NONLINEAR DYNAMIC ANALYSIS OF MICRO CANTILEVER BEAM UNDER ELECTROSTATIC LOADING

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ABSTRACT

A hybrid differential transformation / finite difference scheme is used to analyze the complex nonlinear behavior of an electrostatically-actuated micro cantilever beam which high aspect ratios (length/width). The validity of the proposed method is confirmed by comparing the numerical results obtained for the tip displacement and pull-in voltage of the cantilever beam with the analytical and experimental results presented in the literature. The hybrid scheme is then applied to analyze both the steady-state and the dynamic deflection behavior of the cantilever beam as a function of the applied voltage. Overall, the results confirm that the hybrid method provides an accurate and computationally-efficient means of analyzing the complex nonlinear behavior of both the current micro cantilever beam system and other micro-scale electrostatically-actuated structures.

Keywords: Pull-in voltage, Micro cantilever beam, Electrostatic actuator, MEMS, Hybrid method, Differential transformation.

1. INTRODUCTION

The Many commercial products based on micro-electro-mechanical systems (MEMS) techniques have been proposed in recent years, ranging from accelerometers and pressure sensors for automotive security systems [1], to miniature actuators for Digital Micromirror Devices (DMDs) [2], electrostatic rotary comb actuators for scanning electron microscope (SME) [3], and so forth. Generally speaking, the actuation mechanisms utilized in such devices are realized using either electrostatic [4], electromagnetic [5], thermal or piezoelectric [6] techniques. Electrostatic actuators are particularly advantageous since they have a simple design, a low manufacturing cost, a rapid response and a low power consumption. In essence, an electrostatic actuator comprises two electrodes, one fixed and one movable, separated by a small gap. The application of a voltage across the two electrodes creates an electrostatic force, which causes the movable electrode to deflect toward the stationary electrode. If the electrostatic force exceeds the elastic restoring force developed within the movable electrode, it collapses and makes transient contact with the stationary electrode [7]. The critical value of the voltage at which this phenomenon occurs is referred to as the pull-in voltage, and is of fundamental importance in the design of many MEMS-based devices. In some applications, the pull-in phenomenon is deliberately induced in order to create a switch or relay-type effect [8]. However, in other applications, *e.g.* MEMS-based micromirrors, micropumps, and so forth, the pull-in phenomenon should be avoided in order to prevent a loss in performance and / or the occurrence of mechanical damage. Consequently, a requirement exists for reliable methods capable of accurately predicting the pull-in voltage of a variety of MEMS-based structures. Electrostatic actuators are commonly modeled using the simple lumped model system shown in Fig. 1, comprising two parallel conductive plates separated by a small gap. As shown, one plate is fixed to a substrate, while the other is suspended via an elastic restoring force. When a voltage V is applied between the two plates, an electrostatic force is produced with a magnitude of [9].

However, while the lump model system is simple and has explicit physical meaning, it ignores the electrostatic

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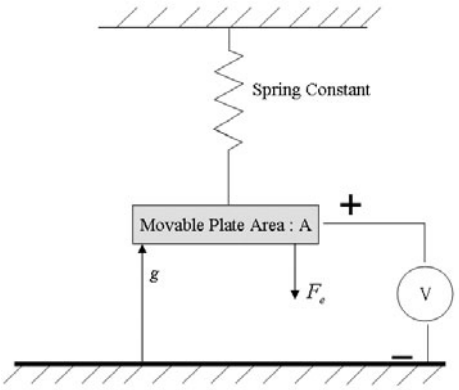


Fig. 1 Simple lumped model representation of micro cantilever beam system

coupling effect, and therefore produces unreliable results. Recently, the finite element method (FEM) has been established as a powerful tool for modeling in the electrostatically actuated systems. The finite element method (MEMCAD software、CoventorWare software) has the advantage of being closer to the reality but has the disadvantages of un-explicit physical meaning and requiring massive numerical calculations.

