Proof of Theorem 2: The proof for Theorem 2 is the same as the above proof but 'parallel' is replaced by 'perpendicular' and the cross-product is replaced by the dot-product.

References

1. C. Kimberling, *Encyclopedia of triangle centers-ETC* https://faculty.evansville.edu/ck6/encyclopedia/ETC.html
2. Stothers' geometry web pages: www.maths.gla.ac.uk/wws/cabripages/cabri0.html
3. Stothers, Results on X-logic Triangles: http://www.maths.gla.ac.uk/wws/cabripages/triangle/ninepoint/xlogic.htm

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105.12 Two solutions to a Sangaku problem

In this Note we present two solutions to the Sangaku problem 1.2.8 from [1].

Sangaku problem 1.2.8

We have three circles, Γ_1, Γ_2 and Γ_3, whose radii are respectively a, b, c. Circle Γ_1 is tangent to circle Γ_2 at P, and tangent to circle Γ_3 at Q. We shall prove that

$$PQ^2 = \frac{a^2d^2}{(a+b)(a+c)},$$

where d is the length of the exterior common tangent to Γ_2 and Γ_3.

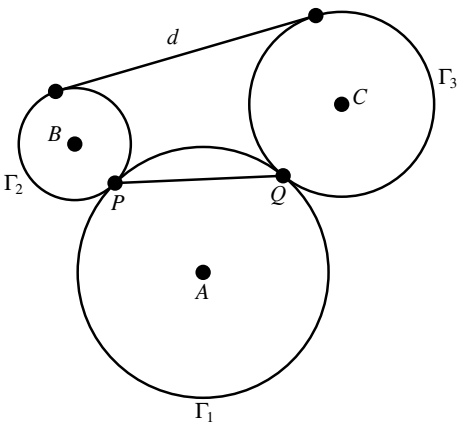


FIGURE 1

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First solution

Let $\angle PAQ = \alpha$. Now, drawing the segments AB, AC, BC the radii of Γ_2 and Γ_3 that are perpendicular to d. Last, we draw a segment from B perpendicular to the radius of Γ_3. Clearly this new segment has length d. See Figure 2.

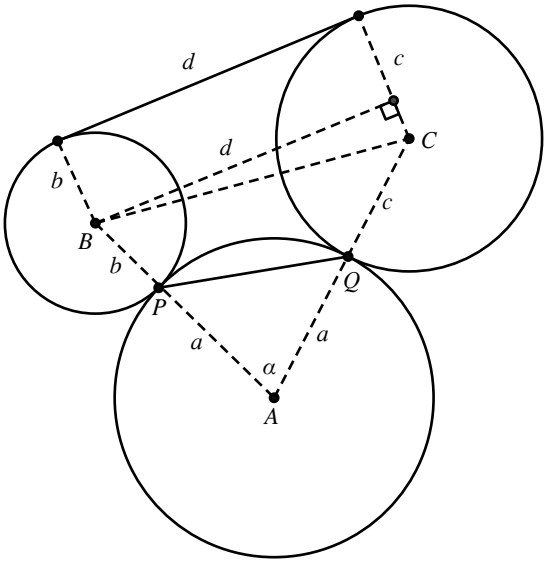


FIGURE 2

We have,

$$PQ^2 = 2a^2(1 - \cos\alpha), \tag{1}$$

$$BC^2 = (a + b)^2 + (a + c)^2 - 2(a + b)(a + c)\cos\alpha, \tag{2}$$

by the cosine rule in the triangles PAQ and BAC respectively. We need to prove the relation

$$d^2 = 2(1 - \cos\alpha)(a + b)(a + c).$$

By the Pythagorean theorem

$$BC^2 = d^2 + (c - b)^2,$$

equating with (2) follows that

$$\begin{aligned} d^2 &= (a + b)^2 + (a + c)^2 - (c - b)^2 - 2(a + b)(a + c)\cos\alpha \\ &= 2a^2 + 2ab + 2bc + 2ca - 2(a + b)(a + c)\cos\alpha \\ &= 2(a + b)(a + c) - 2(a + b)(a + c)\cos\alpha \\ &= 2(1 - \cos\alpha)(a + b)(a + c). \end{aligned}$$

