensions of C_2 by $C_2 \times C_2$ satisfy the conditions of the theorem apart from the condition that G is of odd order, it is seen that this restriction is necessary.

Corollary 3.2 is a restatement of Theorem 3.1 in a form directly applicable to the examples constructed in [**16**] and [**17**] (compare with [**16**, Corollary 6]).

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

COROLLARY 3.2. *Suppose the extensions*

$$N_i \rightarrowtail G_i \longrightarrow K_i$$

are given, for $i = 1, 2$, *where* N_i *is an abelian group and* K_i *is of odd order, such that*

(i) *there exists a group isomorphism* $\bar{\alpha}: K_1 \longrightarrow K_2$;

(ii) *there exists a* K_1*-module isomorphism* $\tilde{\alpha}: N_1 \longrightarrow N_2$ *where the action of* K_1 *on* N_2 *is via* $\bar{\alpha}$;

(iii) (G_1, N_1) *and* (G_2, N_2) *are Camina pairs.*

Then G_1 *and* G_2 *have the same weak Cayley table.*

COROLLARY 3.3. *Under the assumption of Corollary* 3.2, G_1 *and* G_2 *have the same character tables.*

In [**16**] two groups G, H are constructed which have the same character table but different derived lengths. More precisely, the groups G, H have order $p^{3(q-1)}q$ where p and q are odd primes, $q \geqslant 5$ and the multiplicative order of p is $q-1$ modulo q. Both G and H have a normal Sylow p-subgroup with an elementary abelian centre of order p^{q-1}, such that all of its quotients modulo a maximal subgroup of its centre are extraspecial p-groups. (Such p-groups P are sometimes called *semi-extraspecial*; they satisfy $P' = \mathbf{Z}(P) = \mathbf{\Phi}(P)$, and (P, P') is a Camina pair.) The Sylow p-subgroup P of G (respectively $\bar{P}$ of H) has a cyclic complement Q (respectively $\bar{Q}$) which acts irreducibly on P' (respectively $\bar{P}'$) and on A/P' (respectively $X/\bar{P}'$), where $A = [P, Q]$ (respectively $X = [\bar{P}, \bar{Q}]$). It follows that $G' = A$ and $H' = X$. However, the construction is such that A is abelian (in fact, P is a Sylow p-subgroup of $\mathrm{SL}(3, p^{q-1})$, and A is one of its maximal abelian subgroups), while X is not (in fact, $X' = \bar{P}'$). Therefore, G is metabelian and H is of derived length 3.

It was proved in [**16**] that (G, P') and $(H, \bar{P}')$ are Camina pairs, so that we can apply Corollary 3.2 using $G_1 = G$, $N_1 = P'$, $G_2 = H$, $N_2 = \bar{P}'$ to conclude that G and H have the same weak Cayley table.

COROLLARY 3.4. *The pair* G, H *of groups given in* [**16**] *have the same weak Cayley table.*

As stated in [**16**], G and H have the same power maps on conjugacy classes, and hence they form a Brauer pair. This is easy to prove and depends on the fact that both P and $\bar{P}$ have exponent p. However, it is possible to modify the construction to obtain similar groups G_1 and H_1, again of derived lengths 2 and 3, whose Sylow p-subgroups have both exponent p^2, and G_1, H_1 form a Brauer pair. All four groups G, H, G_1, H_1 have the same weak Cayley table. Another variation of this construction is described in [**15**, pp. 90–97], and leads to more pairs of groups of derived lengths 2 and 3, to which our Corollary 3.2 applies.

In [**17**], a pair G_1, G_2 of p-groups of order p^{11} for $p \geqslant 5$ is constructed. These groups have derived lengths 2 and 3 respectively and form a Brauer pair, because they have the same character table and both have exponent p. It is shown in [**17**] that the pair satisfies the conditions of Corollary 3.2.

COROLLARY 3.5. *The pair of groups of order* p^{11}, $p \geqslant 5$ *constructed in* [**17**] *have the same weak Cayley table.*

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

COROLLARY 3.6. *The derived length of a group cannot be deduced from its weak Cayley table.*

COROLLARY 3.7. *If a set of conjugacy classes of a group which make up a normal subgroup N is given, then it cannot be determined from the weak Cayley table of G whether N is abelian.*

Proof. As indicated in the discussion on [**16**, p. 92], if Corollary 3.7 were false it would follow that the derived length could be determined from the weak Cayley table, contradicting Corollary 3.4. □

COROLLARY 3.8. *If p is odd, any pair of extraspecial p-groups of the same order have the same weak Cayley table.*

Proof. It is easy to check that these groups satisfy the conditions of Theorem 3.1 where N is the centre of each group. □

In [**3**] the first example appeared of a pair of groups with the same character tables and the same power maps on conjugacy classes. It is interesting to note that these groups are also extensions which satisfy the conditions of Theorem 3.1.