Accordingly, many researchers have proposed more sophisticated techniques for modeling the nonlinear behavior of electrostatic actuators. For example, Legtenberg *et al*. [10] performed a one-dimensional static analysis of curved electrostatically-actuated electrodes using the Rayleigh–Ritz energy method. Similarly, Gretillat *et al*. [11] examined the nonlinear dynamic response of electrostatic relays using a finite element method (FEM). Hung and Senturia [12] used leveraged bending and strain-stiffening methods to explore the feasibility of extending the travel range of analog-tuned electrostatic actuators prior to the pull-in event. Chan *et al*. [13] characterized the contact behavior of electrostatically-actuated polysilicon beams by performing 2D simulations, and extracted the material properties of the beams utilizing a hybrid approach combining pull-in voltage and capacitance-voltage measurements and 2D numerical simulations, respectively. Nemirovsky [14] presented a model for deriving the pull-in parameters of electrostatic actuators with a single input, *i.e.* charge or voltage, and showed that the model enabled the analysis of both nonlinear mechanical effects and nonlinear, non-ideal and parasitic electrical effects.

Differential transformation theory was originally proposed by Zhou [15] in 1986 as a means of solving linear and nonlinear initial value problems in the circuit analysis field. However, later researchers extended its use to the analysis of initial value problems in the mechanical engineering domain [16,17]. For example, Chen and Ho [18,19] utilized differential transformation theory to solve general eigenvalue problems, while Yu and Chen [20] combined differential transformation theory with the finite difference approximation method to obtain numerical solutions for the Blasius equation. The same hybrid method was also used to analyze the nonlinear transient thermal stress response of isotropic annular fins [21] and to solve the Burgers' equation for flow systems with different Reynolds numbers [22].

In the present study, the hybrid differential transformation / finite difference method is provided an accurate and computationally-efficient means to analyze the complex nonlinear behavior of an electostatically-actuated micro cantilever, but hybrid method has the disadvantages as follow: 1. The original function in the time domain must be an analyzable function. 2. To solve partial differential equation with hybrid method, the finite difference scheme causes slightly rational error. The remainder of this paper is organized as follows. Section 2 reviews the basic principles of differential transformation theory, while Section 3 derives the governing equation of the micro cantilever system and demonstrates the use of the hybrid scheme in solving this equation in a computationally-efficient manner. Section 4 validates the proposed method by comparing the numerical results obtained for the tip displacement and pull-in voltage of the cantilever beam with the analytical and experimental results presented in the literature. In addition, the hybrid scheme is used to analyze both the dynamic and the static response of the cantilever beam as a function of the applied voltage. Finally, Section 5 presents some brief concluding remarks.

2. DIFFERENTIAL TRANSFORMATION THEORY

This section introduces the basic principles of the differential transformation method. If $x(t)$ is an analyzable function in the time domain T, the differential transformation of x at time $t = t_0$ in the K domain is defined as

$$X(k;t_0)=M(k)\left(\frac{d^k}{dt^k}\left(q(t)\,x(t)\right)\right)_{t=t_0},\quad k\in K \tag{1}$$

where k belongs to a set of non-negative integers which collectively form the K domain, $X(k; t_0)$ is the transformed function in the transformation domain, otherwise called the spectrum of $x(t)$ at $t = t_0$ in the K domain, $M(k)$ is the weighting factor, and $q(t)$ is a kernel corresponding to $x(t)$. Both $M(k)$ and $q(t)$ are non-zero and $q(t)$ is an analyzable function in T.

The inverse differential transformation of $X(k; t_0)$ is given as

$$x(t)=\frac{1}{q(t)}\sum_{k=0}^{\infty}\frac{(t-t_0)^k}{k!}\frac{X(k;t_0)}{M(k)},\quad t\in T \tag{2}$$

where $M(k)=\dfrac{H^k}{k!}$ and $q(t) = 1$. Note that H is the time interval.

Let $t_0 = 0$. Equation (1) then becomes

$$X(k)=\frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0},\quad k\in K \tag{3}$$

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The inverse differential transformation of $X(k)$ can then be expressed as follows:

$$x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^{k}X(k)\,,\quad t\in T \tag{4}$$

Substituting Eq. (3) into Eq. (4) gives

$$x(t)=\sum_{k=0}^{\infty}\frac{t^{k}}{k!}\left[\frac{d^{k}x(t)}{dt^{k}}\right]_{t=0}\,,\quad t\in T \tag{5}$$

Equation (5) can be derived as a Taylor series expansion. The basic operational properties of the differential transformation method can be summarized as follows:

(a) Linearity operation

$$T\left[\alpha x(t)+\beta y(t)\right]=\alpha X(k)+\beta Y(k)\,, \tag{6}$$

where T denotes differential transformation and α and β are any real number.