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Second solution

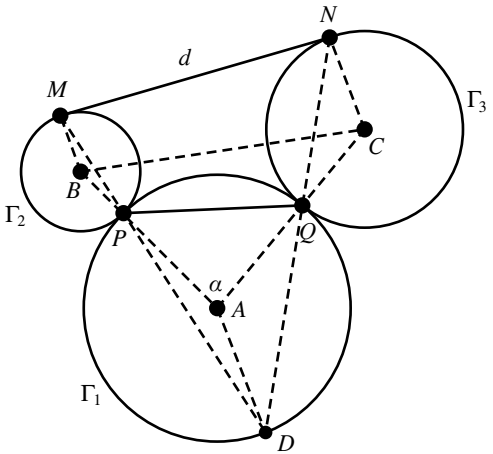


FIGURE 3

If $D = MP \cap NQ$, then $D \in \Gamma_1$. We have,

$$\angle QNM = \frac{\angle QCB + \angle BCN}{2}, \tag{3}$$

$$\angle PMN = \frac{\angle PBC + \angle CBM}{2}, \tag{4}$$

$$\angle NQP = \frac{\alpha + \angle QCB + \angle BCN}{2}, \tag{5}$$

$$\angle QPM = \frac{\alpha + \angle PBC + \angle CBM}{2}. \tag{6}$$

Since $PMNQ$ is a quadrilateral,

$$\angle QNM + \angle PMN + \angle NQP + \angle QPM = 360°.$$

So, from the simultaneous equations (3), (4), (5), (6), we obtain

$$\alpha + \angle QCB + \angle BCN + \angle PBC + \angle CBM = 360°. \tag{7}$$

In triangle DMN , the sum of interior angles is

$$\angle DMN + \angle DNM + \angle MDN = 180°,$$

thus,

$$\frac{360° - \alpha}{2} + \angle MDN = 180°,$$

and then $\angle MDN = \frac{1}{2}\alpha$. As claimed.

From the equations (3), (6) and (7), we deduce

$$\angle QNM + \angle QPM = 180°,$$

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hence, the quadrilateral $PMNQ$ is cyclic. As a consequence, triangles DPQ and DMN are similar. Therefore,

$$\frac{PQ}{d} = \frac{DP}{DN} = \frac{DQ}{DM}.$$

From $AD = AP = AQ = a$, triangles APD and MPB are similar as are triangles DAQ and CQN. Finally, the ratios are

$$\begin{aligned}\frac{d^2}{PQ^2} &= \frac{DN \times DM}{DP \times DQ} \\ &= \frac{(DQ + QN) \times (DP + PM)}{DP \times DQ} \\ &= 1 + \frac{PM}{DP} + \frac{QN}{DQ} + \frac{PM}{DP} \times \frac{QN}{DQ} \\ &= 1 + \frac{b}{a} + \frac{c}{a} + \frac{bc}{a^2} \\ &= \frac{(a + b)(a + c)}{a^2}.\end{aligned}$$

Reference

1. Hidetosi Fukagawa, Dan Pedoe, *Japanese Temple Geometry Problems, Sangaku*, Charles Babbage Research Centre, Winnipeg, Canada, (1989).

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105.13 Proximity of the incentre to the inarc centres

The inarc centres of a triangle

In a triangle ABC with sides a, b, c, semiperimeter s, circumradius R, inradius r and area Δ, let L_A, L_B, L_C denote the inarc centres of the triangle. An *inarc centre* of a triangle is a centre of a circle tangent to two sides of a triangle and internally to the circumcircle of the triangle. The corresponding inarc radii are denoted by ρ_A, ρ_B, ρ_C.

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