EXACT ANALYSIS OF STRESS FIELDS IN COMPOSITE LAMINATES UNDER EXTENSION

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ABSTRACT

Exact analysis of displacements and stresses in 2-D orthotropic laminates under extension is conducted. On the basis of the Hamiltonian state space approach and the transfer matrix method, a complete solution, in the context of generalized strain, which exactly satisfies the state space equation, the traction-free BC on the top and bottom surfaces of the rectangular section, the interfacial continuity conditions in multi-layered laminates, and the end conditions on free edges, is obtained by combing the eigensolutions and the particular solution. Evaluating of the stresses in the boundary layer for verification shows that the stress decay in laminates under uniform extension may be slow and the edge effects may be pronounced.

Keywords: Composite laminates, Orthotropic materials, Free edges, State space formalism.

1. INTRODUCTION

Various types of bounder layer problems can occur in layered composite structures. In the problems of laminates under extension, bounder layer effects can be of particular importance for they are responsible for local damages due to locally stress fields. Since the stress disturbance in anisotropic laminates may not be local, the edge effects of the edge boundary layers may be far reaching, which depend on the material property, the lamination scheme, the boundary conditions, and the applied load [1].

Based on the theory of anisotropic elasticity, exact analysis of stress fields in composite laminates [2] is difficult. Because the interfacial continuity conditions and the edge boundary conditions are hard to exactly satisfy. Then either the exact edge boundary conditions or the interfacial continuity conditions have been relaxed. On this subject, numerous works [2-6] of analytic solutions and finite element solutions based on various simplifications and approximations have been published, such as Becker's work [5], using an assumed displacement model with a particular warp deformation mode. As in Wang *et al*. [6], these solutions are not really exact since rather than in a pointwise fashion, the edge boundary conditions were satisfied in an average sense. Tarn [7] has developed state space formalism for elastic anisotropic materials. Without eliminating field variables or recourse to stress functions, the basic equations of anisotropic elasticity are rewritten to a state equation and an output equation in terms of the state vector that consists of the displacements and conjugate stresses as the dual variables. The output equation, the boundary conditions, and the interfacial continuity conditions are directly expressed in terms of the state vector. So the formalism is well suited for analysis of stress fields in composite laminates.

In this paper we derive an *exact* solution for extension of orthotropic cross-ply laminates based on the refined state space formalism without a priori assumptions. For analysis of stresses in multilayered laminates, the state space approach combined with the transfer matrix method is simple and systematic.

By using a transfer matrix transmitting the state variables from the bottom to the top layer, the interfacial continuity conditions and the traction-free lateral boundary conditions can be satisfied. Regardless the number of layers, the eigenvalue determination requires only the solution of 4×4 determinants. After combining the particular solution with the eigensolutions that both satisfy the prescribed BC on the top and bottom surfaces, a complete solution valid in the whole laminate is obtained. By means of eigenfunction expansion, we delineate the symplectic orthogonality relations of the eigenvectors. Then the edge boundary conditions through the thickness are satisfied, point by

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point, and the solution obtained herein is *exact*, and we shall examine the edge stress decay in laminates subjected to uniform extension. Evaluating of the stresses in the boundary layer shows that the stress decay in laminates may be slow and the edge effects may be pronounced. It must be underlined that such a method cannot account for possible singularities. However, it provides an quite efficient investigation of local effects.

2. STATE SPACE FORMULATION OF THE PROBLEM

2.1 Basic Equations in Matrix Forms

The formulation begins with a refinement of the state space formalism for anisotropic elasticity [7] in which the basic equations are formulated into the state space setting by grouping the stress components judiciously. Herein, the displacement vector and the 3 stress vectors are taken as the basic field variables since the traction and displacement boundary conditions for problems in Cartesian coordinate system are directly associated with them. Upon expressing the 3D equations of anisotropic elasticity in terms of the displacement vector and the stress vectors, we arrive at the following expressions for the generalized Hooke's law, the strain and displacement relations, and the equilibrium equations.

Generalized Hooke's law:

$$\begin{bmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\\ \boldsymbol{\sigma}_3\end{bmatrix}=\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13}\\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23}\\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\\ \boldsymbol{\varepsilon}_3\end{bmatrix}, \tag{1}$$

where $\boldsymbol{\sigma}_i$ denotes the stress vector in the x_i direction, $\boldsymbol{\varepsilon}_i$ consists of the corresponding strain components, $\mathbf{C}_{ij}$ are matrices of the elastic constants given below,

$$\boldsymbol{\sigma}_i=[\sigma_{1i}\ \ \sigma_{2i}\ \ \sigma_{3i}]^T,\quad \boldsymbol{\varepsilon}_i=[\varepsilon_{1i}\ \ \varepsilon_{2i}\ \ \varepsilon_{3i}]^T,$$

$$\mathbf{C}_{21}=\mathbf{C}_{12}^T,\quad \mathbf{C}_{31}=\mathbf{C}_{13}^T,\quad \mathbf{C}_{32}=\mathbf{C}_{23}^T,$$

$$\mathbf{C}_{11}=\begin{bmatrix}c_{11} & c_{16} & c_{15}\\ c_{16} & c_{66} & c_{56}\\ c_{15} & c_{56} & c_{55}\end{bmatrix},\ \mathbf{C}_{12}=\begin{bmatrix}c_{16} & c_{12} & c_{14}\\ c_{66} & c_{26} & c_{46}\\ c_{56} & c_{25} & c_{45}\end{bmatrix},$$

$$\mathbf{C}_{13}=\begin{bmatrix}c_{15} & c_{14} & c_{13}\\ c_{56} & c_{46} & c_{36}\\ c_{55} & c_{45} & c_{35}\end{bmatrix},\ \mathbf{C}_{22}=\begin{bmatrix}c_{66} & c_{26} & c_{46}\\ c_{26} & c_{22} & c_{24}\\ c_{46} & c_{24} & c_{44}\end{bmatrix},$$

$$\mathbf{C}_{23}=\begin{bmatrix}c_{56} & c_{46} & c_{36}\\ c_{25} & c_{24} & c_{23}\\ c_{45} & c_{44} & c_{34}\end{bmatrix},\ \mathbf{C}_{33}=\begin{bmatrix}c_{55} & c_{45} & c_{35}\\ c_{45} & c_{44} & c_{34}\\ c_{35} & c_{34} & c_{33}\end{bmatrix},$$

Strain-displacement relations:

$$\begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\\ \boldsymbol{\varepsilon}_3\end{bmatrix}=\frac{1}{2}\begin{bmatrix}\mathbf{I}\partial_1\\ \mathbf{I}\partial_2\\ \mathbf{I}\partial_3\end{bmatrix}\mathbf{u}+\frac{1}{2}\begin{bmatrix}\nabla u_1\\ \nabla u_2\\ \nabla u_3\end{bmatrix}, \tag{2}$$

where $\mathbf{u}$ is the displacement vector, $\mathbf{I}$ is the identity matrix, ∂_i denotes the partial derivative with respect to the coordinate $\boldsymbol{x}_i$, and

$$\mathbf{u}=[u_1\ \ u_2\ \ u_3]^T,\quad \nabla=[\partial_1\ \ \partial_2\ \ \partial_3]^T.$$

Equilibrium equations:

$$\partial_1\boldsymbol{\sigma}_1+\partial_2\boldsymbol{\sigma}_2+\partial_3\boldsymbol{\sigma}_3+\mathbf{F}=0, \tag{3}$$

where $\mathbf{F}$ is the body force vector.

Equations (1) ~ (2) can be combined into

$$\begin{bmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\\ \boldsymbol{\sigma}_3\end{bmatrix}=\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{C}_{13}\\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{C}_{23}\\ \mathbf{C}_{31} & \mathbf{C}_{32} & \mathbf{C}_{33}\end{bmatrix}\begin{bmatrix}\mathbf{I}\partial_1\\ \mathbf{I}\partial_2\\ \mathbf{I}\partial_3\end{bmatrix}\mathbf{u}. \tag{4}$$

with the basic equations of anisotropic elasticity so expressed, the individual elastic constants and the displacement and stress components are no longer in view; they are replaced by $\mathbf{u}$, $\boldsymbol{\sigma}_i$ and $\mathbf{C}_{ij}$, which play the principal roles hereafter.

2.2 State Equation and Output Equation

The problem under study is a composite laminate of rectangular section (as shown in Fig. 1) subjected to an axial force at the end sections. The origin of the Cartesian coordinates is located at the center of the middle plane. The free-edge boundaries are defined by $x_1=\pm a/2$. The x_2-axis is pointing in the thickness direction such that the top and bottom planes and the interfaces between adjacent layers are defined by $x_2=$ constant. The thickness of laminate is h. The end sections are defined by $x_3=0$, l.

In formulating Eqs. (3) ~ (4) into a state equation and an output equation it is important to choose judiciously from the unknown state variables to form the state vector. Since satisfaction of the free-edge boundary conditions is essential for the problem, it is expedient to choose the displacement vector $\mathbf{u}$ and the stress vector $\boldsymbol{\sigma}_1$ to form the state vector in order to derive the solution that satisfies the free-edge boundary conditions exactly.

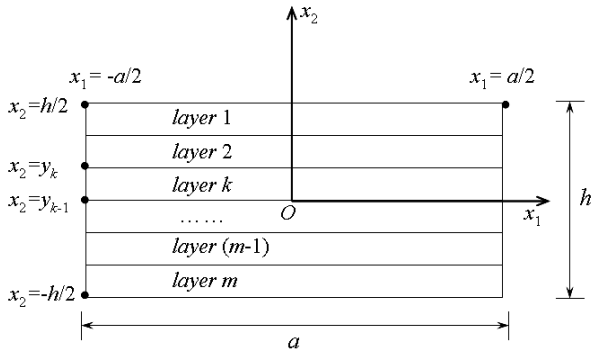


Fig. 1 A composite laminate of rectangular section.