(b) Convolution operation

$$T\left[x(t)y(t)\right]=X(k)\otimes Y(k)=\sum_{\ell=0}^{k}X(\ell)Y(k-\ell)\,,$$

$$T\left[x^{m}(t)\right]=kX(0)X^{m}(k)$$
$$=\sum_{\ell=1}^{k}\left[(m+1)\ell\right]X(\ell)X^{m}(k-\ell)\,,\quad m\in N \tag{7}$$

where $\otimes$ denotes the convolution operation.

(c) Differential operation

$$T\left[\frac{d^{n}x(t)}{dt^{n}}\right]=\frac{(k+n)!}{k!H^{n}}X(k+n)\,, \tag{8}$$

where n is the order of differentiation [20-22].

3. MODELING OF MICRO-CANTILEVER SYSTEM

In this section, the governing equation of the micro cantilever system is derived in the form of a partial differential equation using Hamilton's principle. The electrostatic force term in the governing equation is then expanded as a Taylor series and simplified to a fifth-order model by omitting the higher-order terms. For analytical convenience, the governing equation may be normalized. Finally, the governing equation is normalized and then solved using the hybrid differential transformation / finite difference method.

3.1 Governing Equation

Figure 2 presents a schematic illustration of the micro cantilever beam considered in the present analysis. The strain energy U_m developed by the micro cantilever beam under bending and the electrical potential energy U_e stored between the upper and the lower electrodes in the cantilever system are given respectively by

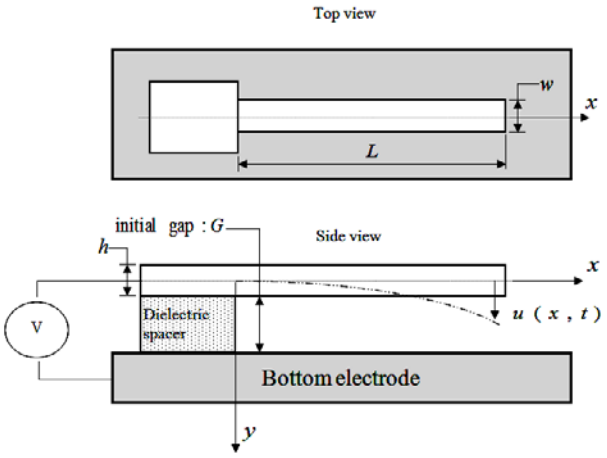


Fig. 2 Schematic illustration of micro cantilever beam system

$$U_{m}=\int_{0}^{L}\frac{\tilde{E}I}{2}\left(\frac{\partial^{2}u}{\partial x^{2}}\right)dx\,, \tag{9}$$

$$U_{e}=-\int_{0}^{L}\frac{\varepsilon_{0}wV^{2}}{2(G-u)}\,dx\,, \tag{10}$$

where $\tilde{E}$, ε_0, L, w, V and G represent the effective Young's modulus of the beam, the permittivity of free space, the length of the micro cantilever beam, the width of the micro cantilever beam, the voltage applied across the electrodes, and the initial gap between the electrodes, respectively. In accordance with Osterberg and Senturia [7], the micro cantilever beam is considered to be wide when $w \geq 5h$, where h is the thickness of the beam. Under this constraint, the effective beam modulus $\tilde{E}$ is equivalent to the plate modulus $E/(1-\nu^2)$, where E is the Young's modulus of the beam material, ν is the Poisson ratio and I is the moment of inertia. The transverse displacement of the beam, u, is a function of both the axial position x and the time t and is therefore denoted as $u = u(x, t)$. Meanwhile, the total potential energy U of the cantilever system can be written as

$$U=U_{m}+U_{e}=\int_{0}^{L}\frac{\tilde{E}I}{2}\left(\frac{\partial^{2}u}{\partial x^{2}}\right)^{2}dx-\int_{0}^{L}\frac{\varepsilon_{0}wV^{2}}{2(G-u)}\,dx\,, \tag{11}$$

while the transverse kinetic energy T_e is given by

$$T_{e}=\frac{1}{2}\int_{0}^{L}\rho A\left(\frac{\partial u}{\partial t}\right)^{2}dx\,, \tag{12}$$

where ρ is the density of the micro cantilever beam and A is its cross-sectional area.