COROLLARY 3.9. *The pair of groups of order p^7, $p \geqslant 5$ given in* [**3**] *have the same weak Cayley table.*

4. *Some generalizations*

The following theorem is a further generalization of Theorem 3.1. A mild generalization is that H_1/N and H_2/N are not assumed to be isomorphic, but only to have the same weak Cayley table. The hypothesis that G has odd order can be weakened as follows. Of course, some assumption on the involutions of G is needed in order to guarantee that H_1 and H_2 have the same 2-power maps.

THEOREM 4.1. *Suppose that G_1 and G_2 have the same weak Cayley table via $\alpha: G_1 \longrightarrow G_2$. Let H_i be an extension of G_i by the module N, for $i = 1, 2$, where $n^g = n^{\alpha(g)}$ for all $g \in G_1$, and suppose that (H_1, N) and (H_2, N) are Camina pairs. Finally, having written each H_i as an extension of N by G_i, suppose that for all involutions $x \in G_1$ we have*

$$(e, x)^2 = (e, \alpha(x))^2.$$

(*It is clearly possible to achieve this if and only if the cosets of N which are the preimages of x in H_1 and of $\alpha(x)$ in H_2 each contain an element which squares to the same element of N; this is satisfied, for example, when both preimages contain an involution.*)

Then H_1 and H_2 have the same weak Cayley table.

Proof. We write H_1 and H_2 as group extensions, as in the proof of Theorem 3.1. Since G_1 and G_2 are distinct groups, it will not be necessary here to use different symbols for the group operations of H_1 and H_2.

If I denotes the set of involutions of G_1, then $\alpha(I)$ is the set of involutions of G_2. Partition $G_1 - I - \{e\}$ into two subsets S and S^{-1} (the latter consisting of the inverses

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

of the elements of S), so that G_1 is a disjoint union $G_1 = \{e\} \cup I \cup S \cup S^{-1}$. It follows that $G_2 = \{e\} \cup \alpha(I) \cup \alpha(S) \cup (\alpha(S))^{-1}$. Now define

$$\begin{aligned} \phi(n,g) &= (n, \alpha(g)) \qquad \text{for } g \in \{e\} \cup I \cup S, \\ \phi((n,g)^{-1}) &= (n, \alpha(g))^{-1} \quad \text{for } g \in S. \end{aligned}$$

It is clear that ϕ is a bijection, and one can easily show as in the proof of Theorem 3.1 that ϕ sends each conjugacy class of H_1 onto a conjugacy class of H_2.

We shall begin to prove that ϕ is a weak Cayley table isomorphism by showing that it commutes with taking inverses. In fact, for all $g \in G_1 - I$ we have $(\phi(n,g))^{-1} = (\phi(n,g)^{-1})$ by construction. To show that the same holds for all $g \in I$, we shall prove the more general fact that $\phi(n,g)\,\phi(m,g) = (n,g)\,(m,g)$ for all $n, m \in N$, and for all $g \in I$. This follows from the following computation, which uses our assumption that $(e,x)^2 = (e, \alpha(x))^2$ for all $x \in I$:

$$\begin{aligned} \phi(n,g)\,\phi(m,g) &= (n, \alpha(g))\,(m, \alpha(g)) = (n,e)\,(e, \alpha(g))\,(m,e)\,(e, \alpha(g)) \\ &= (n,e)\,(e, \alpha(g))^2\,(m^{\alpha(g)}, e) = (n,e)\,(e,g)^2\,(m^g, e) = (n,g)\,(m,g). \end{aligned}$$