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By using Eq. (4), the stress vectors $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ can be expressed in terms of $\mathbf{u}$ and $\boldsymbol{\sigma}_1$ as

$$\begin{bmatrix}\boldsymbol{\sigma}_2\\ \boldsymbol{\sigma}_3\end{bmatrix}=\begin{bmatrix}\mathbf{Q}_{22}\,\partial_2+\mathbf{Q}_{23}\,\partial_3 & \mathbf{C}_{21}\,\mathbf{C}_{11}^{-1}\\ \mathbf{Q}_{32}\,\partial_2+\mathbf{Q}_{33}\,\partial_3 & \mathbf{C}_{31}\,\mathbf{C}_{11}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\sigma}_1\end{bmatrix}, \tag{5}$$

where

$$\mathbf{Q}_{ij}=\mathbf{C}_{ij}-\mathbf{C}_{i1}\,\mathbf{C}_{11}^{-1}\mathbf{C}_{1j}\,.$$

Substituting Eq. (5) into Eq. (3) results in

$$\frac{\partial}{\partial x_1}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\sigma}_1\end{bmatrix}=\begin{bmatrix}\mathbf{D}_{11} & \mathbf{C}_{11}^{-1}\\ \mathbf{D}_{21} & \mathbf{D}_{11}^{T}\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \boldsymbol{\sigma}_1\end{bmatrix}-\begin{bmatrix}\mathbf{0}\\ \mathbf{F}\end{bmatrix}, \tag{6}$$

where

$$\mathbf{D}_{11}=-\mathbf{C}_{11}^{-1}[\mathbf{C}_{12}\;\;\mathbf{C}_{13}]\begin{bmatrix}\mathbf{I}\partial_2\\ \mathbf{I}\partial_3\end{bmatrix},$$

$$\mathbf{D}_{21}=-[\mathbf{I}\partial_2\;\;\mathbf{I}\partial_3]\begin{bmatrix}\mathbf{Q}_{22} & \mathbf{Q}_{23}\\ \mathbf{Q}_{32} & \mathbf{Q}_{33}\end{bmatrix}\begin{bmatrix}\mathbf{I}\partial_2\\ \mathbf{I}\partial_3\end{bmatrix}.$$

Equations (5) ~ (6) are the output equation and the state equation for the problem. They embrace in full the 3D equations of anisotropic elasticity in Cartesian coordinates and hold for nonhomogeneous media as well. Once the state vector is solved from the state equation, the remaining unknown stress vectors follow from the output equation.

The lateral bounding planes of the laminate are traction-free so that the lateral boundary conditions are

$$\boldsymbol{\sigma}_1=[\sigma_{11}\;\;\sigma_{12}\;\;\sigma_{13}]^T=\mathbf{0}\;\;\text{on}\;x_1=\pm a/2\,, \tag{7}$$

$$\boldsymbol{\sigma}_2=[\sigma_{12}\;\;\sigma_{22}\;\;\sigma_{23}]^T=\mathbf{0}\;\;\text{on}\;x_2=-h/2\,,\;h/2\,. \tag{8}$$

The conditions of interfacial continuity require

$$[\mathbf{u}\;\;\boldsymbol{\sigma}_2]_{k+1}=[\mathbf{u}\;\;\boldsymbol{\sigma}_2]_k\;\;\text{on}\;\;x_2=y_k\,, \tag{9}$$

for $k = 1, 2, \ldots, m-1$, where m is the total number of layers, y_k denotes the thickness coordinate at the interface between the k-th layer and the $(k + 1)$-th layer.

The end conditions require that the stress resultants over the end surfaces reduce to an axial force P at $x_3 = 0, l$ such that

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\int_{-a/2}^{a/2}\boldsymbol{\sigma}_3\,dx_1\,dx_2=\mathbf{P}\,, \tag{10}$$

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\int_{-a/2}^{a/2}(\mathbf{x}\times\boldsymbol{\sigma}_3)\,dx_1\,dx_2=\mathbf{0}\,, \tag{11}$$

where the notation " × " denotes the vector product, and

$$\mathbf{x}=[x_1\;\;x_2\;\;x_3]^T\,,\;\;\mathbf{P}=[P\;\;0\;\;0]^T\,.$$

At this stage the problem is reduced to determining the unknown state vector that satisfies Eq. (6) subjected to Eqs. (7) ~ (9) and Eqs. (10) ~ (11).

3. LAMINATES UNDER EXTENSION AND TORSION

When the laminate is subjected to prescribed end loads and surface loads independent of x_3, it is in the state of generalized plane strain such that the stresses and strains are independent of x_3, but the displacements may be functions of x_3. It can be shown [8] that the general expression of the displacement field is

$$\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=\begin{bmatrix}u(x_1,\,x_2)\\ v(x_1,\,x_2)\\ w(x_1,\,x_2)\end{bmatrix}+\varepsilon\begin{bmatrix}0\\ 0\\ x_3\end{bmatrix}+\vartheta\begin{bmatrix}-x_2x_3\\ x_1x_3\\ 0\end{bmatrix}, \tag{12}$$

where the rigid body displacement has been excluded; u, v and w are unknown functions of x_1 and x_2, the parameter ε is a uniform extension, ϑ is associated with the curvature due to twisting. For a laminated system the parameters ε and ϑ must be the same for all layers in order to satisfy the interfacial continuity conditions.

Substituting Eq. (12) in Eqs. (5) ~ (6) gives

$$\frac{\partial}{\partial x_1}\begin{bmatrix}\tilde{\mathbf{u}}\\ \boldsymbol{\sigma}_1\end{bmatrix}=\begin{bmatrix}-\mathbf{C}_{11}^{-1}\mathbf{C}_{12}\,\partial_2 & \mathbf{C}_{11}^{-1}\\ -\mathbf{Q}_{21}\,\partial_{22} & -\mathbf{C}_{21}\,\mathbf{C}_{11}^{-1}\partial_2\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{u}}\\ \boldsymbol{\sigma}_1\end{bmatrix}+\begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix}, \tag{13}$$

$$\begin{bmatrix}\boldsymbol{\sigma}_2\\ \boldsymbol{\sigma}_3\end{bmatrix}=\begin{bmatrix}\mathbf{Q}_{22}\,\partial_2 & \mathbf{C}_{21}\,\mathbf{C}_{11}^{-1}\\ \mathbf{Q}_{32}\,\partial_2 & \mathbf{C}_{31}\,\mathbf{C}_{11}^{-1}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{u}}\\ \boldsymbol{\sigma}_1\end{bmatrix}+\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix}, \tag{14}$$

where the material properties are nonhomogeneous through the thickness such that $c_{ij} = c_{ij}(x_2)$, and

$$\tilde{\mathbf{u}}=[u\;\;v\;\;w]^T,\;\;\begin{bmatrix}\mathbf{g}_1\\ \mathbf{g}_2\end{bmatrix}=\varepsilon\begin{bmatrix}\mathbf{Q}_{23}\mathbf{k}_3\\ \mathbf{Q}_{33}\mathbf{k}_3\end{bmatrix}+\vartheta x_1\begin{bmatrix}\mathbf{Q}_{23}\mathbf{k}_2\\ \mathbf{Q}_{33}\mathbf{k}_2\end{bmatrix},$$

$$\mathbf{f}_1=-\mathbf{C}_{11}^{-1}\mathbf{C}_{13}\left[\varepsilon\mathbf{k}_3+\vartheta(x_1\,\mathbf{k}_2-x_2\,\mathbf{k}_1)\right],\;\;\mathbf{f}_2=\mathbf{0}\,,$$

$$\mathbf{k}_1=[1\;\;0\;\;0]^T,\;\;\mathbf{k}_2=[0\;\;1\;\;0]^T,\;\;\mathbf{k}_3=[0\;\;0\;\;1]^T.$$

For orthotropic elastic materials a substitution of the elastic constants $c_{14}=c_{15}=c_{16}=c_{24}=c_{25}=c_{26}=c_{34}=c_{35}=c_{36}=c_{45}=c_{46}=c_{56}=0$ in Eqs. (13) ~ (14) gives rise to two uncoupled sets of equations as follows.

The set of equations for the inplane displacement and stress components is

$$\frac{\partial}{\partial x_1}\begin{bmatrix}u\\ v\\ \sigma_{11}\\ \sigma_{12}\end{bmatrix}=\begin{bmatrix}0 & -\hat{c}_{12}\partial_2 & c_{11}^{-1} & 0\\ -\partial_2 & 0 & 0 & c_{66}^{-1}\\ 0 & 0 & 0 & -\partial_2\\ 0 & -\tilde{c}_{22}\partial_{22} & -\hat{c}_{12}\partial_2 & 0\end{bmatrix}\begin{bmatrix}u\\ v\\ \sigma_{11}\\ \sigma_{12}\end{bmatrix}-\varepsilon\begin{bmatrix}\hat{c}_{13}\\ 0\\ 0\\ 0\end{bmatrix}, \tag{15}$$

$$\begin{bmatrix}\sigma_{22}\\ \sigma_{33}\end{bmatrix}=\begin{bmatrix}\tilde{c}_{22}\,\partial_2 & \hat{c}_{12}\\ \tilde{c}_{23}\,\partial_2 & \hat{c}_{13}\end{bmatrix}\begin{bmatrix}v\\ \sigma_{11}\end{bmatrix}+\varepsilon\begin{bmatrix}\tilde{c}_{23}\\ \tilde{c}_{33}\end{bmatrix}. \tag{16}$$

The set of equations for the antiplane displacement and stress components is

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$$\frac{\partial}{\partial x_1}\begin{bmatrix} w \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} 0 & c_{55}^{-1} \\ -c_{44}\partial_{22} & 0 \end{bmatrix}\begin{bmatrix} w \\ \sigma_{13} \end{bmatrix} + \vartheta x_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{17}$$

$$\sigma_{23} = c_{44}(\partial_2 w + \vartheta x_1). \tag{18}$$

where

$$\hat{c}_{ij} = c_{ij}\, c_{11}^{-1}, \quad \tilde{c}_{ij} = c_{ij} - c_{i1}\, c_{11}^{-1} c_{j1}.$$

Now, the task is to determine the state vector that satisfies Eqs. (15) and (17) under the boundary conditions Eqs. (7) ~ (8) and the interfacial conditions Eq. (9). The end conditions Eqs. (10) ~ (11) are used a posterior to determine the constants ε and ϑ due to the applied end loads. Conversely, these constants may be regarded as given for there is a one to one correspondence between them and the axial force P and torque M_t through Eqs. (10) ~ (11). We remark in passing that Eqs. (17) ~ (18) are useful for problems of torsion of laminates [9].

3.1 Particular Solution of Uniform Extension

When the orthotropic laminate is subjected to an uniform extension, the state equation of the k-th layer of the laminate is

$$\frac{\partial}{\partial x_1}\mathbf{Z}_k^* = \mathbf{H}_k\, \mathbf{Z}_k^* + \mathbf{g}_k \ (k = 1,\ 2,\ 3, ...,\ m), \tag{19}$$

where

$$\mathbf{Z}_k^* = \begin{bmatrix} u^*(x_1, x_2) \\ v^*(x_1, x_2) \\ \sigma_{11}^*(x_1, x_2) \\ \sigma_{12}^*(x_1, x_2) \end{bmatrix}_k, \quad \mathbf{H}_k = \begin{bmatrix} 0 & -\hat{c}_{12}\partial_2 & c_{11}^{-1} & 0 \\ -\partial_2 & 0 & 0 & c_{66}^{-1} \\ 0 & 0 & 0 & -\partial_2 \\ 0 & -\tilde{c}_{22}\partial_{22} & -\hat{c}_{12}\partial_2 & 0 \end{bmatrix}_k,$$

$$\mathbf{g}_k = -\varepsilon \begin{bmatrix} \hat{c}_{13} \\ 0 \\ 0 \\ 0 \end{bmatrix}_k.$$

Let

$$\mathbf{Z}_k^*(x_1,\ x_2) = \sum_{n=-\infty}^{\infty} \mathbf{X}_{kn}^*(x_2)\ e^{in\pi x_1/a},$$
$$\mathbf{X}_{kn}^*(x_2) = \left[U^*(x_2)\ \ V^*(x_2)\ \ S_{11}^*(x_2)\ \ S_{12}^*(x_2)\right]_{kn}, \tag{20}$$

where a is the width of the laminate, this alone cannot satisfy the edge boundary conditions at $x_1 = -a/2,\ a/2$ of the problem.