Using Hamilton's principle gives the following partial differential equation:

$$\delta\int_{t_1}^{t_2}(T_{e}-U+W_{nc})\,dt=0\,, \tag{13}$$

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where δ is the variation, W_{nc} is the virtual work done by the virtual displacement, and t_1 and t_2 are the initial time and final time, respectively. Substituting Eqs. (11) and (12) into Eq. (13) and integrating by parts yields the following governing equation [23]:

$$\tilde{E}I\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\varepsilon_0 wV^2}{2(G-u)^2}. \tag{14}$$

In analyzing the micro cantilever beam system, the fringing field effect should be taken into account. The first-order fringing field correction is formulated as [25]

$$F_f=0.65\frac{G-u}{w}. \tag{15}$$

Thus, the governing equation given in Eq. (14) is reformulated as follows:

$$\tilde{E}I\frac{\partial^4 u}{\partial x^4}+\rho A\frac{\partial^2 u}{\partial t^2}=\frac{\varepsilon_0 wV^2}{2(G-u)^2}\left(1+0.65\frac{G-u}{w}\right). \tag{16}$$

The corresponding boundary conditions are given by

$$\begin{aligned}&u(x,t)=\frac{\partial u(x,t)}{\partial x}=0 \quad \text{at} \quad x=0,\\ &\frac{\partial^2 u(x,t)}{\partial x^2}=\frac{\partial^3 u(x,t)}{\partial x^3}=0 \quad \text{at} \quad x=L.\end{aligned} \tag{17}$$

Finally, the initial condition is defined as

$$u(x,0)=\frac{\partial u(x,0)}{\partial t}=0 \tag{18}$$

3.2 Normalized Equation

For analytical convenience, the transverse displacement of the cantilever (u) is normalized with respect to the initial gap between the electrodes, the axial length position (x) is normalized with respect to the total beam length, and the time (t) is normalized with respect to the constant T_c, *i.e.*

$$\bar{u}=\frac{u}{G},\quad \bar{x}=\frac{x}{L},\quad \bar{t}=\frac{t}{T_c},\quad \bar{w}=\frac{w}{G}, \tag{19}$$

where $T_c=\sqrt{\frac{\rho AL^4}{\tilde{E}I}}$. In addition, the non-dimensional quantity is defined as

$$\bar{V}=\sqrt{\frac{\varepsilon_0 wL^4}{2\tilde{E}IG^3}}V. \tag{20}$$

Substituting Eq. (19) into Eqs. (16), (17) and (18), the governing equation of the micro cantilever beam is obtained in a normalized form as

$$\frac{\partial^4\bar{u}}{\partial\bar{x}^4}+\frac{\partial^2\bar{u}}{\partial\bar{t}^2}=\frac{\bar{V}^2}{(1-\bar{u})^2}+0.65\frac{\bar{V}^2}{\bar{w}(1-\bar{u})}. \tag{21}$$

The corresponding dimensionless boundary conditions are given by

$$\begin{aligned}&\bar{u}(\bar{x},\bar{t})=\frac{\partial\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}}=0 \quad \text{at} \quad \bar{x}=0,\\ &\frac{\partial^2\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}^2}=\frac{\partial^3\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}^3}=0 \quad \text{at} \quad \bar{x}=1.\end{aligned} \tag{22}$$

Finally, the dimensionless initial condition is defined as

$$\bar{u}(\bar{x},0)=\frac{\partial\bar{u}(\bar{x},0)}{\partial\bar{t}}=0. \tag{23}$$

The nonlinear electrostatic force term in Eq. (21), *i.e.* $\frac{\bar{V}^2}{(1-\bar{u})^2}$, can be approximated by the following Taylor series expansion:

$$\frac{\bar{V}^2}{(1-\bar{u})^2}=\bar{V}^2(1+2\bar{u}+3\bar{u}^2+4\bar{u}^3+5\bar{u}^4+6\bar{u}^5\cdots\cdots). \tag{24}$$

Neglecting orders higher than the fifth order and substituting Eq. (24) into Eq. (21) yields the following nonlinear governing equation:

$$\begin{aligned}\frac{\partial^4\bar{u}}{\partial\bar{x}^4}+\frac{\partial^2\bar{u}}{\partial\bar{t}^2}&=\bar{V}^2+2\bar{V}^2\bar{u}+3\bar{V}^2\bar{u}^2+4\bar{V}^2\bar{u}^3\\ &\quad+5\bar{V}^2\bar{u}^4+6\bar{V}^2\bar{u}^5+0.65\frac{\bar{V}^2}{\bar{w}(1-\bar{u})}\end{aligned} \tag{25}$$