It remains to prove that $\phi((n,g)\,(m,h))$ and $\phi(n,g)\,\phi(m,h)$ lie in the same conjugacy class of H_2, for all $n, m \in N$, and for all $g, h \in G_1$. Equivalently, but more conveniently, we shall prove that $\phi((n,g)\,(m,h)^{-1})$ and $\phi(n,g)\,\phi((m,h)^{-1})$ lie in the same conjugacy class of H_2. As in the proof of Theorem 3.1, the only delicate case is when $g = h \neq e$, and we may assume that $g \in I \cup S$ (if $g \in S^{-1}$, consider the inverses of (n,g) and (m,g) and use the fact that ϕ preserves inverses and conjugacy). Then we have

$$\begin{aligned} \phi((n,g)\,(m,g)^{-1}) &= \phi(nm^{-1}, e) = (nm^{-1}, e), \\ \phi(n,g)\,\phi((m,g)^{-1}) &= \phi(n,g)\,(\phi(m,g))^{-1} = (n, \alpha(g))\,(m, \alpha(g))^{-1} = (nm^{-1}, e), \end{aligned}$$

where the first of the second row of equalities is because ϕ commutes with taking inverses. The proof is complete. □

For an example where the above theorem applies, take H_i to be a (split) extension of an extraspecial group of odd order p^3 and exponent p^i (for $i = 1, 2$) by an automorphism of order 2 which acts non-trivially on the centre N of the extraspecial group.

We have stated that Theorem 3.1, and hence the slightly more general Theorem 4.1, are weak Cayley table analogues for [**16**, Corollary 6], which dealt with character tables. However that corollary was a rather special case of a more general result, namely [**16**, Theorem 5]. We shall not attempt to reach such a generality here, but one step in that direction will be worth taking, namely the removal from Theorem 4.1 of the hypothesis that G_1 and G_2 act in the same way on N (via α). In fact, such a hypothesis precludes the application of Theorem 4.1 to an important class of Camina pairs (H, N), consisting of Frobenius groups H, where N is the Frobenius kernel (because the hypotheses of Theorem 4.1 would force an isomorphism between the Frobenius groups H_1 and H_2).

In analogy with [**16**, Theorem 5], the milder assumption on the actions of G_1 and G_2 on N would be that G_1 and G_2 merely have the same orbits on N (that is to say, the conjugacy classes of H_1 contained in N coincide with the conjugacy classes of H_2 contained in N). However, this is too weak an assumption in dealing with weak Cayley tables (when $|G_i|$ is even), because the weak Cayley table contains more

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

information on the action of G_i on N than the character table. We shall try to justify this assertion.

For any element $g \in G$ we denote by $\lfloor g, G \rfloor$ the set of commutators $[g, h]$ for h running over G (whence $[g, G] = \langle \lfloor g, G \rfloor \rangle$). It is well known that the cardinality of the set $\lfloor g, G \rfloor \cap C$ can be read from the character table of a group G, for any $g \in G$, and for any conjugacy class C of G. The weak Cayley table of G tells us more, that is, it determines the set $\lfloor g, G \rfloor$ as the set of all $h \in G$ such that $gh \in g^G$. A similar conclusion obviously holds for the sets $\lfloor G, g \rfloor = \lfloor g, G \rfloor^{-1}$.

Now let $x \in G$ be an involution. Then all elements of $\lfloor G, x \rfloor$ are inverted by x (by conjugation), because $[g, x]^x = (g^{-1}g^x)^x = g^{-x}g = [g, x]^{-1}$. Conversely, if $h \in G$ is inverted by x, then $[h, x] = h^{-1}h^x = h^{-2}$, or better, $[h^{-1}, x] = h^2$ (noting that $[h, x]$ commutes with h), and hence h^2 belongs to $\lfloor G, x \rfloor$. If in addition h has odd order r, then $[h^{-(r-1)/2}, x] = h^{-(r-1)} = h$, and hence $h \in \lfloor G, x \rfloor$ in this case. We conclude that, for any involution $x \in G$, the weak Cayley table of G tells us which elements of odd order in G are inverted by x.

For example, let N be an abelian normal subgroup of odd order of G, and let $x \in G$ be an involution. Then N decomposes as $N = N_1 \times N_{-1}$, where $N_1 = \mathbf{C}_N(x)$ and $N_{-1} = \lfloor N, x \rfloor = [N, x]$ are the homogeneous components with respect to the action of x. We have seen that N_{-1} can be determined from the weak Cayley table of G.

PROBLEM 4.2. Can N_1 be determined from the weak Cayley table of G?

In the following generalization of Theorem 4.1 we shall thus assume that every involution of G_1 acts in the same way on N as the corresponding involution of G_2, and in addition that the conjugacy classes of H_1 and H_2 inside N coincide. This will not be a restriction in the case of H_i a Frobenius group with kernel N, because a Frobenius complement can contain at most one involution, and this acts by inverting all elements of N (which will be abelian in this case).