Substituting (20) into (19) yields:

(1) $n = 0$

$$\mathbf{H}_k\, \mathbf{X}_{k(n=0)}^* = -\mathbf{g}_k, \tag{21}$$

Solve (21) and use the output equation, we have

$$\mathbf{Y}_{k0}^*(x_2) = \mathbf{T}_{k0}(x_2;\ y_{k-1})\, \mathbf{Y}_{k0}^*(y_{k-1}) + \mathbf{q}_{k0},$$

$$\mathbf{q}_{k0} = \mathbf{T}_{k0}(x_2;\ y_{k-1})\, \mathbf{Y}_{p0}^*(y_{k-1}) + \mathbf{Y}_{p0}^*(x_2),$$

where

$$\mathbf{Y}_{kn}^*(x_2) = \left[U(x_2)\ \ V(x_2)\ \ S_{12}(x_2)\ \ S_{22}(x_2)\right]_{kn}^T,$$
$$\mathbf{T}_{k0}(x_2;\ y_{k-1}) = \begin{bmatrix} \mathbf{I} & \mathbf{L}_{05} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\mathbf{L}_{05} = \begin{bmatrix} (x_2 - y_{k-1})/c_{66} & 0 \\ 0 & (x_2 - y_{k-1})/c_{22} \end{bmatrix}_k,$$
$$\mathbf{Y}_{p0}^* = \varepsilon[0\ \ 0\ \ 0\ \ c_{23}]_k^T.$$

(2) $n \neq 0$

$$\left(\mathbf{H}_k - \frac{in\pi}{a}\mathbf{I}\right)\mathbf{X}_{kn}^* = 0. \tag{22}$$

Let

$$\mathbf{X}_{kn}^* = \mathbf{D}_k\, e^{\alpha n\pi x_2/a}, \tag{23}$$

where

$$\mathbf{D}_k = [d_1\ \ d_2\ \ d_3\ \ d_4]_k^T,$$

Substituting (23) into (22) yields

$$\begin{bmatrix} -i & -\hat{c}_{12}\alpha & c_{11}^{-1} & 0 \\ -\alpha & -i & 0 & c_{66}^{-1} \\ 0 & 0 & -i & -\alpha \\ 0 & -\tilde{c}_{22}\alpha^2 & -\hat{c}_{12}\alpha & -i \end{bmatrix}_k \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{24}$$

and nontrivial solution requires

$$\left[(\hat{c}_{22})_k \alpha_k^4 - (c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12})_k \alpha_k^2 + 1\right] = 0, \tag{25}$$

which has four roots in pairs:

$$\alpha_{k1} = -\alpha_{k3} = \left[2\hat{c}_{22}^{-1}\left(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12} + \sqrt{(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12})^2 - 4\hat{c}_{22}}\right)\right]_k^{1/2}, \tag{26}$$

$$\alpha_{k2} = -\alpha_{k4} = \left[2\hat{c}_{22}^{-1}\left(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12} - \sqrt{(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12})^2 - 4\hat{c}_{22}}\right)\right]_k^{1/2}. \tag{27}$$

We can obtain the state vector and the output vector as

$$\begin{bmatrix} u^* \\ v^* \\ \sigma_{11}^* \\ \sigma_{12}^* \end{bmatrix}_k = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{4}\left(c_j \begin{bmatrix} f_1(\alpha_j,\ n) \\ f_2(\alpha_j,\ n) \\ f_3(\alpha_j,\ n) \\ f_4(\alpha_j,\ n) \end{bmatrix} e^{\alpha_j n\pi x_2/a} e^{in\pi x_1/a}\right)_k, \tag{28}$$

$$\begin{bmatrix} \sigma_{22}^* \\ \sigma_{33}^* \end{bmatrix}_k = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{4}\left(c_j \begin{bmatrix} f_5(\alpha_j,\ n) \\ f_6(\alpha_j,\ n) \end{bmatrix} e^{\alpha_j n\pi x_2/a} e^{in\pi x_1/a}\right)_k, \tag{29}$$

where

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$$f_1(\alpha_j, n) = c_{66}^{-1}\tilde{c}_{22}\,\alpha_j^2 - \hat{c}_{12}\,\alpha_j^2 + 1, \quad f_2(\alpha_j, n) = \hat{c}_{12}\,\alpha_j^3 - \alpha_j,$$
$$f_3(\alpha_j, n) = \tilde{c}_{22}\,\alpha_j^3 (in\pi/a), \quad f_4(\alpha_j, n) = -\tilde{c}_{22}\,\alpha_j^2 (in\pi/a),$$
$$f_5(\alpha_j, n) = -\tilde{c}_{22}\,\alpha_j^4 (in\pi/a),$$
$$f_6(\alpha_j, n) = \alpha_j^2 (in\pi/a)\left[(\hat{c}_{12}c_{23} - \hat{c}_{13}c_{22})\,\alpha_j^2 - \tilde{c}_{23}\right].$$

Therefore we can obtain

$$\mathbf{Y}_{kn}^*(x_2) = \mathbf{P}(x_2;\ y_{k-1})_k\,\mathbf{Y}_{kn}^*(y_{k-1}), \tag{30}$$

where

$$\mathbf{P}(x_2;\ y_{k-1})_k = [\mathbf{m}_1\ \ \mathbf{m}_2\ \mathbf{m}_3\ \ \mathbf{m}_4]_k < e^{\kappa_{kj}\,\mu(x_2 - y_{k-1})} > [\mathbf{m}_1\ \ \mathbf{m}_2\ \ \mathbf{m}_3\ \ \mathbf{m}_4]_k^{-1},$$
$$\mathbf{m}_j = \left[\, f_1(\alpha_j,\ n)\ \ f_2(\alpha_j,\ n)\ \ f_3(\alpha_j,\ n)\ f_4(\alpha_j,\ n)\ \right]^T,$$

Satisfaction of the interfacial continuity conditions at $x_2 = y_k$ requires

$$\mathbf{Y}_{(k+1)n}^*(y_k) = \mathbf{Y}_{kn}^*(y_k). \tag{31}$$

It follows that

$$\mathbf{Y}_{(k+1)n}^*(y_k) = \mathbf{P}_{kn}(y_k;\ y_{k-1})\,\mathbf{Y}_{kn}^*(y_{k-1}) + \mathbf{q}_k(y_k), \tag{32}$$

$$\mathbf{q}_{k(n\neq 0)}(y_k) = 0, \quad \mathbf{q}_{k(n=0)} = \mathbf{P}_{k0}(y_k;\ y_{k-1})\,\mathbf{Y}_p^*(y_{k-1}) + \mathbf{Y}_p^*(y_k), \tag{33}$$

which transfers the state vector from the k-th layer to the $(k + 1)$-th layer.

Transferring the state vector recursively from the bottom plane downward yields

$$\mathbf{Y}_{kn}^*(x_2) = \mathbf{T}_{kn}(x_2)\,\mathbf{Y}_{1n}^*(h/2) + \boldsymbol{\varphi}_{kn}(x_2), \tag{34}$$

where

$$\mathbf{T}_{kn}(x_2) = \begin{cases} \mathbf{P}_{1n}(x_2; h/2), & k = 1 \\ \mathbf{P}_{kn}(x_2;\ y_{k-1})\mathbf{T}_{(k-1)n}(y_{k-1}), & k = 2,\ 3,\ ...,\ m; \end{cases}$$

$$\boldsymbol{\varphi}_{kn}(x_2) = \begin{cases} \mathbf{q}_{1n}(x_2;\ h/2), & k = 1 \\ \mathbf{P}_{kn}(x_2;\ y_{k-1})\,\boldsymbol{\varphi}_{(k-1)n}(y_{k-1}) + \mathbf{q}_{kn}(x_2;\ y_{k-1}) & k = 2,\ 3,\ ...,\ m. \end{cases}$$

Transferring the state vector recursively from the bottom plane downward and imposing the traction-free boundary conditions at $x_2 = -h/2$ and $x_2 = h/2$ (h: the thickness of the laminate), we have:

(1) $n = 0$

$$\begin{bmatrix} U^*(-h/2) \\ V^*(-h/2) \\ 0 \\ 0 \end{bmatrix}_0 = \begin{bmatrix} 1 & 0 & P_{13}^{(0)} & 0 \\ 0 & 1 & 0 & P_{24}^{(0)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_m \begin{bmatrix} U^*(h/2) \\ V^*(h/2) \\ 0 \\ 0 \end{bmatrix}_0 + \begin{bmatrix} 0 \\ \varphi_2 \\ 0 \\ 0 \end{bmatrix}_{m0}, \tag{35}$$

where

$$\left[P_{13}^{(0)}\right]_m = \frac{h}{m}\sum_{j=1}^{m}(c_{66}^{-1})_j, \quad \left[P_{24}^{(0)}\right]_m = \frac{h}{m}\sum_{j=1}^{m}(c_{22}^{-1})_j,$$
$$(\varphi_2)_{m0} = -\varepsilon\frac{h}{m}\sum_{j=1}^{m}\left(\frac{c_{23}}{c_{22}}\right)_j.$$

(2) $n \neq 0$

$$\begin{bmatrix} U^*(-h/2) \\ V^*(-h/2) \\ 0 \\ 0 \end{bmatrix}_n = \begin{bmatrix} P_{11}(-h/2) & P_{12}(-h/2) & P_{13}(-h/2) & P_{14}(-h/2) \\ P_{21}(-h/2) & P_{22}(-h/2) & P_{23}(-h/2) & P_{24}(-h/2) \\ P_{31}(-h/2) & P_{32}(-h/2) & P_{33}(-h/2) & P_{34}(-h/2) \\ P_{41}(-h/2) & P_{42}(-h/2) & P_{43}(-h/2) & P_{44}(-h/2) \end{bmatrix}_m \begin{bmatrix} U^*(h/2) \\ V^*(h/2) \\ 0 \\ 0 \end{bmatrix}_n. \tag{36}$$

It follows that

$$\left[U^*(h/2)\right]_0 = \left[V^*(h/2)\right]_0 = \left[U^*(h/2)\right]_n = \left[V^*(h/2)\right]_n = 0. \tag{37}$$

As a result, we obtain

$$\begin{bmatrix} u^* \\ v^* \\ \sigma_{11}^* \\ \sigma_{12}^* \end{bmatrix}_k = \varepsilon\begin{bmatrix} 0 \\ -\dfrac{h}{k}\displaystyle\sum_{n=1}^{n=k}\left(\frac{c_{23}}{c_{22}}\right)_n \\ (c_{13} - c_{12}c_{22}^{-1}c_{23})_k \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_{22}^* \\ \sigma_{33}^* \end{bmatrix}_k = \varepsilon\begin{bmatrix} 0 \\ (c_{33} - c_{23}^2 c_{22}^{-1})_k \end{bmatrix}, \tag{38}$$

which satisfies the nonhomogeneous state Eq. (19), the interfacial continuity conditions, and the BC at $x_2 = -h/2$ and $x_2 = h/2$, but not the edge conditions at $x_1 = -a/2,\ a/2$.