3.3 Application of Hybrid Method to Solution of Governing Equation

In solving the governing equation of the micro cantilever beam using the hybrid differential transformation / finite difference method, the solution procedure commences by applying differential transformation to each of the terms in Eq. (25) with respect to the time domain t, *i.e.*

$$\begin{aligned}&T\left[\frac{\partial^4\bar{u}}{\partial\bar{x}^4}\right]=\frac{d^4U(\bar{x},k)}{d\bar{x}^4},\\ &T\left[\frac{\partial^2\bar{u}}{\partial\bar{t}^2}\right]=\frac{(k+1)(k+2)}{H^2}U(\bar{x},k+2),\\ &T\left[2\bar{V}^2\bar{u}\right]=2\bar{V}^2U(\bar{x},k),\\ &T\left[3\bar{V}^2\bar{u}^2\right]=3\bar{V}^2\left(U(\bar{x},k)\otimes U(\bar{x},k)\right)\\ &\qquad=3\bar{V}^2\left(\sum_{\ell=0}^{k}U(\bar{x},\ell)\,U(\bar{x},k-\ell)\right),\\ &T\left[4\bar{V}^2\bar{u}^3\right]=4\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(3+1)\ell\right]U(\bar{x},\ell)\,U^3(\bar{x},k-\ell)\right),\\ &T\left[5\bar{V}^2\bar{u}^4\right]=5\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(4+1)\ell\right]U(\bar{x},\ell)\,U^4(\bar{x},k-\ell)\right),\end{aligned} \tag{26a}$$

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$$T\left[6\bar{V}^2\bar{u}^5\right]=4\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(5+1)\ell\right]U(\bar{x},\ell)\,U^5(\bar{x},k-\ell)\right),$$
$$T\left[0.65\frac{\bar{V}^2}{\bar{w}(1-\bar{u})}\right]=\frac{0.65\bar{V}^2}{\bar{w}}\left(\frac{1}{1-U(\bar{x},k)}\right),$$
$$T\left[\bar{V}^2\right]=\bar{V}^2\delta(k)\,. \tag{26b}$$

Note that $\delta(k)$ has the form

$$\delta(k)=\begin{cases}1 & \text{for} \quad k=0\\ 0 & otherwise\end{cases} \tag{27}$$

The governing equation can therefore be rewritten as

$$\begin{aligned}&\frac{d^4U(\bar{x},k)}{d\bar{x}^4}+\frac{(k+1)(k+2)}{H^2}U(\bar{x},k+2)\\&-2\bar{V}^2U(\bar{x},k)-3\bar{V}^2\left(\sum_{\ell=0}^{k}U(\bar{x},\ell)\,U(\bar{x},k-\ell)\right)\\&-4\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(3+1)\ell\right]U(\bar{x},\ell)\,U^3(\bar{x},k-\ell)\right)\\&-5\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(4+1)\ell\right]U(\bar{x},\ell)\,U^4(\bar{x},k-\ell)\right)\\&-6\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(5+1)\ell\right]U(\bar{x},\ell)\,U^5(\bar{x},k-\ell)\right)\\&-\frac{0.65\bar{V}^2}{\bar{w}}\left(\frac{1}{1-U(\bar{x},k)}\right)=\bar{V}^2\delta(k)\end{aligned} \tag{28}$$

Similarly, the boundary conditions given in Eq. (22) can be rewritten as

$$\begin{aligned}&T\left[\bar{u}(\bar{x},\bar{t})\right]=U(\bar{x},k)=0 \quad \text{at} \quad \bar{x}=0\,,\\&T\left[\frac{\partial\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}}\right]=\frac{dU(\bar{x},k)}{d\bar{x}}=0 \quad \text{at} \quad \bar{x}=0\,,\\&T\left[\frac{\partial^2\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}^2}\right]=\frac{d^2U(\bar{x},k)}{d\bar{x}^2}=0 \quad \text{at} \quad \bar{x}=1\,,\\&T\left[\frac{\partial^3\bar{u}(\bar{x},\bar{t})}{\partial\bar{x}^3}\right]=\frac{d^3U(\bar{x},k)}{d\bar{x}^3}=0 \quad \text{at} \quad \bar{x}=1\,.\end{aligned} \tag{29}$$

