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NOTE ON A DYNAMICAL GENERALISATION OF THE PRIME NUMBER THEOREM FOR ARITHMETIC PROGRESSIONS

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Abstract

Bergelson and Richter ['Dynamical generalizations of the prime number theorem and disjointness of additive and multiplicative semigroup actions', *Duke Math. J.* **171**(15) (2022), 3133–3200] established a new dynamical generalisation of the prime number theorem (PNT) and the PNT for arithmetic progressions. Let $h \geq 1, k \geq 2$. Mirsky ['Note on an asymptotic formula connected with r-free integers', *Quart. J. Math. Oxford Ser.* **18** (1947), 178–182] showed that the numbers n such that $n + l_1, \ldots, n + l_h$ are k-free have a natural density for any given nonnegative integers $l_1, \ldots, l_h$. In this note, we show that the Bergelson–Richter theorem holds for the numbers studied by Mirsky.

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1. Introduction

For any $n \in \mathbb{N}$, let $\Omega(n)$ be the number of prime divisors of n counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ be the Liouville function. It is well known (see, for example, [5]) that the prime number theorem (PNT) is equivalent to the assertion that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(n) = 0. \tag{1.1}$$

In 2022, Bergelson and Richter [1] established a new dynamical generalisation of the PNT by placing (1.1) in a dynamical framework, showing that given any uniquely ergodic topological dynamical system (X, μ, T),

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\Omega(n)}x) = \int_X f \, d\mu \tag{1.2}$$

holds for any $f \in C(X)$ and $x \in X$, where $C(X)$ denotes the set of continuous functions defined on X. The equivalent form (1.1) of the PNT is recovered from (1.2) by taking (X, T) as a rotation on two points.

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The Bergelson–Richter theorem (1.2) is a dynamical generalisation of some other number theoretic results. For example, taking (X, T) as a rotation on m points ($m \geq 2$) in (1.2) recovers a classical result of Pillai and Selberg [8, 9], which states that $\Omega(n)$ distributes evenly over all residue classes modulo m. Moreover, taking (X, T) as irrational circle rotations in (1.2) implies that the sequence $\{\Omega(n)\alpha\}_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 for any irrational number α, a result mentioned by Erdős in [3] without proof and later proved by Delange [2]. We refer to [1] for more interesting applications of (1.2).

In [1], Bergelson and Richter also established the following dynamical generalisation of the PNT for arithmetic progressions.

THEOREM 1.1 [1, Corollary 1.16]. *Let (X, μ, T) be a uniquely ergodic topological dynamical system. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} f(T^{\Omega(mn+r)}x) = \int_X f \, d\mu \tag{1.3}$$

for any $f \in C(X)$, $x \in X$, $m \in \mathbb{N}$ and $r \in \{0, 1 \ldots, m-1\}$.

Indeed, taking rotation on two points in (1.3) gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda(mn + r) = 0 \quad \text{for all } m \in \mathbb{N} \text{ and } r \in \{0, 1 \ldots, m-1\},$$

which is an equivalent form of the PNT along arithmetic progressions (see [10]).

Let $k \geq 2$ be an integer. A natural number n is said to be k-free if it has no divisor $d > 1$ which is a perfect kth power. We define $\mu_k(n) = 1$ if n is k-free and zero otherwise. By Theorem 1.1, the orbits $\{f(T^{\Omega(mn+r)}x)\}_{n\in\mathbb{N}}$ are equi-distributed in the space X for any $x \in X$. In this note, we show that the orbits $\{f(T^{\Omega(mn+r)}x)\}_{n\in\mathbb{N}}$ are also equi-distributed along a system of k-free numbers.

2. Main results

Let $h, q \geq 1$ be integers. Let $l_1, \ldots, l_h$ be nonnegative integers. In [7], Mirsky showed that the numbers qn such that $qn + l_1, \ldots, qn + l_h$ are k-free have a natural density $A_q(l_1, \ldots, l_h)$ defined by

$$A_q(l_1, \ldots, l_h) = \frac{1}{q} \prod_{p^k \nmid q} \Big(1 - \frac{(p^k, q)\nu_q(p)}{p^k}\Big), \tag{2.1}$$

and $\nu_q(p) = \#\{1 \leq n \leq p^k : (p^k, q) \mid n, n \equiv l_i \pmod{p^k} \text{ for some } i, 1 \leq i \leq h\}$. The following is our main theorem, which shows that the Bergelson–Richter theorem holds for the numbers studied by Mirsky.