THEOREM 4.3. *Suppose that G_1 and G_2 have the same weak Cayley table via $\alpha: G_1 \longrightarrow G_2$. Let H_i be an extension of G_i by the abelian normal subgroup N, such that the conjugacy classes of H_1 which lie in N are the same as the conjugacy classes of H_2 in N. Suppose that (H_1, N) and (H_2, N) are Camina pairs.*

Finally, having fixed a representation for each H_i as an extension of G_i by N, suppose that for every involution $x \in G_1$ we have

$$(e, x)^2 = (e, \alpha(x))^2,$$
$$n^x = n^{\alpha(x)}, \quad \text{for all } n \in N.$$

Then H_1 and H_2 have the same weak Cayley table.

Proof. Observe that the only place in the proof of Theorem 4.1 where we used the explicit actions of G_1 and G_2 on N was in proving that the bijection ϕ commutes with taking inverses, and only for elements (n, g) with $g \in I$. Hence the proof of Theorem 4.1 is still valid under the present hypotheses. □

Notice that in the case that G_i has odd order, its action on N will not play any role (apart from the orbits) either in Theorem 4.3 or in its proof. In that case, one can drop the assumption that N is abelian from the statement of Theorem 4.3, leaving the proof unchanged. If G_i has even order the situation is different. The statement of Theorem 4.3 would not be meaningful for N not abelian, because there would be no action of

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

G_i on N. It may be possible to modify the statement and the proof of Theorem 4.3 in order to allow both N non-abelian and G_i of even order, but we have not undertaken this task, especially because these two possibilities are mutually exclusive in the case of Frobenius groups H_i with Frobenius kernel N.

It is a consequence of Theorem 4.3, and of the above remark, that two Frobenius groups G_1 and G_2 have the same weak Cayley table provided that G_1 and G_2 have isomorphic complements and isomorphic (but not necessarily abelian) kernels N_1 and N_2 such that the orbits of G_1 on N_1 (or, in other words, the conjugacy classes of G_1 contained in N_1) coincide with the orbits of G_2 on N_2 (after N_1 and N_2 have been identified). In particular, two doubly-transitive Frobenius groups have the same weak Cayley table provided they have isomorphic kernels and isomorphic complements. This was known in the solvable case, and is the hard part of [**12**, Theorem 2.1], together with [**2**, Theorem 1.4 and Theorem 2.2]. Note that the solvable case exhausts all possibilities for pairs of doubly-transitive Frobenius groups with the same weak Cayley table, because, according to the classification of finite near-fields due to Zassenhaus, there are exactly three non-solvable doubly-transitive Frobenius groups, and they have distinct orders (namely $p^2(p^2-1)$ for $p = 11, 29, 59$).

We shall now generalize Theorem 4.1 in a different direction, by weakening the condition that (H, N) is a Camina pair. In fact, it is known that the Camina condition imposes rather severe restrictions on the structure of H with respect to its normal subgroup N. A condition which is much more frequently satisfied is that H has two normal subgroups N and M, with $N \leqslant M$, such that $h^H \cdot N = h^H$ for all $h \in H - M$. We shall then say that (H, M, N) is a *Camina triple*. For example, if H is a p-group with an abelian maximal subgroup M, and $N = [M, G]$, then (H, M, N) is a Camina triple. The Camina triple condition clearly reduces to the condition that (H, N) is a Camina pair when $N = M$. A generalization of [**16**, Theorem 5] with this weaker condition replacing the Camina pair condition was formulated in [**15**, Theorem 4.4.1]. Its conclusion is that H_1 and H_2 have the same character table, where (H_1, M_1, N_1) and (H_2, M_2, N_2) are Camina triples. Its hypotheses are similar to those of [**16**, Theorem 5], and require that H_1/N_1 has the same character table as H_2/N_2, and that M_1 has the same character table as M_2, in such a way that H_1-orbits of irreducible characters of M_1 correspond to H_2-orbits of irreducible characters of M_2. An additional hypothesis is then necessary, and requires that the character table isomorphisms of the H_i/N_i and of the M_i are compatible where they overlap, namely on the normal sections M_i/N_i. This last hypothesis becomes harder to deal with when one tries to adapt [**15**, Theorem 4.4.1] to weak Cayley tables, as we do in Theorem 4.4 below (with the additional condition that H/M has odd order, for the sake of simplicity).