Finally, the initial condition given in Eq. (23) can be rewritten as

$$\begin{aligned}&T[\bar{u}(\bar{x},0)]=U(\bar{x},0)=0 \quad , \; k=0\\&T\left[\frac{\partial\bar{u}(\bar{x},0)}{\partial\bar{t}}\right]=\frac{k+1}{H}U(\bar{x},k+1)=0 \quad , \; k=0\end{aligned} \tag{30}$$

where $U(\bar{x},k)$ is the spectrum of $\bar{u}(\bar{x},\bar{t})$, k and ℓ are transformation parameters and H is the time interval.

Having applied the differential transformation method, the finite difference method is then applied to the transformed governing equation and its corresponding boundary and initial conditions (given in Eqs. (28), (29) and (30), respectively). Applying the fourth-order accurate central difference scheme to Eq. (28) yields

$$\begin{aligned}&\frac{U_{i+2}(k)-4U_{i+1}(k)+6U_i(k)-4U_{i-1}(k)+U_{i-2}(k)}{\Delta\bar{x}^4}\\&+\frac{(k+1)(k+2)}{H^2}U_i(k+2)-2\bar{V}^2U_i(k)\\&-3\bar{V}^2\left(\sum_{\ell=0}^{k}U_i(\ell)\,U_i(k-\ell)\right)\\&-4\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(3+1)\ell\right]U_i(\ell)\,U_i^3(k-\ell)\right)\\&-5\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(4+1)\ell-k\right]U_i(\ell)\,U_i^4(k-\ell)\right)\\&-6\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(5+1)\ell-k\right]U_i(\ell)\,U_i^5(k-\ell)\right)\\&-\frac{0.65\bar{V}^2}{\bar{w}}\left(\frac{1}{1-U_i(k)}\right)=\bar{V}^2\delta(k)\,,\end{aligned} \tag{31}$$

where $\Delta\bar{x}$ is the incremental change in the axial beam position and i is a position index in the x-direction. In Eq. (31), $U_i(k+2)$ is the only unknown and can therefore be computed as follows:

$$\begin{aligned}&U_i(k+2)=\frac{H^2}{(k+1)(k+2)}\\&\times\left\{\bar{V}^2\delta(k)+2\bar{V}^2U_i(k)+3\bar{V}^2\left(\sum_{\ell=0}^{k}U_i(\ell)\,U_i(k-\ell)\right)\right.\\&+4\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(3+1)\ell\right]U_i(\ell)\,U_i^3(k-\ell)\right)\\&+5\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(4+1)\ell-k\right]U_i(\ell)\,U_i^4(k-\ell)\right)\\&+6\bar{V}^2\left(\sum_{\ell=1}^{k}\left[(5+1)\ell-k\right]U_i(\ell)\,U_i^5(k-\ell)\right)+\frac{0.65\bar{V}^2}{\bar{w}}\left(\frac{1}{1-U_i(k)}\right)\\&\left.-\frac{U_{i+2}(k)-4U_{i+1}(k)+6U_i(k)-4U_{i-1}(k)+U_{i-2}(k)}{\Delta\bar{x}^4}\right\}\end{aligned} \tag{32}$$

Applying the first-order, second-order and third-order accurate central difference formulae to Eqs. (29) and (30), the boundary

$$\begin{aligned}&U_0(k)=0\,,\\&\frac{U_1(k)-U_{-1}(k)}{2\Delta\bar{x}}=0\,,\\&\frac{U_{j+1}(k)-2U_j(k)+U_{j-1}(k)}{\Delta\bar{x}^2}=0\,,\\&\frac{U_{j+2}(k)-2U_{j+1}(k)+2U_{j-1}(k)-U_{j-2}(k)}{2\Delta\bar{x}^2}=0\,,\end{aligned} \tag{33}$$

where j is the final index position in the x-direction.

