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Tame systems and scrambled pairs under an Abelian group action

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Abstract. A dynamical version of the Bourgain–Fremlin–Talagrand dichotomy shows that the enveloping semigroup of a dynamical system is either very large and contains a topological copy of $\beta\mathbb{N}$, or it is a 'tame' topological space whose topology is determined by the convergence of sequences. In the latter case, Glasner (On tame enveloping semigroups. *Colloq. Math.* **105** (2006), 283–395) calls the system tame. In this paper, we study the tame system under an Abelian group action. We introduce the notion of scrambled pairs for a dynamical system under an Abelian group action and show that a tame system has no scrambled pair. At the same time, we give some sufficient conditions such that a pair is a scrambled one. Moreover, using these sufficient conditions we prove that a minimal tame system under an Abelian group action is almost automorphic and uniquely ergodic. This gives a positive answer to Problem 2.5 in the paper cited above. Finally, we prove that for $\mathbb{Z}$-actions a tame system is M-null, that is, the metric sequence entropy is zero for any invariant measure and any increasing sequence of natural numbers. We also give examples to show that tameness is strictly weaker than nullness for $\mathbb{Z}$-actions.

1. *Introduction*

In order to study the asymptotic behavior of a compact (metrizable) G-dynamical system (X, G), where G is a topological group, Ellis introduced in 1960 [**4**] the enveloping (or Ellis) semigroup $E = E(X, G)$. It is defined as the closure in X^X of the set G considered as a subset of X^X, and is an indispensable tool in the abstract theory of topological dynamical system (see, for example, Ellis [**5**], Glasner [**8**], Veech [**30**], Auslander [**1**] and de Vries [**31**]). However, explicit computations of enveloping semigroups are quite rare.

In [**21**], Köhler pointed out the relevance of a theorem of Bourgain, Fremlin and Talagrand [**2**] to the study of enveloping semigroups. Recently, Glasner and Megrelishvili in [**12**] obtained a dynamical Bourgain–Fremlin–Talagrand (BFT) dichotomy: the enveloping semigroup of a dynamical system is either very large and contains a topological

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copy of $\beta\mathbb{N}$, or it is a 'tame' topological space whose topology is determined by the convergence of sequences. In the latter case, Glasner in [**11**] calls the dynamical system tame. Examples of tame dynamical systems include metric minimal equicontinuous systems, topologically transitive weakly almost periodic (WAP) systems (see [**12**]), topologically transitive hereditarily non-sensitive (HNS) systems (see [**12**]) and null systems under $\mathbb{Z}$-action (see [**20**]).

Glasner in [**11**] proved that (1) a minimal distal metric system is tame if and only if it is equicontinuous, (2) a tame minimal cascade (X, T) has zero topological entropy, (3) for an Abelian acting group a metric minimal tame system is a PI-flow, hence in particular a non-trivial minimal weakly mixing G-system is never tame. It is an open question (see [**11**, Problem 2.5]) whether every minimal metrizable tame system (X, G) with an Abelian acting group is a proximal extension of an equicontinuous system. Answering the question affirmatively, in this paper we show that a minimal tame system under an Abelian group action is uniquely ergodic and almost automorphic, that is, an almost one-to-one extension of a minimal equicontinuous system. At the same time, we prove that for $\mathbb{Z}$-actions a tame system is M-null, that is, the metric sequence entropy is zero for any invariant measure and any increasing sequence of natural numbers. We also give examples to show that tameness is strictly weaker than nullness for $\mathbb{Z}$-actions.

2. *Preliminary and tame systems*

Let X be a compact metric space and G be a topological group. A *G-dynamical system* (for short, *G-system*) (X, G) is a continuous mapping $\Pi : G \times X \to X$ which satisfies the following properties:

(1) $\Pi(e, x) = x$ for $x \in X$, where e is the unit element of G;

(2) $\Pi(g, \Pi(h, x)) = \Pi(gh, x)$ for $x \in X$ and $g, h \in G$.

For convenience, we sometimes denote $\Pi(g, x)$ by $\Pi_g(x)$ or simply by $g(x)$. When X is a singleton, we say (X, G) is *a trivial G-system*. In this paper, the topological group G is always Abelian.