3.2 Eigensolution

Consider the homogeneous equations of (19):

$$\frac{\partial}{\partial x_1}\mathbf{Z}_k(x_1,\ x_2) = \mathbf{H}_k(x_2)\,\mathbf{Z}_k(x_1,\ x_2), \quad (k = 1,\ 2,\ 3...,\ m) \tag{39}$$

which, together with the interfacial continuity conditions and the traction-free boundary conditions at $x_2 = -h/2,\ h/2$, constitutes an eigenvalue problem.

Let

$$\mathbf{Z}_k(x_1,\ x_2) = \mathbf{X}_k(x_2)\ e^{\mu x_1}, \tag{40}$$

where μ is the eigenvalue to be determined, and

$$\mathbf{X}_k(x_2) = \left[U(x_2)\ \ V(x_2)\ \ S_{11}(x_2)\ \ S_{12}(x_2)\right]_k^T.$$

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Substituting (40) into (39) yields

$$\mathbf{H}_k \mathbf{X}_k = \mu \mathbf{X}_k \,. \tag{41}$$

Let

$$\mathbf{X}_k = \mathbf{C}_k \, e^{\lambda_k x_2} \,, \tag{42}$$

where

$$\mathbf{C}_k = [c_1 \;\; c_2 \;\; c_3 \;\; c_4]_k^T \,.$$

Substituting (42) into (41) yields

$$\begin{bmatrix} -\mu & -\hat{c}_{12}\lambda & c_{11}^{-1} & 0 \\ -\lambda & -\mu & 0 & c_{66}^{-1} \\ 0 & 0 & -\mu & -\lambda \\ 0 & -\tilde{c}_{22}\lambda^2 & -\hat{c}_{12}\lambda & -\mu \end{bmatrix}_k \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{43}$$

and nontrivial solution requires

$$\left[(\hat{c}_{22})_k \kappa_k^4 - (c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12})_k \kappa_k^2 + 1 \right] = 0 \,, \quad (\kappa_k = \lambda_k / \mu) \tag{44}$$

which has four roots in pairs:

$$\kappa_{k1} = -\kappa_{k3} = \left[2\hat{c}_{22}^{-1} \left(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12} + \sqrt{(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12})^2 - 4\hat{c}_{22}} \right) \right]_k^{1/2} , \tag{45}$$

$$\kappa_{k2} = -\kappa_{k4} = \left[2\hat{c}_{22}^{-1} \left(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12} - \sqrt{(c_{66}^{-1}\tilde{c}_{22} - 2\hat{c}_{12})^2 - 4\hat{c}_{22}} \right) \right]_k^{1/2} . \tag{46}$$

and we obtain the state vector and the output vector:

$$\begin{bmatrix} u \\ v \\ \sigma_{11} \\ \sigma_{12} \end{bmatrix}_k = \begin{bmatrix} U(x_2) \\ V(x_2) \\ S_{11}(x_2) \\ S_{12}(x_2) \end{bmatrix}_k e^{\mu x_1} \,, \quad \begin{bmatrix} \sigma_{22} \\ \sigma_{33} \end{bmatrix}_k = \begin{bmatrix} S_{22}(x_2) \\ S_{33}(x_2) \end{bmatrix}_k e^{\mu x_1} \,, \tag{47}$$

where

$$\begin{bmatrix} U(x_2) \\ V(x_2) \\ S_{11}(x_2) \\ S_{12}(x_2) \end{bmatrix}_k = \sum_{j=1}^{4} c_j \begin{bmatrix} f_1(\kappa_j, \mu) \\ f_2(\kappa_j, \mu) \\ f_3(\kappa_j, \mu) \\ f_4(\kappa_j, \mu) \end{bmatrix}_k e^{\kappa_{kj}\mu x_2} \,,$$

$$\begin{bmatrix} S_{22}(x_2) \\ S_{33}(x_2) \end{bmatrix}_k = \sum_{j=1}^{4} c_j \begin{bmatrix} f_5(\kappa_j, \mu) \\ f_6(\kappa_j, \mu) \end{bmatrix}_k e^{\kappa_{kj}\mu x_2} \,, \tag{48}$$

$$f_1(\kappa_j, \mu) = c_{66}^{-1}\tilde{c}_{22}\kappa_j^2 - \hat{c}_{12}\kappa_j^2 + 1, \quad f_2(\kappa_j, \mu) = \hat{c}_{12}\kappa_j^3 - \kappa_j \,,$$

$$f_3(\kappa_j, \mu) = -\tilde{c}_{22}\kappa_j^4 \mu \,, \quad f_4(\kappa_j, \mu) = \tilde{c}_{22}\kappa_j^3 \mu \,,$$

$$f_5(\kappa_j, \mu) = -\tilde{c}_{22}\kappa_j^2 \mu \,,$$

$$f_6(\kappa_j, \mu) = \kappa_j^2 \mu \left[(\hat{c}_{12}c_{23} - \hat{c}_{13}c_{22})\kappa_j^2 - \tilde{c}_{23} \right].$$

Therefore we can obtain

$$\hat{\mathbf{Y}}_k(x_2) = \mathbf{P}_k(x_2; y_{k-1}) \hat{\mathbf{Y}}_k(y_{k-1}) \,, \tag{49}$$

where

$$\hat{\mathbf{Y}}_k(x_2) = [U(x_2) \;\; V(x_2) \;\; S_{12}(x_2) \;\; S_{22}(x_2)]_k^T \,,$$

$$\mathbf{P}(x_2; y_{k-1})_k = [\mathbf{m}_1 \;\; \mathbf{m}_2 \;\; \mathbf{m}_3 \;\; \mathbf{m}_4]_k < e^{\kappa_{kj}\mu(x_2 - y_{k-1})} > [\mathbf{m}_1 \;\; \mathbf{m}_2 \;\; \mathbf{m}_3 \;\; \mathbf{m}_4]_k^{-1} ,$$

$$\mathbf{m}_j = \left[f_1(\kappa_j, \mu) \;\; f_2(\kappa_j, \mu) \;\; f_3(\kappa_j, \mu) \;\; f_4(\kappa_j, \mu) \right]^T .$$

Specifically, the interfacial continuity conditions at $x_2 = y_k$ require

$$\hat{\mathbf{Y}}_{k+1}(y_k) = \hat{\mathbf{Y}}_k(y_k) \,. \tag{50}$$

It follows that

$$\hat{\mathbf{Y}}_{k+1}(x_2) = \mathbf{P}_{k+1}(x_2; y_k) \, \mathbf{P}_k(y_k; y_{k-1}) \, \hat{\mathbf{Y}}_k(y_{k-1}) \,, \tag{51}$$

which transfers the state vector from the k-th layer to the $(k + 1)$-th layer.

Transferring the state vector recursively from the top plane downward yields

$$\hat{\mathbf{Y}}_k(x_2) = \mathbf{T}_k(x_2) \hat{\mathbf{Y}}_1(h/2) \,, \tag{52}$$

where

$$\mathbf{T}_k(x_2) = \begin{cases} \mathbf{P}_1(x_2; \, h/2), & k = 1 \,; \\ \mathbf{P}_k(x_2; \, y_{k-1}) \mathbf{T}_{k-1}(y_{k-1}) \,, & k = 2, \, 3, \ldots, \, m \,. \end{cases}$$

4. SATISFACTION OF THE BOUNDARY CONDITIONS AT $x_2 = -h/2, \, h/2$

Following the same line by using the transfer matrix, the eigenvalue μ_j (j = 1, 2, ⋯) and the associated eigenvector are determined by satisfying the interfacial continuity conditions and the traction-free conditions on the top and bottom planes.

Setting $x_2 = -h/2$ in Eq. (52) gives

$$\hat{\mathbf{Y}}_m(-h/2) = \mathbf{T}_m(-h/2) \, \hat{\mathbf{Y}}_1(h/2) \,, \tag{53}$$

which can be expressed as

$$\begin{bmatrix} \mathbf{U}_m(-h/2) \\ \mathbf{S}_m(-h/2) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{uu}(-h/2) & \mathbf{T}_{us}(-h/2) \\ \mathbf{T}_{su}(-h/2) & \mathbf{T}_{ss}(-h/2) \end{bmatrix}_m \begin{bmatrix} \mathbf{U}_1(h/2) \\ \mathbf{S}_1(h/2) \end{bmatrix}. \tag{54}$$

The traction-free BC on $x_2 = -h/2$ and $x_2 = h/2$ demand

$$\mathbf{S}_1(h/2) = 0 \,, \quad \mathbf{S}_m(-h/2) = 0 \,. \tag{55}$$

Imposing them on Eq. (54) yields

$$\left| [\mathbf{T}_{su}(-h/2)]_m \right| = 0 \,, \tag{56}$$

from which the eigenvalues μ are determined.