Similarly, the initial condition can be expressed as

$$\begin{aligned}&U_i(0)=0\\&\frac{1}{H}U_i(1)=0 \quad \rightarrow U_i(1)=0\end{aligned} \tag{34}$$

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4. NUMERICAL RESULTS AND DISCUSSION

The performance of the hybrid method in analyzing the complex nonlinear behavior of the micro cantilever system was evaluated by comparing the numerical solutions obtained for the tip displacement and pull-in voltage, respectively, with the analytical and experimental results presented in the literature. The computations were performed using MATLAB and considered two different micro cantilever systems, denoted as Case 1 and Case 2 in Table 1. Table 2 compares the numerical results obtained by the hybrid method for the end gap of the Case 1 cantilever system for four different values of the applied voltage with the equivalent analytical and experimental results presented by Hu *et al*. [23]. It is observed that the numerical solutions are in broad general agreement with the analytical results and deviate from the experimental measurements by no more than 3.9%. Table 3 compares the pull-in voltage predicted by the hybrid method for the Case 2 cantilever system with the results obtained using the closed-Form expression, CoventorWare (CW) Architect^{TM} and FEA [24], respectively. From inspection the pull-in voltage obtained using the hybrid method deviates by no more than 1.075% from the solutions obtained using the closed- Form expression and CW Architect^{TM}, respectively. Thus, overall, the results presented in Tables 2 and 3 confirm the validity of the hybrid differential transformation / finite difference solution scheme.

Figure 3 compares the results obtained by the hybrid method for the end gap of the Case 1 micro cantilever with the experimental results presented in [23] From inspection, the pull-in voltage predicted by the hybrid method has a value of 66.7 V while that measured experimentally has a value of 68.5 V. Thus, the deviation between the two results is just 2.6%.

Figure 4 illustrates the deflection of the Case 1 cantilever beam at various voltages in the range 20 ~ 66.7 V. The results confirm that at all values of the applied voltage less than the pull-in voltage, the cantilever system remains in a stable condition and a finite gap is maintained between the tip of the beam and the lower substrate. However, when a voltage equivalent to the pull-in voltage (66.7V) is applied between the two electrodes, it can be seen that the cantilever collapses, causing the tip of the beam to make contact with the fixed substrate.

The hybrid method was also used to analyze the time-based variation of the beam deflection for voltages difference. Figure 5 illustrates the results obtained by the hybrid scheme for the dynamic deflection behavior of the Case 1 cantilever beam under an applied voltage of 60V. From this figure, it can be seen that different times have different beam deflection values until a steady state is reached. The results clearly show that the beam evolves from an initial transient state to a final steady state. In other words, the hybrid method provides the means to analyze both the dynamic response of the cantilever beam and its steady-state behavior.

Table 1 Material and geometry parameters used in micro cantilever beam models

Parameters	Case 1.	Case 2.
Young's modulus (E) (GPa)	155.8	77
Poisson's Ratio (ν)	0.06	0.33
Density (ρ) (Kg/m^3)	2.33×10^3	2.33×10^3
Permittivity of free space (ε_o) (F/m)	8.85×10^{-12}	8.85×10^{-12}
Width of cantilever beam (w) (μm)	5000	50
Thickness of cantilever beam (h) (μm)	57	1.0
Initial gap (G) (μm)	92	2.5
Length of cantilever beam (L) (μm)	20000	500

Table 2 Comparison of results obtained using hybrid method for end gap with analytical and experimental results presented in the literature. (Case 1.)

Voltage (V)	End gap (Hybrid Method)	End gap (Modal Expansion [23])	Experimental [23]	Δe (%)
20	90.18	90.2	90.5	0.3
40	84.21	84.3	84.6	0.46
60	69.86	71.5	70.0	0.2
65	61.51	67.2	64.0	3.9

* $\Delta e = \dfrac{(\text{Experimental} - \text{H.M})}{\text{Experimental}} \times 100\%$

Table 3 Comparison of results obtained for pull-in voltage. (Case 2.)

Prediction Method	Pull-in Voltage (V)	percent error with H.M
Closed- Form expression [24]	0.8188	1.075
CW Architect^{TM} [24]	0.8125	0.3077
CW FEA [24]	0.75	8
Hybrid Method (H.M)	0.81	

5. CONCLUSIONS

This study has utilized a hybrid method comprising the differential transformation scheme and the finite difference approximation method to analyze the nonlinear behavior of a micro cantilever beam. In the proposed approach, each term in the normalized governing equation of the cantilever beam is processed