Let (X, G) be a G-system. A subset $M \subset X$ is *G-invariant* if for each $x \in M$, its orbit $Gx = \{gx : g \in G\}$ lies in M. Let M be a closed G-invariant set of X, then (M, G) is also a G-system; we say (M, G) is a *subsystem* of (X, G). A non-empty closed G-invariant set $M \subset X$ is *minimal* if it contains no non-empty, proper, closed G-invariant subset. (X, G) is *minimal* if X itself is a minimal set. A point x of (X, G) is *minimal point*, if the closure of its orbit is a minimal set of (X, G). Moreover, as a consequence of Zorn's lemma, a G-system always contains a minimal set.

A continuous map $\pi : X \to Y$ is called a *homomorphism* of systems (X, G) and (Y, G) if it is onto and $\pi g = g\pi$ for any $g \in G$. In this case we say (X, G) is an *extension* of (Y, G) or (Y, G) is a *factor* of (X, G). If π is also injective then it is called an *isomorphism*.

The *enveloping (or Ellis) semigroup* $E = E(X, G)$ of a G-system (X, G) is defined as the closure in X^X (with its compact, usually non-metrizable, pointwise convergence topology) of the set $\{\Pi_g : X \to X\}_{g \in G}$ considered as a subset of X^X. We identify here the elements of G with the corresponding homeomorphisms of X. With the operation of composition of maps this is a right topological semigroup. Moreover, the map $i : G \to E$, $g \to \Pi_g$ is a right topological semigroup compactification of G.

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In the following, we recall the notion of tame system which was first studied in [**11**]. Recall that a topological space K is said to be *Rosenthal compact* [**14**] if it is homeomorphic to a pointwise compact subset of the space $B_1(X)$ of functions of the first Baire class on a Polish space X. All metric compact spaces are Rosenthal compact. An example of a separable non-metrizable Rosenthal compact is the Helly compact of all (not only strictly) increasing self-maps of $[0, 1]$ in the pointwise topology.

A topological space K is a *Fréchet space* if for every $A \subset K$ and every $x \in \bar{A}$ there exists a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$ (see Engelking [**6**]). A topological space K is *angelic* if (i) every relatively countably compact set in K is relatively compact and (ii) for every relatively compact subset A of X and every $x \in \bar{A}$ there exists a sequence $x_n \in A$ with $\lim_{n\to\infty} x_n = x$. Thus a compact space is angelic if and only if it is Fréchet. Clearly, $\beta\mathbb{N}$, the Stone–Cech compactification of the natural numbers $\mathbb{N}$, cannot be embedded into a Fréchet space.

The following dynamical BFT dichotomy is derived in [**12**].

THEOREM 2.1. (A dynamical BFT dichotomy) *Let (X, G) be a G-system and let $E = E(X, G)$ be its enveloping semigroup. We have the following alternative. Either:*

(1) *E is a separable Rosenthal compact, hence with cardinality* $\mathrm{card}(E) \leq 2^{\aleph_0}$*; or*

(2) *the compact space E contains a homeomorphic copy of $\beta\mathbb{N}$, hence* $\mathrm{card}(E) = 2^{2^{\aleph_0}}$.

Definition 2.2. [**11**] A dynamical system (X, G) is called tame, if its enveloping semigroup $E(X, G)$ is separable and Fréchet.

Remark 2.3. (1) [**11**] If we embed the group G in the Polish group $\mathrm{Homeo}(X)$ of homeomorphisms of X equipped with the topology of uniform convergence, then it is easy to deduce from this fact that the enveloping semigroup $E(X, G)$ is separable. Hence Theorem 2.1 can be rephrased as saying that a G-system (X, G) is either tame or $E(X, G)$ contains a topological copy of $\beta\mathbb{N}$. When (X, G) is tame, $E(X, G)$ is Fréchet and every element $p \in E(X, G)$ is a limit of a sequence of elements of G, that is, $p = \lim_{n\to\infty} g_n$ for some $\{g_n\}_{n\in\mathbb{N}} \subset G$.

(2) Note that in a Fréchet space, compact sets are sequentially compact, and we have that if (X, G) is tame, then $E(X, G)$ is sequentially compact, that is, any sequence of $E(X, G)$ has a convergent subsequence.

A G-system (X, G) is called *equicontinuous*, if for any $\epsilon > 0$, there exists $\delta > 0$ such that when $d(x, y) < \delta$ one has $d(gx, gy) < \epsilon$ for each $g \in G$. A minimal equicontinuous G-system is called *almost periodic*. It is well known that if (X, G) is almost periodic, then $E(X, G)$ is a compact metric topological group, hence is separable and Fréchet, therefore (X, G) is tame. Other examples of tame systems are topologically transitive weakly almost periodic (WAP) systems (see [**12**]), topologically transitive hereditarily non-sensitive (HNS) systems (see [**12**]) and null systems under $\mathbb{Z}$-actions (see [**20**]).