THEOREM 4.4. *Let H_1 and H_2 be extensions of two groups G_1 and G_2 of odd order by the (not necessarily abelian) normal subgroup M, and let $N \leqslant M$ be a normal subgroup of both H_1 and H_2. Suppose that (H_1, M, N) and (H_2, M, N) are Camina triples. Suppose that H_1/N and H_2/N have the same weak Cayley table via*

$$\alpha : H_1/N \longrightarrow H_2/N,$$

and assume that $\alpha(g)\,\alpha(h)^{-1} = gh^{-1}$ whenever $gh^{-1} \in M/N$ (in particular, α extends the identity map of M/N), or, equivalently, that α preserves left-multiplication by elements of M/N, in the sense that $\alpha(mg) = m\alpha(g)$ for all $m \in M/N$ and for all $g \in H_1/N$. Finally,

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

suppose that the conjugacy classes of H_1 inside M are the same as the conjugacy classes of H_2 inside M. Then H_1 and H_2 have the same weak Cayley table.

REMARK 4.5. Note that the hypothesis that α preserves left-multiplication by elements of M/N is obviously satisfied when α is a group isomorphism which extends the identity map of M/N, as in the example which we shall see below. We know more examples of weak Cayley table isomorphisms α satisfying this condition, for instance, the weak Cayley table isomorphism $\phi: G_1 \longrightarrow G_2$ constructed in the proof of Theorem 3.1 preserves left-multiplication (but not right-multiplication, in general) by elements of N. Furthermore, if $\alpha: G_1 \longrightarrow G_2$ is a weak Cayley table isomorphism, and G_1, G_2 have a common *central* subgroup N with $\alpha(N) = N$, then α necessarily preserves (left-) multiplication by elements of N.

Proof of Theorem 4.4. The proof is similar to the proofs of previous theorems, though some additional difficulties arise. The weak Cayley table isomorphism $\alpha: H_1/N \longrightarrow H_2/N$ induces a weak Cayley table isomorphism $\bar{\alpha}: G_1 \longrightarrow G_2$.

Write each H_i as an extension of N by H_i/N. This is done by choosing transversals $t_i: H_i/N \longrightarrow H_i$ (all transversals will tacitly be assumed to satisfy $t_i(e) = e$, as is customary and convenient), so that every element of H_i can be written as $(n, g) := nt_i(g)$ for some uniquely determined $n \in N$ and $g \in H_i/N$. We shall choose the transversals t_1 and t_2 in a special way with respect to the common subgroup M. Start by choosing transversals $\bar{t}_i: G_i \longrightarrow H_i$ for M in H_i, compatible with α, in the sense that $\alpha(\pi_1(\bar{t}_1(x))) = \pi_2(\bar{t}_2(\bar{\alpha}(x)))$ for all $x \in G_1$, where $\pi_i: H_i \longrightarrow H_i/N$ are the canonical maps. Then choose a transversal $t: M/N \longrightarrow M$ for N in M, and extend it to transversals $t_i: H_i/N \longrightarrow H_i$ for N in H_i by putting

$$t_i(m\pi_i(\bar{t}_i(x))) := t(m)\,\bar{t}_i(x), \quad \text{for } m \in M/N \text{ and for } x \in G_i.$$

One consequence of our choice for the transversals t_i is that $t_1(g) = t_2(g)$ for all $g \in M/N$. Consequently, the symbol (n, g) is well defined, because for $g \in M/N$ it denotes the same element independently of $i = 1, 2$.

A second consequence is that, whenever $gM = hM$ for some $g, h \in H_1/N$, we shall have

$$t_1(g)\,t_1(h)^{-1} = t_2(\alpha(g))\,t_2(\alpha(h))^{-1},$$

or, written in our notation for group extensions, $(e, g)(e, h)^{-1} = (e, \alpha(g))(e, \alpha(h))^{-1}$. Indeed, write $g = m\pi_1(\bar{t}_1(x))$ and $h = n\pi_1(\bar{t}_1(x))$, where $x = gM = hM \in G_1$, and for some $m, n \in M/N$. Then

$$\alpha(g) = \alpha(m\pi_1(\bar{t}_1(x))) = m\alpha(\pi_1(\bar{t}_1(x))) = m\pi_2(\bar{t}_2(\bar{\alpha}(x)))$$