THEOREM 2.1. *Let $h, q \geq 1$ be integers. Let $l_1, \ldots, l_h$ be nonnegative integers. Let (X, μ, T) be a uniquely ergodic topological dynamical system. Then,*

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$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\le qn\le N}\mu_k(qn+l_1)\cdots\mu_k(qn+l_h)f(T^{\Omega(mn+r)}x)=A_q(l_1,\ldots,l_h)\int_X f\,d\mu \quad (2.2)$$

for any $f\in C(X)$, $x\in X$, $m\in\mathbb{N}$ and $r\in\{0,1\ldots,m-1\}$, where $A_q(l_1,\ldots,l_h)$ is defined by (2.1).

By Theorem 2.1, the orbits $\{f(T^{\Omega(mn+r)}x)\}_{n\in\mathbb{N}}$ are equi-distributed along the numbers n such that $qn+l_1,\ldots,qn+l_h$ are k-free for any $x\in X$. As a corollary, we obtain the following special case by taking $q=1,m=1,r=0$ in (2.2).

COROLLARY 2.2. *Let $h\ge 1$ be an integer. Let $l_1,\ldots,l_h$ be nonnegative integers. Let (X,μ,T) be a uniquely ergodic topological dynamical system. Then,*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\le n\le N}\mu_k(n+l_1)\cdots\mu_k(n+l_h)f(T^{\Omega(n)}x)=A(l_1,\ldots,l_h)\int_X f\,d\mu$$

for any $f\in C(X)$ and $x\in X$, where

$$A(l_1,\ldots,l_h)=\prod_p\left(1-\frac{\nu(p)}{p^k}\right)$$

and $\nu(p)=\#\{l_i \pmod{p^k} : 1\le i\le h\}$.

In [6], Mirsky showed that the numbers n such that $n+l_1,\ldots,n+l_h$ are k-free have a natural density $A(l_1,\ldots,l_h)$. Moreover, he obtained an asymptotic formula for the density with a better error term than [7]. By taking $k=2,h=1$ and $l_1=0$ in Corollary 2.2, we obtain the result in [1, Corollary 1.8]: for any uniquely ergodic topological dynamical system (X,μ,T),

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{1\le n\le N\\ n\ \text{squarefree}}}f(T^{\Omega(n)}x)=\frac{6}{\pi^2}\int_X f\,d\mu \quad (2.3)$$

for any $f\in C(X)$ and $x\in X$. Taking (X,T) as a rotation on two points in (2.3) gives another well-known equivalent form of the PNT:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu(n)=0,$$

where $\mu(n)$ is the Möbius function, which is defined to be 1 if $n=1$, $(-1)^k$ if n is the product of k distinct primes and zero otherwise. Therefore, Corollary 2.2 gives a dynamical extension of the PNT. Moreover, we obtain the following variant of the Bergelson–Richter theorem along twins of squarefree numbers by taking $k=h=2$, $l_1=0,l_2=1$ in Corollary 2.2.

COROLLARY 2.3. *Let (X,μ,T) be a uniquely ergodic topological dynamical system. Then,*

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$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{1\le n\le N\\ n,\,n+1\ squarefree}} f(T^{\Omega(n)}x)=\prod_p\Big(1-\frac{2}{p^2}\Big)\int_X f\,d\mu$$

for any $f\in C(X)$ and $x\in X$.

REMARK 2.4. In [7], Mirsky also studied the following three systems of positive integers relating to k-free numbers:

(1) positive integers n such that $n+a_1,\dots,n+a_l$ are k-free while $n+b_1,\dots,n+b_m$ are not, where $a_1,\dots,a_l,b_1,\dots,b_m$ are distinct positive integers;
(2) blocks of s k-free numbers, $s\ge 1$;
(3) blocks of s numbers that are not k-free, $s\ge 1$.

Similar to the arguments in [7], one can apply Corollary 2.2 to establish the variants of the Bergelson–Richter theorem along these three systems of numbers.

In the next section, we will prove Theorem 2.1 by applying some ideas of Mirsky in [6, 7], Hall in [4, Theorem 1], and Bergelson and Richter in [1, Corollary 1.8].

3. Proof of Theorem 2.1

To prove Theorem 2.1, we first cite some lemmas in [4, 7]. For positive integers $m_1,\dots,m_s$ and $a_1,\dots,a_s$, it is well known that the system of congruences in n,

$$n+a_i\equiv 0\pmod{m_i},\quad 1\le i\le s, \tag{3.1}$$

is soluble if and only if $(m_i,m_j)\mid(a_i-a_j)$ for $1\le i<j\le s$. If (3.1) is soluble, then the solutions form exactly one residue class modulo $[m_1,\dots,m_s]$. Let

$$E\binom{m_1,\dots,m_s}{a_1,\dots,a_s}=1\text{ or }0$$

according to whether (3.1) is or is not soluble.