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By inspection on Eq. (49), if μ is a root of Eq. (56), so is −μ. Moreover, μ = 0 is a repeated root, which can be verified by replacing μ by −μ in Eq. (56). The case of $\mu_0 = 0$ requires special consideration. Since $\mu_0 = 0$ is a repeated root of Eq. (56), linearly independent eigenvectors associated with $\mu_0 = 0$ can be determined by considering the Jordan-chain solution given as

$$\begin{cases} \boldsymbol{\psi}_{k01} = \boldsymbol{\psi}_{k0}^{(0)}, \\ \boldsymbol{\psi}_{k02} = \boldsymbol{\psi}_{k0}^{(1)} + x_1 \boldsymbol{\psi}_{k0}^{(0)}, \end{cases}, \tag{57}$$

and are determined from the Jordan chain:

$$\begin{cases} \mathbf{H}_k \boldsymbol{\psi}_{k0}^{(0)} = 0, \\ \mathbf{H}_k \boldsymbol{\psi}_{k0}^{(1)} = \boldsymbol{\psi}_{k0}^{(0)}, \end{cases} \tag{58}$$

with the interfacial continuity conditions and the traction-free BC at $x_2 = -h/2, h/2$ where

$$\boldsymbol{\psi}_{k0}^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\psi}_{k0}^{(1)} = \begin{bmatrix} 0 \\ \eta_k - \left[(\tilde{c}_{22} + c_{12}\hat{c}_{12})^{-1} c_{12} \right]_k x_2 \\ \left[c_{11} - (\tilde{c}_{22} + c_{12}\hat{c}_{12})^{-1} c_{12}^2 \right]_k \\ 0 \end{bmatrix}, \tag{59}$$

with

$$\eta_k = \sum_{j=1}^{k-1} \left\{ \left[(\tilde{c}_{22} + c_{12}\hat{c}_{12})^{-1} c_{12} \right]_{j+1} - \left[(\tilde{c}_{22} + c_{12}\hat{c}_{12})^{-1} c_{12} \right]_j \right\} y_j .$$

Adding the eigensolution (47) to the particular solution (38), a linear combination produces the complete solution of Eq. (15):

$$\begin{bmatrix} u \\ v \\ \sigma_{11} \\ \sigma_{12} \end{bmatrix}_k = \begin{bmatrix} u^* \\ v^* \\ \sigma_{11}^* \\ \sigma_{12}^* \end{bmatrix}_k + (A_0 \boldsymbol{\psi}_{k01} + A_1 \boldsymbol{\psi}_{k02}) + \sum_{n=1}^{\infty} \left[A_n \boldsymbol{\psi}_{kn} e^{\mu_n (x_1 - a/2)} + A_{-n} \boldsymbol{\psi}_{-kn} e^{-\mu_n (x_1 + a/2)} \right], \tag{60}$$

where A_0, A_1, A_n and A_{-n} are constants of linear combination, and

$$\boldsymbol{\psi}_{kn} = \begin{bmatrix} \psi_1(x_2) \\ \psi_2(x_2) \\ \psi_3(x_2) \\ \psi_4(x_2) \end{bmatrix}_k, \quad \boldsymbol{\psi}_{-kn} = \begin{bmatrix} \psi_{-1}(x_2) \\ \psi_{-2}(x_2) \\ \psi_{-3}(x_2) \\ \psi_{-4}(x_2) \end{bmatrix}_k,$$

with

$$\begin{bmatrix} \psi_1(x_2) \\ \psi_2(x_2) \\ \psi_3(x_2) \\ \psi_4(x_2) \end{bmatrix}_k = \sum_{j=1}^{4} c_{kj} \begin{bmatrix} f_1(\kappa_j, \mu_n) \\ f_2(\kappa_j, \mu_n) \\ f_3(\kappa_j, \mu_n) \\ f_4(\kappa_j, \mu_n) \end{bmatrix}_k e^{\kappa_{kj}\mu_n x_2},$$

$$\begin{bmatrix} \psi_{-1}(x_2) \\ \psi_{-2}(x_2) \\ \psi_{-3}(x_2) \\ \psi_{-4}(x_2) \end{bmatrix}_k = \sum_{j=1}^{4} c_{kj} \begin{bmatrix} f_1(\kappa_j, -\mu_n) \\ f_2(\kappa_j, -\mu_n) \\ f_3(\kappa_j, -\mu_n) \\ f_4(\kappa_j, -\mu_n) \end{bmatrix}_k e^{-\kappa_{kj}\mu_n x_2}.$$

5. SYMPLECTIC ORTHOGONALITY

To facilitate exposition, let us express Eq. (41) with the BC at $x_2 = y_k$ and $x_2 = y_{k+1}$ as

$$\mathbf{H}_k \boldsymbol{\psi}_k = \mu \boldsymbol{\psi}_k, \quad \mathbf{B}_k(\boldsymbol{\psi}_k) = \mathbf{0}, \tag{61}$$

where

$$\mathbf{H}_k = \begin{bmatrix} \mathbf{L}_{11k} & \mathbf{L}_{12k} \\ \mathbf{L}_{21k} & -\mathbf{L}_{11k}^* \end{bmatrix},$$

$$\mathbf{B}_k(\boldsymbol{\psi}_k) = \begin{bmatrix} \hat{\mathbf{Y}}_k(y_k) - \hat{\mathbf{Y}}_{k+1}(y_k) \\ \hat{\mathbf{Y}}_k(y_{k-1}) - \hat{\mathbf{Y}}_{k-1}(y_{k-1}) \end{bmatrix},$$

$$\mathbf{L}_{11k} = \begin{bmatrix} 0 & -\hat{c}_{12}\partial_2 \\ -\partial_2 & 0 \end{bmatrix}_k, \quad \mathbf{L}_{12k} = \begin{bmatrix} c_{11}^{-1} & 0 \\ 0 & c_{66}^{-1} \end{bmatrix}_k,$$

$$\mathbf{L}_{21k} = \begin{bmatrix} 0 & 0 \\ 0 & -\tilde{c}_{22}\partial_{22} \end{bmatrix}_k,$$

$$\boldsymbol{\psi}_k = \begin{bmatrix} \mathbf{U} \\ \mathbf{S} \end{bmatrix}_k, \quad \mathbf{U}_k = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_k, \quad \mathbf{S}_k = \begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix}_k.$$

The matrix $\mathbf{H}_k$ in Eq. (61) exhibits the characteristics of a Hamiltonian matrix such that $\mathbf{L}_{12k}$ and $\mathbf{L}_{21k}$ are symmetric matrix and the operators $\mathbf{L}_{11k}$ and $\mathbf{L}_{11k}^*$ are adjoint. Such characteristics are of fundamental significance in the formulation.

First of all, the adjoint system of Eq. (61) is

$$\mathbf{H}_k \boldsymbol{\varphi}_k = -\mu \boldsymbol{\varphi}_k, \quad \mathbf{B}_k(\boldsymbol{\varphi}_k) = \mathbf{0}, \tag{62}$$

where $\boldsymbol{\phi}$ is the eigenvector associated with the eigenvalue −μ, and

$$\boldsymbol{\varphi}_k = \begin{bmatrix} \mathbf{V} \\ \mathbf{T} \end{bmatrix}_k, \quad \mathbf{V}_k = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}_k, \quad \mathbf{T}_k = \begin{bmatrix} \varphi_3 \\ \varphi_4 \end{bmatrix}_k.$$

Derivation of symplectic orthogonality [10] begins with the equality

$$\sum_{k=1}^{m} \int_{y_{k-1}}^{y_k} \boldsymbol{\varphi}_k^T \mathbf{J} \mathbf{H}_k \boldsymbol{\psi}_k \, dx_2 = \sum_{k=1}^{m} \int_{y_{k-1}}^{y_k} \left[\mathbf{V}_k^T (\mathbf{L}_{21k}\mathbf{U}_k - \mathbf{L}_{11k}^* \mathbf{S}_k) - \mathbf{T}_k^T (\mathbf{L}_{11k}\mathbf{U}_k + \mathbf{L}_{12k}\mathbf{S}_k) \right] dx_2, \tag{63}$$

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where

$$\mathbf{J}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}_{4\times 4}.$$

Integrating by parts the right-hand side of Eq. (63), after manipulation we arrive at

$$\begin{aligned}&\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\varphi}_k^T\mathbf{J}\mathbf{H}_k\boldsymbol{\psi}_k\,dx_2\\&=\sum_{k=1}^{m}\left\{\int_{y_{k-1}}^{y_k}\left[\mathbf{U}_k^T(\mathbf{L}_{21k}\mathbf{V}_k-\mathbf{L}_{11k}^*\mathbf{T}_k)-\mathbf{S}_k^T(\mathbf{L}_{11k}\mathbf{V}_k+\mathbf{L}_{12k}\mathbf{T}_k)\right]dx_2\right.\\&\quad\left.+(\mathbf{U}_k^T\mathbf{T}_k-\mathbf{V}_k^T\mathbf{S}_k)\Big|_{x_2=y_{k-1}}^{x_2=y_k}\right\}\\&=\sum_{k=1}^{m}\left[\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_k^T\mathbf{J}\mathbf{H}_k\boldsymbol{\varphi}_k dx_2+(\mathbf{U}_k^T\mathbf{T}_k-\mathbf{V}_k^T\mathbf{S}_k)\Big|_{x_2=y_{k-1}}^{x_2=y_k}\right],\end{aligned}\tag{64}$$

which shows that $\mathbf{J}\mathbf{H}_k$ is self-adjoint such that $(\mathbf{J}\mathbf{H}_k)^* = \mathbf{J}\mathbf{H}_k$, where the star * denotes the adjoint.

Pre-multiplying Eq. (61) by $\boldsymbol{\varphi}_k^T\mathbf{J}$, integrating it over (y_k, y_{k+1}) and taking transpose, with $\mathbf{J}^T=-\mathbf{J}$, we obtain

$$\begin{aligned}\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\varphi}_k^T\mathbf{J}\mathbf{H}_k\boldsymbol{\psi}_k\,dx_2&=\mu\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\varphi}_k^T\mathbf{J}\boldsymbol{\psi}_k\,dx_2\\&=-\mu\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_k^T\mathbf{J}\boldsymbol{\varphi}_k\,dx_2,\end{aligned}\tag{65}$$

which, upon substituting into Eq. (64), results in

$$\begin{aligned}&\sum_{k=1}^{m}\left[\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_k^T\mathbf{J}(\mathbf{H}_k\boldsymbol{\varphi}_k+\mu\boldsymbol{\varphi}_k)\,dx_2\right.\\&\qquad\left.+(\mathbf{U}_k^T\mathbf{T}_k-\mathbf{V}_k^T\mathbf{S}_k)\Big|_{x_2=y_{k-1}}^{x_2=y_k}\right]=0.\end{aligned}\tag{66}$$

There follows Eq. (66) under the homogeneous BC at $x_2=-h/2$ and $x_2=h/2$.

Equations (61) and (62) reveal that if μ is an eigenvalue of $\mathbf{H}_k$, so is $-\mu$. It follows that $\mu=0$ must be a repeated eigenvalue. Indeed, the eigenvalues determined from Eqs. (56) and (58) verify this important property of a Hamiltonian system.

Since the eigenvalues appear in $(\mu_i, -\mu_i)$ pairs, it is expedient to arrange the order of the eigenvectors in the following sequence:

(1) the eigenvectors $\boldsymbol{\psi}_{ki}$ ($i=1, 2, \cdots$) associated with the eigenvalues μ_i.

(2) the eigenvectors $\boldsymbol{\psi}_{-ki}\equiv\boldsymbol{\phi}$ ($i=1, 2, \cdots$) associated with the eigenvalues μ_{-i} ($=-\mu_i$).

(3) the eigenvectors $\boldsymbol{\psi}_{k0}$ associated with the repeated eigenvalue $\mu_0=0$.