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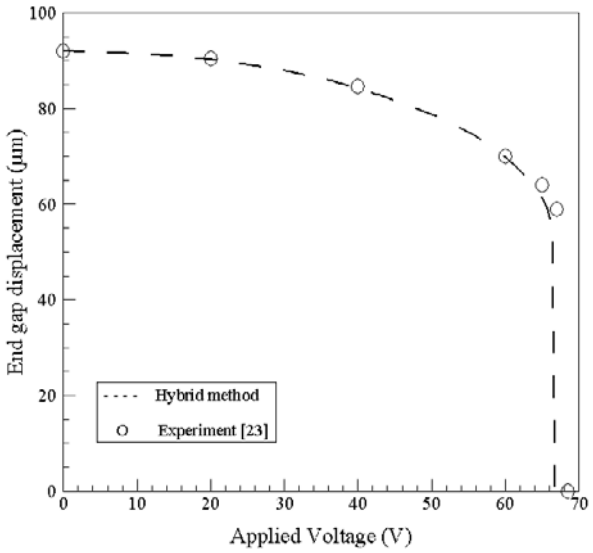


Fig. 3 Variation of end gap with applied voltage

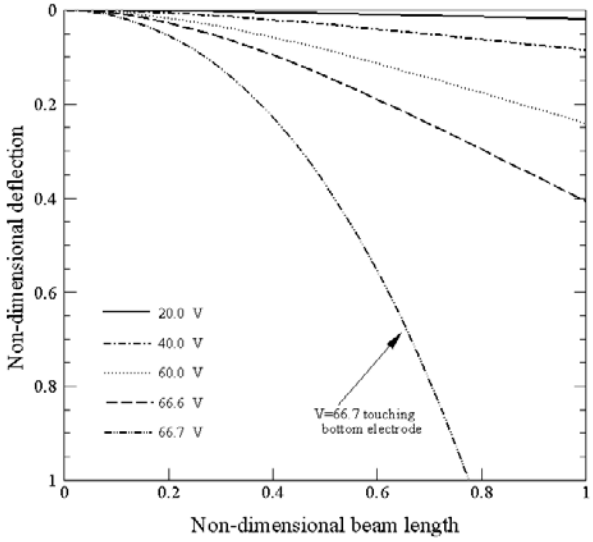


Fig. 4 Variation of micro cantilever beam deflection as function of applied voltage

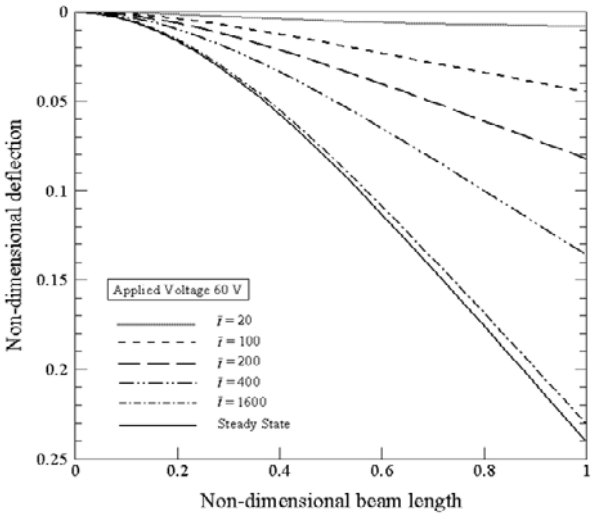


Fig. 5 Transient and steady-state deflection of micro cantilever beam under applied voltage of 60V

using the differential transformation method with respect to the time domain t. The transformed equation is then processed using the finite difference approximation method with respect to $\bar{x}$ (*i.e.* the axial direction of the cantilever beam). Compared to existing methods such as the finite element method (FEM), the hybrid method has the advantages of an explicit physical meaning and a greater computational speed and simplicity. The validity of the proposed approach has been confirmed by comparing the numerical results obtained for the tip deflection and pull-in voltage of the cantilever beam with the analytical and experimental results presented in the literature. The results have shown that the solutions obtained using the hybrid scheme deviate by no more than 3.9% from the experimental results. In addition, it has been shown that the hybrid method provides the means to analyze both the steady-state behavior of the cantilever beam and its initial dynamic response. Overall, the results presented in this study indicate that the hybrid differential transformation / finite difference method successfully models the complex interactions between the electrostatic coupling effect, the fringing field effect and the nonlinear electrostatic force, and is therefore a suitable technique for analyzing the nonlinear response of the micro cantilever system considered in this study.

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