(with a similar equation for h), the second equality being a consequence of our hypothesis that α commutes with left-multiplication by elements of M/N. Hence

$$\begin{aligned} t_1(g)\,t_1(h)^{-1} &= t(m)\,\bar{t}_1(x)\,\bar{t}_1(x)^{-1}\,t(n)^{-1} \\ &= t(m)\,t(n)^{-1}, \\ t_2(\alpha(g))\,t_2(\alpha(h))^{-1} &= t(m)\,\bar{t}_2(\bar{\alpha}(x))\,\bar{t}_2(\bar{\alpha}(x))^{-1}\,t(n)^{-1} \\ &= t(m)\,t(n)^{-1}, \end{aligned}$$

as desired.

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

Now partition G_1 as $G_1 = \{e\} \cup S \cup S^{-1}$, let $\pi: H_1/N \longrightarrow G_1 = H_1/M$ be the canonical epimorphism, and define

$$\begin{aligned} \phi(n,g) &= (n, \alpha(g)) \qquad \text{for } \pi(g) \in \{e\} \cup S, \\ \phi((n,g)^{-1}) &= (n, \alpha(g))^{-1} \quad \text{for } \pi(g) \in S. \end{aligned}$$

The map ϕ is a bijection, extends the identity map of M, and preserves inverses by construction. It is easy to show that ϕ preserves conjugacy classes. In fact, the conjugacy classes of H_i outside M are whole preimages of conjugacy classes of H_i/N, which, in turn, are preserved by α.

It remains to prove that $\phi((n,g)(m,h)^{-1})$ and $\phi(n,g)\phi((m,h)^{-1})$ lie in the same conjugacy class of H_2, for all $n, m \in N$, and for all $g, h \in H_1/N$. We discuss three cases:

(i) $\pi(g) = \pi(h) = e$. Then both (n,g) and (m,h) belong to M, and the conclusion follows because ϕ is the identity map on M.

(ii) $\pi(g) \neq \pi(h)$. Here the products $(n,g)(m,h)^{-1}$ and $\phi(n,g)\phi((m,h)^{-1})$ both lie outside M. Since the conjugacy classes of H_i outside M are preimages of conjugacy classes of H_i/N, the desired conclusion follows from the fact that ϕ induces the map α on the quotient groups H_i/N, and α is a weak Cayley table isomorphism.

(iii) $\pi(g) = \pi(h) \neq e$. We may assume that $\pi(g) \in S$. On the one hand, we have

$$\begin{aligned} \phi((n,g)(m,h)^{-1}) &= \phi((n,e)(e,g)(e,h)^{-1}(m,e)^{-1}) \\ &= (n,e)(e,g)(e,h)^{-1}(m,e)^{-1}, \end{aligned}$$

because $(e,g)(e,h)^{-1} \in M$, and ϕ is the identity map on M. On the other hand, we have

$$\begin{aligned} \phi(n,g)\phi((m,h)^{-1}) &= \phi(n,g)(\phi(m,h))^{-1} \\ &= (n,\alpha(g))(m,\alpha(h))^{-1} \\ &= (n,e)(e,\alpha(g))(e,\alpha(h))^{-1}(m,e)^{-1}, \end{aligned}$$

and the conclusion follows from the fact, proved earlier, that $(e,g)(e,h)^{-1} = (e,\alpha(g))(e,\alpha(h))^{-1}$. □

We give an application of Theorem 4.4 in the following example.

EXAMPLE 4.6. Let p be an odd prime, let $r \leqslant p$ be a positive integer, and let A be an elementary abelian group with basis $a_1, \ldots, a_r$. Let γ be the automorphism of A acting as $a_i^\gamma = a_i a_{i+1}$ for $i < r$, and $a_r^\gamma = a_r$. We construct two extensions H_1, H_2 of the normal subgroup A by a quotient group of order p. Let $H_1 = \langle A, x \rangle$ be a split extension, where x has order p and acts on A as γ. Note that H_1 has exponent p if $r < p$, while it is isomorphic to the regular wreath product of two cyclic groups of order p if $r = p$, and thus it has exponent p^2 in this case. Let $H_2 = \langle A, y \rangle$ be a non-split extension, where $y^p = a_r$, and y acts on A as γ. Observe that H_1 and H_2 are p-groups of maximal class, and that they are never isomorphic.