Consider two linearly independent eigenvectors $\boldsymbol{\psi}_{ki}$ and $\boldsymbol{\psi}_{kj}$ associated with the eigenvalues μ_i and μ_j, respectively,

$$\mathbf{H}_k\boldsymbol{\psi}_{ki}=\mu_i\boldsymbol{\psi}_{ki},\quad \mathbf{B}_k(\boldsymbol{\psi}_{ki})=\mathbf{0},\tag{67}$$

$$\mathbf{H}_k\boldsymbol{\psi}_{kj}=\mu_j\boldsymbol{\psi}_{kj},\quad \mathbf{B}_k(\boldsymbol{\psi}_{kj})=\mathbf{0}.\tag{68}$$

Pre-multiplying Eq. (67) by $\boldsymbol{\psi}_{kj}^T\mathbf{J}$ and Eq. (68) by $\boldsymbol{\psi}_{ki}^T\mathbf{J}$, integrating the resulting equations with respect to x_2 over (y_{k-1}, y_k), we obtain

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{kj}^T\mathbf{J}\mathbf{H}_k\boldsymbol{\psi}_{ki}\,dx_2=\mu_i\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{kj}^T\mathbf{J}\boldsymbol{\psi}_{ki}\,dx_2,\tag{69}$$

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{ki}^T\mathbf{J}\mathbf{H}_k\boldsymbol{\psi}_{kj}\,dx_2=\mu_j\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{ki}^T\mathbf{J}\boldsymbol{\psi}_{kj}\,dx_2.\tag{70}$$

Integrating by parts the left-hand side of (70) as was done in Eq. (65) by setting $\boldsymbol{\phi}_k=\boldsymbol{\psi}_{kj}$, $\boldsymbol{\psi}_k=\boldsymbol{\psi}_{kj}$, using the fact that $\mathbf{J}\mathbf{H}_k$ is self-adjoint and $\mathbf{J}^T=-\mathbf{J}$, we obtain

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{ki}^T\mathbf{J}\mathbf{H}_k\boldsymbol{\psi}_{kj}\,dx_2=-\mu_i\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{ki}^T\mathbf{J}\boldsymbol{\psi}_{kj}\,dx_2.\tag{71}$$

Subtracting Eqs. (70) and (71) gives

$$(\mu_i+\mu_j)\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{ki}^T\mathbf{J}\boldsymbol{\psi}_{kj}\,dx_2=0.\tag{72}$$

Since $\mu_i+\mu_j\neq 0$ ($i\neq -j$) according to the sequence of the eigenvectors defined before, there follows the orthogonal relation for the eigenvectors associated with $\mu_i\neq 0$ ($i=1, 2, \cdots$):

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{ki}^T\mathbf{J}\boldsymbol{\psi}_{kj}\,dx_2\begin{cases}=0 & i\neq -j\\ \neq 0 & i=-j\end{cases},\tag{73}$$

or the simpler form:

$$\int_{-h/2}^{h/2}\boldsymbol{\psi}_i^T\mathbf{J}\boldsymbol{\psi}_j\,dx_2\begin{cases}=0 & i\neq -j\\ \neq 0 & i=-j\end{cases},\tag{74}$$

where the definite integrals derived from the orthogonality relations are piecewise continuous integration with respect to x_2 over the laminate thickness, with the lower limit $x_2=-h/2$ and the upper limit $x_2=h/2$.

The case of the zero eigenvalue needs special consideration. It has been shown that $\mu_0=0$ is a repeated eigenvalue. The two linearly independent eigenvectors $\boldsymbol{\psi}_k^{(0)}$ and $\boldsymbol{\psi}_k^{(1)}$ can be determined from the Jordan chain associated with $\mu_0=0$:

$$\mathbf{H}_k\boldsymbol{\psi}_{k0}^{(0)}=\mathbf{0},\quad \mathbf{B}_k\left(\boldsymbol{\psi}_{k0}^{(0)}\right)=\mathbf{0},\tag{75}$$

$$\mathbf{H}_k\boldsymbol{\psi}_{k0}^{(1)}=\boldsymbol{\psi}_{k0}^{(0)},\quad \mathbf{B}_k\left(\boldsymbol{\psi}_{k0}^{(1)}\right)=\mathbf{0}.\tag{76}$$

Consider the nonhomogeneous system

$$\mathbf{H}_k\boldsymbol{\phi}_{k0}=\boldsymbol{\psi}_{k0}^{(1)},\quad \mathbf{B}_k(\boldsymbol{\phi}_{k0})=\mathbf{0},\tag{77}$$

which is unsolvable because the Jordan chain of $\mu_0=0$ breaks at the second order.

Pre-multiplying Eq. (77) by $\left[\boldsymbol{\psi}_{k0}^{(0)}\right]^T\mathbf{J}$ and integrating it over (y_{k-1}, y_k) gives

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$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(0)}\right]^T\mathbf{JH}\boldsymbol{\phi}_{k0}\,dx_2=\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(0)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(1)}\,dx_2\,, \tag{78}$$

in which, on setting $\boldsymbol{\phi}_{k0}=\boldsymbol{\psi}_{k0}^{(0)}$, the left-hand side is zero because of Eq. (75), so the solvability condition of Eq. (78) is

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(0)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(1)}\,dx_2=0\,. \tag{79}$$

Knowing that Eq. (78) is unsolvable, so the above solvability condition is not satisfied. It follows

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(0)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(1)}\,dx_2=-\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(1)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(0)}\,dx_2\neq 0\,, \tag{80}$$

in which the equality is reached by taking the transpose and using $\mathbf{J}^T=-\mathbf{J}$.

By the same token,

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(\alpha)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(\alpha)}dx_2 =-\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(\alpha)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(\alpha)}dx_2\,,\quad(\alpha=0,\,1) \tag{81}$$

which is possible if and only if

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(\alpha)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(\alpha)}dx_2=0\,,\quad(\alpha=0,\,1)\,. \tag{82}$$

The orthogonal relation of the eigenvectors associated with $\mu_0=0$ is

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\left[\boldsymbol{\psi}_{k0}^{(\alpha)}\right]^T\mathbf{J}\boldsymbol{\psi}_{k0}^{(\beta)}dx_2\begin{cases}=0\,, & (\beta=\alpha\,;\ \alpha,\ \beta=0,\,1)\\ \neq 0\,, & (\beta\neq\alpha\,;\ \alpha,\ \beta=0,\,1)\end{cases}, \tag{83}$$

or the simpler form:

$$\int_{-h/2}^{h/2}\left[\boldsymbol{\psi}_{0}^{(\alpha)}\right]^T\mathbf{J}\boldsymbol{\psi}_{0}^{(\beta)}dx_2\begin{cases}=0\,, & (\beta=\alpha\,;\ \alpha,\ \beta=0,\,1)\\ \neq 0\,, & (\beta\neq\alpha\,;\ \alpha,\ \beta=0,\,1)\end{cases}. \tag{84}$$

Symplectic orthogonality in the Hamiltonian state space holds for the eigenvector derived from the state vector which is composed of the displacement vector and the associated stress vector.

6. SATISFACTION OF THE BOUNDARY CONDITIONS AT $x_1=\pm a/2$

The complete solution of k-th layer is required to satisfy the end conditions at $x_1=-a/2,\ a/2$. To this end, imposing Eqs. (5) ~ (6) on Eq. (51), we obtain

$$\mathbf{F}_{k(-a/2)}=(A_0\boldsymbol{\psi}_{k01}+A_1\boldsymbol{\psi}_{k02})_{x_1=-a/2} +\sum_{n=1}^{\infty}(A_n\boldsymbol{\psi}_{kn}e^{-\mu_n a}+A_{-n}\boldsymbol{\psi}_{-kn})\,, \tag{85}$$

$$\mathbf{F}_{k(a/2)}=(A_0\boldsymbol{\psi}_{k01}+A_1\boldsymbol{\psi}_{k02})_{x_1=a/2} +\sum_{n=1}^{\infty}(A_n\boldsymbol{\psi}_{kn}+A_{-n}\boldsymbol{\psi}_{-kn}\,e^{-\mu_n a})\,, \tag{86}$$

where

$$\mathbf{F}_{k(-a/2)}=\begin{bmatrix}u(-a/2,\,x_2)\\ v(-a/2,\,x_2)\\ 0\\ 0\end{bmatrix}_k-\varepsilon\begin{bmatrix}0\\ -\dfrac{h}{m}\displaystyle\sum_{n=1}^{n=m}\left(\frac{c_{23}}{c_{22}}\right)_n\\ c_{13}-c_{12}\,c_{22}^{-1}c_{23}\\ 0\end{bmatrix}_k,$$

$$\mathbf{F}_{k(a/2)}=\begin{bmatrix}u(a/2,\,x_2)\\ v(a/2,\,x_2)\\ 0\\ 0\end{bmatrix}_k-\varepsilon\begin{bmatrix}0\\ -\dfrac{h}{m}\displaystyle\sum_{n=1}^{n=m}\left(\frac{c_{23}}{c_{22}}\right)_n\\ c_{13}-c_{12}\,c_{22}^{-1}c_{23}\\ 0\end{bmatrix}_k,$$

in which A_0, A_1, A_n and A_{-n} are constants to be determined, $\mathbf{F}_{k(-a/2)}$ contains the unknown displacements $[u(-a/2,\,x_2)]_k$ and $[v(-a/2,\,x_2)]_k$ at the left free end, $\mathbf{F}_{k(a/2)}$ contains the unknown displacements $[u(a/2,\,x_2)]_k$ and $[v(a/2,\,x_2)]_k$ at the right free end, which, in turn, can be expressed in terms of A_0, A_1, A_n and A_{-n} by using Eq. (60) as

$$\begin{bmatrix}u(-a/2,\,x_2)\\ v(-a/2,\,x_2)\end{bmatrix}_k=\begin{bmatrix}u^*(-a/2,\,x_2)\\ v^*(-a/2,\,x_2)\end{bmatrix}_k+A_0\begin{bmatrix}1\\ 0\end{bmatrix} +A_1\begin{bmatrix}-a/2\\ \eta_k-\left[(\tilde{c}_{22}+c_{12}\hat{c}_{12})^{-1}c_{12}\right]_k x_2\end{bmatrix} +\sum_{n=1}^{\infty}\left\{A_n\begin{bmatrix}\psi_1(x_2)\\ \psi_2(x_2)\end{bmatrix}_k e^{-\mu_n a}+A_{-n}\begin{bmatrix}\psi_{-1}(x_2)\\ \psi_{-2}(x_2)\end{bmatrix}_k\right\}, \tag{87}$$

$$\begin{bmatrix}u(a/2,\,x_2)\\ v(a/2,\,x_2)\end{bmatrix}_k=\begin{bmatrix}u^*(a/2,\,x_2)\\ v^*(a/2,\,x_2)\end{bmatrix}_k+A_0\begin{bmatrix}1\\ 0\end{bmatrix} +A_1\begin{bmatrix}a/2\\ \eta_k-\left[(\tilde{c}_{22}+c_{12}\hat{c}_{12})^{-1}c_{12}\right]_k x_2\end{bmatrix} +\sum_{n=1}^{\infty}\left\{A_n\begin{bmatrix}\psi_1(x_2)\\ \psi_2(x_2)\end{bmatrix}_k+A_{-n}\begin{bmatrix}\psi_{-1}(x_2)\\ \psi_{-2}(x_2)\end{bmatrix}_k e^{-\mu_n a}\right\}. \tag{88}$$

To determine A_0, A_1, A_n and A_{-n} in Eqs. (85) and (86) with the symplectic orthogonality given by Eqs. (74) and (84), it is possible to express an arbitrary integrable state vector in terms of a sequence of orthogonal eigenvectors in the state space by means of eigenfunction expansion.