LEMMA 3.1 [7, Lemma 3]. *Let $m_1,\dots,m_s,a_1,\dots,a_s,q$ be given positive integers, and T the number of positive integers n such that*

$$qn+a_i\equiv 0\pmod{m_i},\quad 1\le i\le s\quad\text{and}\quad qn\le N.$$

Then,

$$T=N\frac{E\binom{q,m_1,\dots,m_s}{0,a_1,\dots,a_s}}{[q,m_1,\dots,m_s]}+O(1).$$

LEMMA 3.2 [7, Lemma 6]. *We have*

$$\sum_{d_1=1}^{\infty}\cdots\sum_{d_h=1}^{\infty}\frac{\mu(d_1)\cdots\mu(d_h)}{[q,d_1^k,\dots,d_h^k]}E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h}=\frac{1}{q}\prod_{p^k\nmid q}\Big(1-\frac{(p^k,q)\nu_q(p)}{p^k}\Big),$$

where $\nu_q(p)=\#\{1\le n\le p^k:(p^k,q)\mid n,n\equiv l_i\pmod{p^k}\text{ for some }i,1\le i\le h\}$.

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LEMMA 3.3 [4, Lemma 2]. *For each fixed h and $\sigma > 1$,*

$$\sum_{d_1>H} \cdots \sum_{d_h>H} \frac{1}{[d_1, \ldots, d_h]^\sigma} \ll_{h,\sigma} H^{1-\sigma}.$$

Now, we prove Theorem 2.1 by an elementary method.

PROOF OF THEOREM 2.1. Put $a(n) = f(T^{\Omega(mn+r)}x)$ and $\alpha = \int_X f\, d\mu$. By Theorem 1.1,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\le n\le N} a(dn + t) = \alpha \tag{3.2}$$

for any $d \in \mathbb{N}$ and $t \in \{0, 1 \ldots, d-1\}$. Clearly, $a(n)$ is bounded. We may assume that $|a(n)| \le 1$ for all n. It is well known that $\mu_k(n) = \sum_{d^k|n} \mu(d)$. Let

$$\mu_{k,H}(n) = \sum_{\substack{d^k|n \\ d\le H}} \mu(d) \quad \text{and} \quad e_{k,H}(n) = \sum_{\substack{d^k|n \\ d>H}} \mu(d)$$

for $H \ge 2$. Then, $\mu_k = \mu_{k,H} + e_{k,H}$. Define

$$S_H(N) = \sum_{1\le qn\le N} \mu_{k,H}(qn + l_1) \cdots \mu_{k,H}(qn + l_h) a(n).$$

By using $\mu_k = \mu_{k,H} + e_{k,H}$ and expanding the left-hand side of (2.2),

$$\begin{aligned} &\Big| \sum_{1\le qn\le N} \mu_k(qn + l_1) \cdots \mu_k(qn + l_h) a(n) - S_H(N) \Big| \\ &\le \sum_{1\le i\le h} \sum_{qn\le N} |e_{k,H}(qn + l_i)| + \sum_{1\le i,j\le h} \sum_{qn\le N} |e_{k,H}(qn + l_i) e_{k,H}(qn + l_j)| + \cdots \\ &\quad + \sum_{qn\le N} |e_{k,H}(qn + l_1) \cdots e_{k,H}(qn + l_h)|. \end{aligned} \tag{3.3}$$

Let $l = \max\{l_i : 1 \le i \le h\}$. For $1 \le r \le h$ and $1 \le i_1, \ldots, i_r \le h$, by Hölder's inequality,

$$\sum_{qn\le N} |e_{k,H}(qn + l_{i_1}) \cdots e_{k,H}(qn + l_{i_r})| \le \prod_{1\le j\le r} \Big(\sum_{qn\le N} |e_{k,H}(qn + l_{i_j})|^r \Big)^{1/r} \le \sum_{n\le N+l} |e_{k,H}(n)|^r.$$

Moreover, by Lemma 3.3,

$$\sum_{n\le N+l} |e_{k,H}(n)|^r \le \sum_{n\le N+l} \sum_{\substack{d_1^k|n \\ d_1>H}} \cdots \sum_{\substack{d_r^k|n \\ d_r>H}} 1 \le \sum_{d_1>H} \cdots \sum_{d_r>H} \frac{N+l}{[d_1, \ldots, d_r]^k} \ll \frac{N+l}{H^{k-1}}.$$

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It follows that the right-hand side of (3.3) is bounded by

$$\sum_{r=1}^{h}\binom{h}{r}\frac{N+l}{H^{k-1}} = (2^h-1)\frac{N+l}{H^{k-1}}.$$

This implies that

$$\sum_{1\le qn\le N}\mu_k(qn+l_1)\cdots\mu_k(qn+l_h)a(n) = S_H(N) + O_{h,k,l}\Big(\frac{N}{H^{k-1}}\Big). \tag{3.4}$$