The following proposition was proved in [**11**].

PROPOSITION 2.4. *Tameness is preserved by taking:*

(1) *subsystems;*

(2) *countable products; and*

(3) *factors.*

Every G-system (X, G) admits a maximal tame factor.

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3. *Scrambled pairs and strong scrambled pairs*

In the section, we will introduce the notions of scrambled pairs and strongly scrambled pairs. Then we will prove that a tame system has no scrambled pair. This fact will be used in the next section to characterize the structure of a tame system.

Definition 3.1. Let (X, G) be a G-system.

(1) A pair $(x_1, x_2) \in X \times X$ is called scrambled, if $x_1 \neq x_2$ and there exist closed neighborhoods U_i of x_i, $i = 1, 2$ and a sequence $\{g_1, g_2, \dots\}$ of G with $U_1 \cap U_2 = \emptyset$ and $\bigcap_{i=1}^{\infty} \Pi_{g_i^{-1}} U_{s(i)} \neq \emptyset$ for any $s = (s(1), s(2), \dots) \in \{1, 2\}^{\mathbb{N}}$.

(2) A pair $(x_1, x_2) \in X \times X$ is called strongly scrambled, if $x_1 \neq x_2$ and for any neighborhood U_i of x_i, $i = 1, 2$ there exists a sequence $\{g_1, g_2, \dots\}$ of G with $\bigcap_{i=1}^{\infty} \Pi_{g_i^{-1}} U_{s(i)} \neq \emptyset$ for any $s = (s(1), s(2), \dots) \in \{1, 2\}^{\mathbb{N}}$.

Denote by $S(X, G)$ the set of all scrambled pairs of (X, G) and by $SS(X, G)$ the set of all strongly scrambled pairs of (X, G). Clearly, $SS(X, G) \subseteq S(X, G)$.

THEOREM 3.2. *Let (X, G) be a G-system. If (X, G) is tame, then $S(X, G) = \emptyset$.*

Proof. Let (X, G) be tame and $(x_1, x_2) \in S(X, G)$ when $S(X, G) \neq \emptyset$. Then there exist closed neighborhoods U_i of x_i, $i = 1, 2$ and a sequence $\{g_1, g_2, \dots\}$ of G such that $U_1 \cap U_2 = \emptyset$ and $\bigcap_{i=1}^{\infty} \Pi_{g_i^{-1}} U_{s(i)} \neq \emptyset$ for any $s = (s(1), s(2), \dots,) \in \{1, 2\}^{\mathbb{N}}$. Since $E(X, G)$ is sequentially compact, there exists a subsequence $\{n_1 < n_2 < \cdots\}$ such that $\Pi_{g_{n_i}} \to p \in E(X, G)$ in the pointwise topology of X^X.

It is easy to see that for any $t = (t(1), t(2), \dots) \in \{1, 2\}^{\mathbb{N}}$, $\bigcap_{i=1}^{\infty} \Pi_{g_{n_i}^{-1}} U_{t(i)} \neq \emptyset$. In particular, it is true for $t^* = (1, 2, 1, 2, 1, 2, \dots)$. Then $\bigcap_{i=1}^{\infty} \Pi_{g_{n_i}^{-1}} U_{t^*(i)} \neq \emptyset$ and there is $x \in \bigcap_{i=1}^{\infty} \Pi_{g_{n_i}^{-1}} U_{t^*(i)}$. Thus, $g_{n_{2k}}(x) \in U_2$ and $g_{n_{2k-1}}(x) \in U_1$ for each $k \in \mathbb{N}$.

Hence, we have $p(x) = \lim_{k \to +\infty} g_{n_{2k}}(x) \in U_1$ and $p(x) = \lim_{k \to +\infty} g_{n_{2k-1}}(x) \in U_2$, that is, $p(x) \in U_1 \cap U_2$, which contradicts $U_1 \cap U_2 = \emptyset$. This shows that $S(X, G) = \emptyset$. □

In the rest of the section, we will give some sufficient conditions such that a pair can be a strongly scrambled one. To this aim, we need to introduce the notion of weakly mixing pairs (which notion was first introduced and studied by Huang *et al* in [**16**]).