Multiplying both sides of Eqs. (85) and (86) by $\boldsymbol{\psi}_{-kn}^T\mathbf{J}$, $\boldsymbol{\psi}_{kn}^T\mathbf{J}$, $[\boldsymbol{\psi}_{k0}^{(1)}]^T\mathbf{J}$ and $[\boldsymbol{\psi}_{k0}^{(0)}]^T\mathbf{J}$, respectively, and integrating them over $(y_{k-1},\,y_k)$ and then summating from 1-st layer to m-th layer, making use of the symplectic orthogonality, we obtain

$$A_n=e^{\mu_n a}\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{-kn}^T\mathbf{J}\mathbf{F}_{k(-a/2)}\,dx_2\,,$$
$$A_{-n}=-\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k}\boldsymbol{\psi}_{kn}^T\mathbf{J}\mathbf{F}_{k(-a/2)}\,dx_2\,, \tag{89}$$

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$$A_0 - \frac{a}{2}A_1 = \sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} [\boldsymbol{\psi}_{k0}^{(1)}]^T \mathbf{J}\mathbf{F}_{k(-a/2)}\, dx_2,$$
$$A_1 = -\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} [\boldsymbol{\psi}_{k0}^{(0)}]^T \mathbf{J}\mathbf{F}_{k(-a/2)}\, dx_2, \tag{90}$$

$$A_n = \sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} \boldsymbol{\psi}_{-kn}^T \mathbf{J}\mathbf{F}_{k(a/2)} dx_2\ ,$$
$$A_{-n} = -e^{\mu_n a}\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} \boldsymbol{\psi}_{kn}^T \mathbf{J}\mathbf{F}_{k(a/2)} dx_2\ , \tag{91}$$

$$A_0 + \frac{a}{2}A_1 = \sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} [\boldsymbol{\psi}_{k0}^{(1)}]^T \mathbf{J}\mathbf{F}_{k(a/2)} dx_2\ ,$$
$$A_1 = -\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} [\boldsymbol{\psi}_{k0}^{(0)}]^T \mathbf{J}\mathbf{F}_{k(a/2)} dx_2\ . \tag{92}$$

In Eqs. (89) ~ (92) each of the vectors $\mathbf{F}_{k(-a/2)}$ and $\mathbf{F}_{k(a/2)}$ contains four components, of which the displacements $\left[u(-a/2,\ x_2)\right]_k$ and $\left[v(-a/2,\ x_2)\right]_k$ are unknown at the left end $x_1 = -a/2$, the displacements $\left[u(a/2,\ x_2)\right]_k$ and $\left[v(a/2,\ x_2)\right]_k$ are unknown at the right free end $x_1 = a/2$. Substituting Eqs. (89) ~ (90) in Eqs. (91) ~ (92) gives

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} \boldsymbol{\psi}_{-kn}^T \mathbf{J}\left[\mathbf{F}_{k(-a/2)} - e^{-\mu_i a}\mathbf{F}_{k(a/2)}\right] dx_2 = 0\,, \tag{93}$$

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} \boldsymbol{\psi}_{kn}^T \mathbf{J}\left[e^{-\mu_i a}\mathbf{F}_{k(a/2)} - \mathbf{F}_{k(a/2)}\right] dx_2 = 0\,, \tag{94}$$

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} [\boldsymbol{\psi}_{k0}^{(0)}]^T \mathbf{J}\left[\mathbf{F}_{k(-a/2)} - \mathbf{F}_{k(a/2)}\right] dx_2 = 0\,, \tag{95}$$

$$\sum_{k=1}^{m}\int_{y_{k-1}}^{y_k} \left\{[\boldsymbol{\psi}_{k0}^{(1)}]^T \mathbf{J}\left[\mathbf{F}_{k(-a/2)} - \mathbf{F}_{k(a/2)}\right]\right.$$
$$\left. -\frac{a}{2}[\boldsymbol{\psi}_{k0}^{(0)}]^T \mathbf{J}\left[\mathbf{F}_{k(-a/2)} + \mathbf{F}_{k(a/2)}\right]\right\} dx_2 = 0\,, \tag{96}$$

in which the four unknowns in $\mathbf{F}_{k(-a/2)}$ and $\mathbf{F}_{k(a/2)}$ are related to the constants A_0, A_1, A_n and A_{-n} by Eqs. (87) ~ (88); the terms on the right-hand sides are completely known.

Equation (93) ~ (96) are, in fact, a system of four integral equations for the four unknown functions of x_2 (2 displacement components at the left free end: $\left[u(-a/2,\ x_2)\right]_k$ and $\left[v(-a/2,\ x_2)\right]_k$; 2 displacement components at the right free end: $\left[u(a/2,\ x_2)\right]_k$ and $\left[v(a/2,\ x_2)\right]_k$, which in turn are given by Eqs. (87) ~ (88) in series forms). Thus, substituting Eqs. (87) ~ (88) in Eqs. (93) ~ (96), taking N terms of the series for computation, we obtain a system of $2N + 2$ algebraic equations ($2N$ equations from Eqs. (93) and (94) for $n = 1, 2, \cdots, N$; 2 equations from Eqs. (95) and (96) for $n = 0$) for the $2N + 2$ unknowns: A_0, A_1, A_n and A_{-n} ($n = 1, 2, \cdots, N$), which can be solved by using a standard method for simultaneous linear algebraic equations.

By imposing the edge boundary conditions on Eq. (51), the constants are determined by using the symplectic orthogonality of the eigenvectors. Following the solution procedure, we obtain a solution which satisfies the nonhomogeneous state Eq. (15), the interfacial continuity conditions, and all the boundary conditions on $x_2 = -h/2$, $h/2$ and $x_1 = -a/2$, $a/2$. It is an *exact* solution.

7. RESULTS AND DISCUSSIONS

We computed the results using a set of data for a [0/90]s laminate, for which the results based on various approximate solutions [2 ~ 6] are available for comparisons. The dimensionless width of the laminate is taken to be $a/h = 10$, where a is the width and h is the thickness. The elastic constants for layer 1 and layer 4 are $c_{11} = c_{22} = 15{,}300\text{N/mm}^2$, $c_{33} = 140{,}000\text{N/mm}^2$, $c_{44} = c_{55} = 5900\text{N/mm}^2$, $c_{12} = c_{23} = 3{,}900\text{N/mm}$, $c_{13} = 3{,}300\ \text{N/mm}^2$. For layer 2 and layer 3, c_{22} and c_{33} are $15{,}300\text{N/mm}^2$ and c_{11} is $140{,}000\text{N/mm}^2$. The numerical results for the lowest twenty eigenvalues $\mu_n h$ are listed in Table 1, which may be identified as symmetric mode about the middle plane.

To compare with the published results [2,4-6], Fig. 2 shows the interlaminar normal stress $\sigma_{22}/\varepsilon_0$ (ε_0 being the uniform strain in the x_3 direction) at the 0/90-interface by setting $x_1 = x$ and taking different N terms of the series for computation. The results by taking 20, 40, ... and 100 terms in the eigenfunction expansions are presented to show the convergence. Compared with the published results, it is found that the difference exists only in a very narrow region ($x > 4.9h$) near the free edge where the stress disturbance occurs. At $x = 5h$ the present result, $\sigma_{22}/\varepsilon_0 = 2100\text{N/mm}^2$, are relatively close to the results obtained by Pagano [2] (2070N/mm^2), Becker [5] (2020N/mm^2) and Wang *et al.* [6] (2000N/mm^2). The percentages of difference for the cases are 1%, 4% and 5%, respectively.

In Fig. 3, we further compare the distributions of normalized shear stress σ_{12}/c_{33} (c_{33} being the stiffness in the direction of uniform extension) at the 0/90-interface. We can see that the present result and the result by Becker [5] are reasonably consistent in the region $x < 4.75h$, but marked discrepancies exist within $x > 4.75h$, near the free edge. According to CLT [11] the normal stress σ_{22} and shear stress σ_{12} vanish everywhere in symmetric laminates under extension, but Figs. 2 and 3 show that these stresses approach to zero only for $x < 3.5h$, where Saint-Venant's principle is applicable. The boundary layer zone in this case is localized to the region of a distance $1.5h$ from the edge. The computed values of interlaminar stresses σ_{22}/c_{33} and σ_{12}/c_{33}, as $x \to 5h$, grow with the terms taken in the expansion.

Figures 4 and 5 show the in-plane stress distribution σ_{11}/c_{33} at the middle surface of the laminate and at the middle surface of the 0-layer, respectively. The normalized stress σ_{11}/c_{33} approaches to a constant value at about $x < 2h \sim 3h$. The traction-free edge condition

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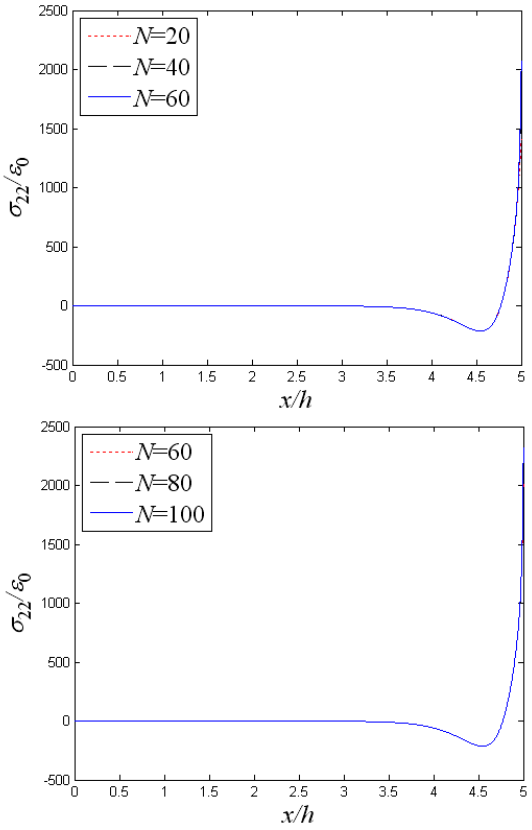


Fig. 2 Interlaminar normal stress σ_{22} at the 0/90-interface for a [0/90]s laminate under extension.