Now we put $M = A$, and $N = \langle a_s, \ldots, a_r \rangle$, for any s with $2 \leqslant r$. Then (H_i, M, N) are Camina triples, for $i = 1, 2$, and Theorem 4.4 applies, with $\alpha: H_1/N \longrightarrow H_2/N$ any isomorphism which extends the identity map of M/N. Hence H_1 and H_2 have the same weak Cayley table.

It is instructive to see what happens to our example if we allow $p = 2$. Both H_1 and H_2 are then isomorphic to the dihedral group of order 8. In particular, they do have the same weak Cayley table, but our theorem fails, the map given in its proof being a bijection $\phi: H_1 \longrightarrow H_2$, which is not a weak Cayley table isomorphism.

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press

References

1. R. Brauer, 'Representations of finite groups', *Lectures on modern mathematics – Vol.* 1 (ed. T. L. Saaty, John Wiley, 1963) 133–175.
2. G. H. Cliff and S. K. Sehgal, 'On groups having the same character tables', *Comm. Algebra* 9 (1981) 627–640.
3. E. C. Dade, 'Answer to a question of R. Brauer', *J. Algebra* 1 (1964) 1–4.
4. J. Denes and A. D. Keedwell, 'Latin squares', *Ann. Discrete Math.* 46 (1991).
5. E. Formanek and D. Sibley, 'The group determinant determines the group', *Proc. Amer. Math. Soc.* 112 (1991) 649–656.
6. G. Frobenius, 'Über die Primfaktoren der Gruppendeterminante', *Sber. Preuss. Akad. Wiss. Berlin* (1896) 1343–1382.
7. H.-J. Hoehnke and K. W. Johnson, 'The 1-, 2- and 3-characters determine a group', *Bull. Amer. Math. Soc.* 27 (1992) 243–245.
8. H.-J. Hoehnke and K. W. Johnson, '3-characters are sufficient for the group determinant', Proceedings of the Second International Conference on Algebra, *Contemp. Math.* 184 (1995) 193–206.
9. H.-J. Hoehnke and K. W. Johnson, 'K-characters and group invariants', *Comm. Algebra* 26 (1997) 1–27.
10. K. W. Johnson, 'On the group determinant', *Math. Proc. Cambridge Philos. Soc.* 109 (1991) 299–311.
11. K. W. Johnson, 'The group determinant and k-characters, a survey and a conjecture', *Group theory. Proceedings of the Biennial Ohio-State-Denison Conference* (eds. Surinder K. Sehgal and R. Solomon, World Scientific Press, Singapore, 1993) 181–197.
12. K. W. Johnson and Surinder K. Sehgal, 'The 2-character table does not determine a group', *Proc. Amer. Math. Soc.* 119 (1993) 1021–1027.
13. K. W. Johnson and Surinder K. Sehgal, 'The 2-characters and the group determinant', *European J. Combin.* 16 (1995) 623–631.
14. K. W. Johnson and J. D. H. Smith, 'Characters of finite quasigroups', *European J. Combin.* 5 (1984) 43–50.
15. S. Mattarei, 'Retrieving information about a group from its character table', PhD Thesis, University of Warwick, 1992.
16. S. Mattarei, 'Character tables and metabelian groups', *J. London Math. Soc.* (2) 46 (1992) 92–100.
17. S. Mattarei, 'An example of p-groups with identical character tables and different derived lengths', *Arch. Math.* 62 (1994) 12–20.
18. J. McKay and D. Sibley, 'Brauer pairs with the same 2-characters', preprint.
19. A. I. Saksonov, 'An answer to a question of R. Brauer', *Vesci Akad. Navuk BSSR, Ser. Fiz. Mat. Navuk* I (1967) 129–130.

Kenneth W. Johnson
Abington College
Pennsylvania State University
1600 Woodland Road
Abington
PA 19001
USA

kwj1@psu.edu

Surinder K. Sehgal
Department of Mathematics
Ohio State University
231 W 18th Avenue
Columbus
Ohio 43210
USA

sehgal@math.ohio-state.edu

Sandro Mattarei
Dipartimento di Matematica Pura ed Applicata
via Belzoni 7
Università di Padova
I-35131 Padova
Italy

mattarei@math.unipd.it

Current address:
Dipartimento di Matematica
Università di Trento
I-38050 POVO (*Trento*)
Italy

mattarei@science.unitn.it

https://doi.org/10.1112/S0024610799008571 Published online by Cambridge University Press