Definition 3.3. For a G-system (X, G), $(x_1, x_2) \in X \times X$ is a weakly mixing pair if $x_1 \neq x_2$ and for any open neighborhood U_i of x_i, $i = 1, 2$, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$, where $N(U, V) = \{g \in G : U \cap \Pi_{g^{-1}} V \neq \emptyset\}$.

The set of all weakly mixing pairs is denoted by $WM(X, G)$. Recall that the *regionally proximal relation* $RP(X, G)$ is the set of all points $(x_1, x_2) \in X \times X$ such that for each $\epsilon > 0$ and each open neighborhood U_i of x_i, $i = 1, 2$ there are $x_i' \in U_i$, $i = 1, 2$ and $g \in G$ with $d(g(x_1'), g(x_2')) < \epsilon$.

PROPOSITION 3.4. *Let (X, G) be a G-system and $\Delta_X = \{(x, x) : x \in X\}$. Then $WM(X, G)$ has the following properties:*

(1) *$WM(X, G) \cup \Delta_X$ is a closed G-invariant subset of $X \times X$;*

(2) *$SS(X, G) \subseteq WM(X, G)$ and $WM(X, G) \subseteq RP(X, G) \setminus \Delta_X$;*

(3) *If (X, G) is a minimal G system, then $WM(X, G) \cup \Delta_X = RP(X, G)$.*

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Proof. (1) is obvious.

Now we prove (2) Assume that $x_1 \neq x_2$ and $(x_1, x_2) \notin WM(X, T)$. Then there are open neighborhoods U_i of x_i, $i = 1, 2$ such that $N(U_1, U_1) \cap N(U_1, U_2) = \emptyset$. Therefore, $U_1 \cap \Pi_{g^{-1}} U_1 = \emptyset$ or $U_1 \cap \Pi_{g^{-1}} U_2 = \emptyset$ for any $g \in G$. This implies that $(x_1, x_2) \notin SS(X, T)$ and we have proved that $SS(X, T) \subseteq WM(X, T)$.

Let $(x_1, x_2) \in WM(X, G)$. Then for any open neighborhood U_i of x_i, $i = 1, 2$, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$. Thus, there exists $g \in G$ such that $U_1 \cap \Pi_{g^{-1}} U_1 \neq \emptyset$ and $U_1 \cap \Pi_{g^{-1}} U_2 \neq \emptyset$. This implies that there are $x_1' \in U_1$ and $x_2' \in U_2$ with $\Pi_{g^{-1}} x_1', \Pi_{g^{-1}} x_2' \in U_1$. Thus, we have $(x_1, x_2) \in RP(X, G) \setminus \Delta_X$.

(3) is [**16**, Theorem 2.4] and for completeness a proof is included. Assume that (X, G) is minimal. For $(x_1, x_2) \in RP(X, G) \setminus \Delta_X$, let $L(x_1, x_2)$ be the subset of X such that for each x_3 in the set, each neighborhood U_i of x_i, $i = 1, 2, 3$, there are $x, y \in U_3$ and $g \in G$ with $\Pi_g(x) \in U_1$ and $\Pi_g(y) \in U_2$. It is easy to see that $L(x_1, x_2)$ is non-empty, closed and G-invariant. As (X, G) is minimal, we have $X = L(x_1, x_2)$. In particular, $x_1 \in L(x_1, x_2)$. Thus, $(x_1, x_2) \in WM(X, G)$ and we have proved that $RP(X, G) \setminus \Delta_X \subseteq WM(X, G)$. Finally by (2), one has $WM(X, G) = RP(X, G) \setminus \Delta_X$. □

The following lemma gives a sufficient condition such that a weakly mixing pair could be a strongly scrambled one. Recall that for topological spaces X and Y, a map $f : X \to Y$ is *semi-open* if for any non-empty open subset A of X, the interior of $f(A)$ is not empty.

LEMMA 3.5. *Let (X, G) be a G-system, $(x_1, x_2) \in X \times X \setminus \Delta_X$ and $A = \overline{\{(\Pi_g(x_1), \Pi_g(x_2)) : g \in G\}}$. If the projection to the first coordinate $\pi_1 : A \to X$ is semi-open, then $(x_1, x_2) \in SS(X, G)$ if and only if $(x_1, x_2) \in WM(X, G)$.*

Proof. We follow the arguments in the proof of [**16**, Lemma 3.1]. By Proposition 3.4, $SS(X, G) \subseteq WM(X, G)$. Now assume $(x_1, x_2) \in WM(X, G)$ and $\pi_1 : A \to X$ is semi-open. We are going to prove that $(x_1, x_2) \in SS(X, G)$. Since $(x_1, x_2) \in WM(X, G)$, we have $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$ for any open neighborhoods U_1, U_2 of x_1, x_2. Moreover, we have the following.