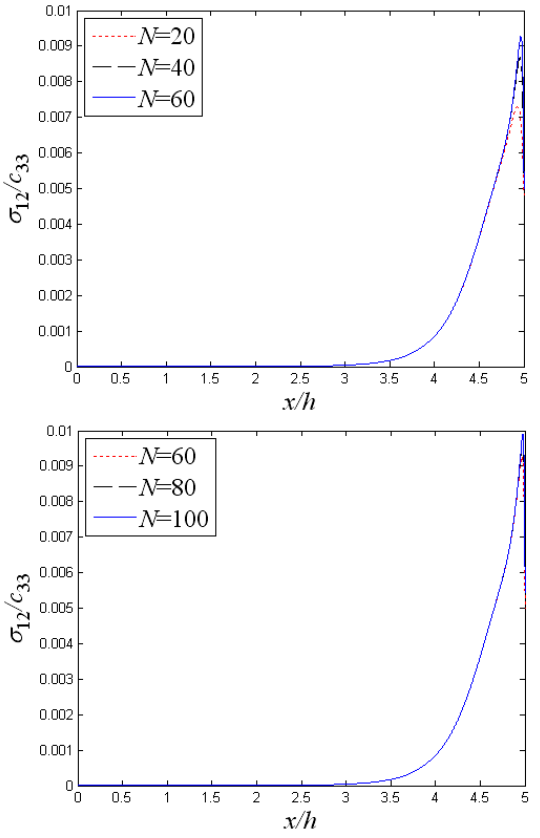


Fig. 3 Interlaminar normal stress σ_{12} at the 0/90-interface for a [0/90]s laminate under extension.

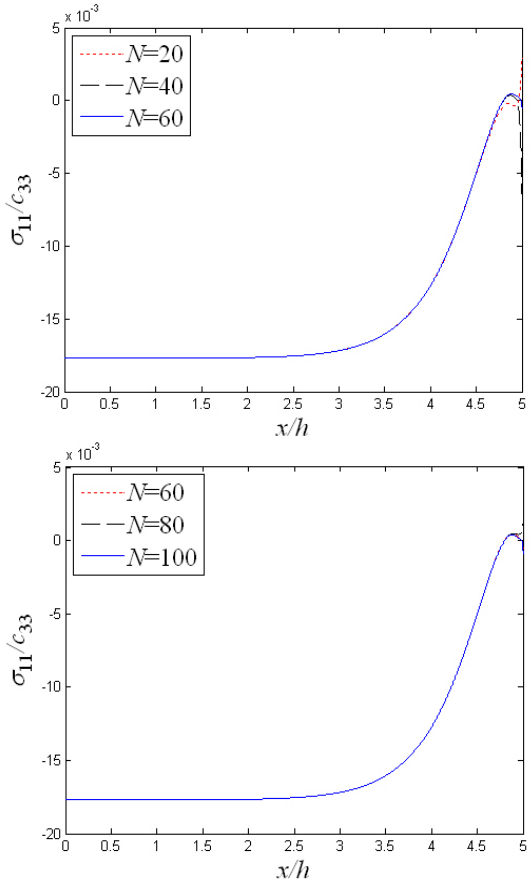


Fig. 4 In-plane stress σ_{12} at the middle surface of the laminate under extension.

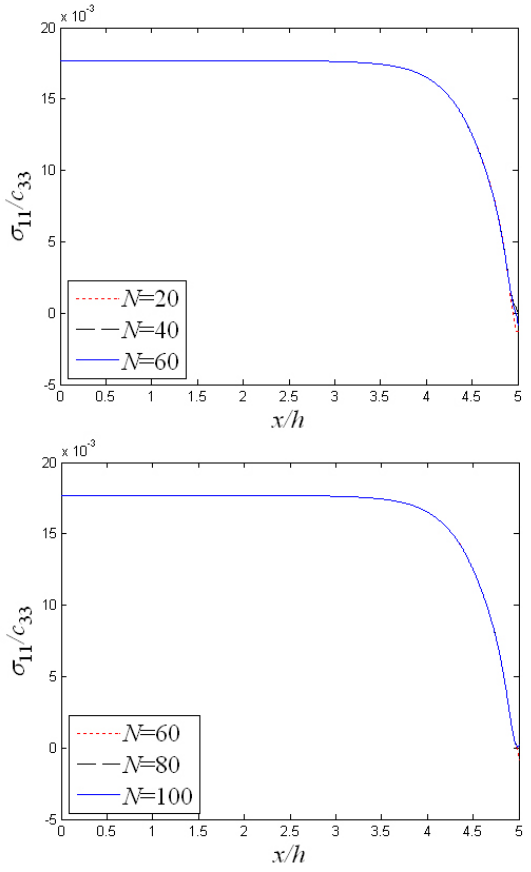


Fig. 5 In-plane stress σ_{11} at the middle surface of the 0-layer of the laminate under extension.

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Table 1 The dimensionless eigenvalues ($\mu_n h$) for a [0/90]s laminate.

mode	$\mu_n h$
1	0
2	2.4047
3	4.8382 + 1.3184i
4	4.8382 - 1.3184i
5	5.0570
6	7.9168
7	10.5279
8	12.9979 + 4.0733i
9	12.9979 - 4.0733i
10	13.2253
11	15.9337
12	18.5857
13	21.2594
14	24.0235
15	24.1721 + 5.6237i
16	24.1721 - 5.6237i
17	27.1007
18	28.5040 + 0.9782i
19	28.5040 - 0.9782i
20	31.5260

requires σ_{11}/c_{33} be zero at $x = 5h$. The condition is satisfied as the expansion terms increase. The stress near the free edge shows numerical oscillations which are common to solutions obtained via eigenfunction expansions involving complex eigenvalues.

Setting $x_2 = y$, Figs. 6 and 7 depict variations of normal stress σ_{11} and shear stress σ_{12} along the free edge $x = 5h$ of the laminate under extension. The results by taking 20, 40, ... and 100 terms in the eigenfunction expansions are presented to show the convergence. Rather than in an average sense in Wang *et al.* (2000), the edge boundary conditions are really satisfied in a pointwise fashion except near the 0/90 interface ($y = 0.25h$), where the normalized stresses are small but nonzero. This is expected because due to stress singularities the free-edge normal and shear stresses at the 0/90 interface are not vanishing. In Figs. 6 in layer 1 ($0.25 < y/h < 0.5$) with the material constant $c_{11} = c_{22}$, κ_{kj} in Eq. (44) becomes $\pm i$ so that $exp(\kappa_{kj}\mu x_2)$ in Eq. (48) becomes $exp(\pm i\mu x_2)$, thus in the region $0.25 < y/h < 0.5$ the stress oscillation is more obvious than that in $0 < y/h < 0.25$ and it always needs much more terms in the expansion to describe the stress distribution. For the shear stress is smaller in magnitude than the normal stress, in the problem of lamina under extension, its variation shown in Fig.7 is not so obvious compared to the normal stress shown in Fig. 6 while they are drawn to the same scale. Since the singular behavior at the free edge interfaces was not considered in the analysis, the exact nature of the singularity cannot be determined numerically in the present context.

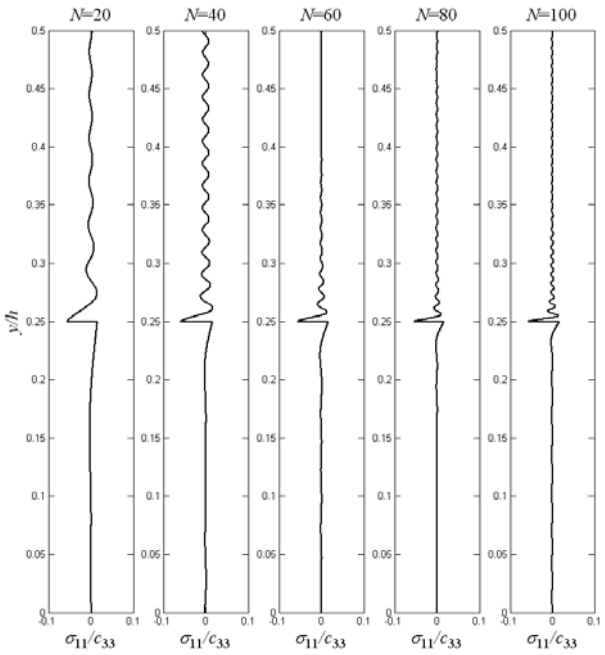


Fig. 6 Variation of normal stress σ_{11} along the free edge $x_1 = a/2$ of the laminate under extension.

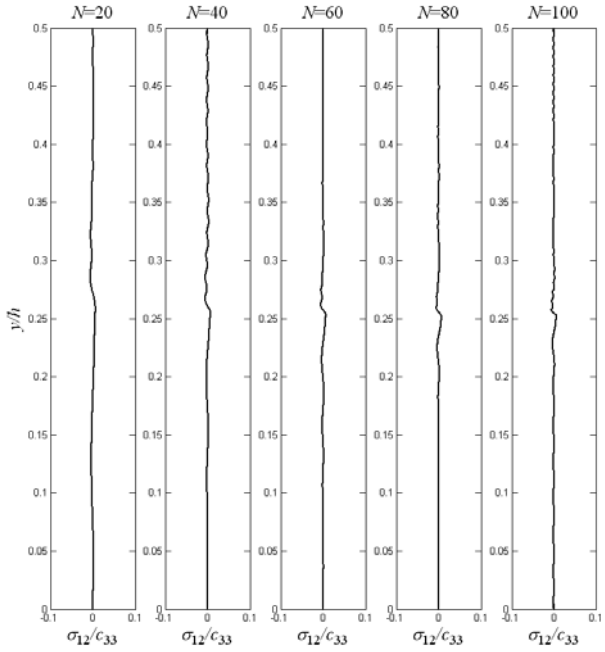


Fig. 7 Variation of shear stress σ_{12} along the free edge $x_1 = a/2$ of the laminate under extension.

8. CONCLUSIONS

In this paper we have formulated the problems of generalized plane strain of cross-ply laminates in a state space setting and developed a systematic method for stress analysis of the laminates subjected to extension and the end effect is considered. On the basis of Hamiltonian state space formulation, an exact analysis of the deformations and stress distributions in the laminates of orthotropic elastic materials is conducted.

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The state space approach can treat multilayered systems in an elegant and effective way. By means of the transfer matrix we have derived Eq. (52) which transmits the state vector in thickness direction from bottom surface to top surface. On $x_2 = -h/2$ and $x_2 = h/2$ either the traction or the displacement, or a mixed boundary conditions may be prescribed. Equation (52) consists of 4 equations in 4 unknowns. Among the 4 components of displacements and stresses, 2 are prescribed and other 2 are unknown. For a well-posed problem, we can always find a unique solution for the unknowns following the same solution procedure. By using the transfer matrix, once displacements and stresses on $x_2 = h/2$ are found, the displacements and stresses in the successive layers can be determined via Eq. (51).

As compared with the solutions of those existing methods, our solution satisfies the edge boundary conditions through the thickness point by point, besides finding the interlaminar stresses of the classical cross-play laminate problem. After Examination of the applicability of Saint-Venant's principle and boundary layer effects in laminates, from the edge a boundary layer zone of width $1.5h$ exists thus the stresses σ_{22} and σ_{12} are not zero. So the stress disturbance is not local and the edge effects are far reaching. In studying the stress singularities in multilayered laminates, applications of the state space approach require a continuing study.

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