As regards $S_H(N)$, by the definition of $\mu_{k,H}$,

$$S_H(N) = \sum_{d_1\le H}\cdots\sum_{d_h\le H}\mu(d_1)\cdots\mu(d_h)\sum_{\substack{qn\le N\\ d_i^k\mid n+l_i \text{ for all } i}} a(n).$$

Let $T_q(N)$ be the number of positive integers n such that $qn\le N$ and $d_i^k \mid n+l_i$ for all i with $1\le i\le h$. Then, by Lemma 3.1,

$$T_q(N) = N\frac{E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h}}{[q,d_1^k,\dots,d_h^k]} + O(1).$$

If $E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h} = 1$, we write $n = q'm+t$ for some t with $0\le t<q'$, where $qq' = [q,d_1^k,\dots,d_h^k]$.

If B is a finite nonempty set, we define

$$\mathop{\mathbb{E}}_{x\in B} b(x) := \frac{1}{|B|}\sum_{x\in B} b(x)$$

for any function $b: B\to\mathbb{C}$ on B. Then,

$$\begin{aligned}
S_H(N) &= \sum_{d_1\le H}\cdots\sum_{d_h\le H}\mu(d_1)\cdots\mu(d_h)E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h}\sum_{q(q'm+t)\le N} a(q'm+t)\\
&= \sum_{d_1\le H}\cdots\sum_{d_h\le H}\mu(d_1)\cdots\mu(d_h)E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h}T_q(N)\mathop{\mathbb{E}}_{m\le (N-qt)/qq'} a(q'm+t)\\
&= \sum_{d_1\le H}\cdots\sum_{d_h\le H}\mu(d_1)\cdots\mu(d_h)E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h}\Big(\frac{N}{[q,d_1^k,\dots,d_h^k]}+O(1)\Big)\\
&\quad\cdot\mathop{\mathbb{E}}_{m\le (N-qt)/qq'} a(q'm+t)\\
&= N\sum_{d_1\le H}\cdots\sum_{d_h\le H}\frac{\mu(d_1)\cdots\mu(d_h)}{[q,d_1^k,\dots,d_h^k]}E\binom{q,d_1^k,\dots,d_h^k}{0,l_1,\dots,l_h}\mathop{\mathbb{E}}_{m\le (N-qt)/qq'} a(q'm+t) + O(H^h).
\end{aligned} \tag{3.5}$$

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Combining (3.4) and (3.5),

$$\frac{1}{N}\sum_{1\le qn\le N}\mu_k(qn+l_1)\cdots\mu_k(qn+l_h)a(n)$$
$$=\sum_{d_1\le H}\cdots\sum_{d_h\le H}\frac{\mu(d_1)\cdots\mu(d_h)}{[q,d_1^k,\dots,d_h^k]}E\begin{pmatrix}q,d_1^k,\dots,d_h^k\\0,l_1,\dots,l_h\end{pmatrix}\mathbb{E}_{m\le((N-qt)/qq')}a(q'm+t)+O\Big(\frac{H^h}{N}\Big)$$
$$+O\Big(\frac{1}{H^{k-1}}\Big) \tag{3.6}$$

for $2\le H\le N$.

Now, we fix H first. By (3.2), taking $N\to\infty$ in (3.6),

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\le qn\le N}\mu_k(qn+l_1)\cdots\mu_k(qn+l_h)a(n)$$
$$=\alpha\sum_{d_1\le H}\cdots\sum_{d_h\le H}\frac{\mu(d_1)\cdots\mu(d_h)}{[q,d_1^k,\dots,d_h^k]}E\begin{pmatrix}q,d_1^k,\dots,d_h^k\\0,l_1,\dots,l_h\end{pmatrix}+O\Big(\frac{1}{H^{k-1}}\Big). \tag{3.7}$$

Then, taking $H\to\infty$ in (3.7) gives

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\le n\le N}\mu_k(n+l_1)\cdots\mu_k(n+l_h)a(n)$$
$$=\alpha\sum_{d_1=1}^{\infty}\cdots\sum_{d_h=1}^{\infty}\frac{\mu(d_1)\cdots\mu(d_h)}{[q,d_1^k,\dots,d_h^k]}E\begin{pmatrix}q,d_1^k,\dots,d_h^k\\0,l_1,\dots,l_h\end{pmatrix},$$

which is equal to $A_q(l_1,\dots,l_h)\alpha$ by Lemma 3.2. This completes the proof of Theorem 2.1. □

REMARK 3.4. In Theorem 1.1, the rate of convergence in (1.3) is $o_{N\to\infty}(1)$. If one can show a better bound than $o_{N\to\infty}(1)$ in Theorem 1.1, then one can also obtain better bounds on the rate of convergence in Theorem 2.1 and Corollary 2.3 by using (3.6) and the arguments similar to [4, Theorem 1].

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