CLAIM. *For any open neighborhoods U_1, U_2 of x_1, x_2 in X, there exists a sequence $\{g_n\}$ of G such that for any $\ell \in \mathbb{N}$ and $s \in \{1, 2\}^\ell$, we can find $M_s \in G$ with $\Pi_{M_s}(x_1) \in \bigcap_{i=1}^{\ell} \Pi_{g_i^{-1}} U_{s(i)}$ and $\Pi_{M_s}(x_j) \in U_j$, $j = 1, 2$.*

Proof of claim. The claim is proved by induction.

Fix open neighborhoods U_1, U_2 of x_1, x_2 and let $W_1 = \text{int}(\pi_1((U_1 \times U_2) \cap A))$. Then, W_1 is a non-empty open set and $W_1 \subset U_1$. Set $W_2 = U_2$.

Since $(W_1 \times W_2) \cap A$ is a non-empty open set of A, there exists $g \in G$ such that $(\Pi_g x_1, \Pi_g x_2) \in (W_1 \times W_2) \cap A$. Since $(\Pi_g x_1, \Pi_g x_2) \in WM(X, G)$ (see Proposition 3.4(1)), so there exists $g_1 \in G$ such that $W_1 \cap \Pi_{g_1^{-1}} W_1 \neq \emptyset$ and $W_1 \cap \Pi_{g_1^{-1}} W_2 \neq \emptyset$. As $\{(\Pi_g(x_1), \Pi_g(x_2)) : g \in G\}$ is dense in A, there exist $M_1, M_2 \in G$ such that $(\Pi_{M_i}(x_1), \Pi_{M_i}(x_2)) \in ((W_1 \cap \Pi_{g_1^{-1}} W_i) \times U_2) \cap A$, $i = 1, 2$. Thus, $\Pi_{M_i}(x_1) \in U_1 \cap \Pi_{g_1^{-1}} U_i$ and $\Pi_{M_i}(x_j) \in U_j$, for $i, j = 1, 2$.

Now assume $g_1, g_2, \ldots, g_\ell, \ell \geq 1$ are defined and for any $s \in \{1, 2\}^\ell$, there exists $M_s \in G$ such that $\Pi_{M_s}(x_1) \in \bigcap_{i=1}^{l} \Pi_{g_i^{-1}} U_{s(i)}$ and $\Pi_{M_s}(x_j) \in U_j$, for $j = 1, 2$.

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We are going to define $g_{\ell+1}$. Take $\delta > 0$ such that for any $z_1, z_2 \in X$ with $d(z_1, x_1) < \delta$, and $d(z_2, x_2) < \delta$, one has $\Pi_{M_s}(z_1) \in \bigcap_{i=1}^{\ell} \Pi_{g_i^{-1}} U_{s(i)}$ and $\Pi_{M_s}(z_j) \in U_j$, $j = 1, 2$ for any $s \in \{1, 2\}^{\ell}$.

Let $U_1^{\delta} = B(x_1, \delta)$ and $U_2^{\delta} = B(x_2, \delta)$. So $W_1^{\delta} = \mathrm{int}(\pi_1((U_1^{\delta} \times U_2^{\delta}) \cap A))$ is a non-empty open set and $W_1^{\delta} \subset U_1^{\delta}$. Set $W_2^{\delta} = U_2^{\delta}$. Then $N(W_1^{\delta}, W_1^{\delta}) \cap N(W_1^{\delta}, W_2^{\delta}) \neq \emptyset$. Hence there exists $g_{\ell+1}$ such that $W_1^{\delta} \cap \Pi_{g_{\ell+1}^{-1}} W_i^{\delta} \neq \emptyset$, $i = 1, 2$. Thus we can choose $p_1, p_2 \in G$ such that for $i, j = 1, 2$, $\Pi_{p_i}(x_1) \in W_1^{\delta} \cap \Pi_{g_{\ell+1}^{-1}} W_i^{\delta}$, $\Pi_{p_i}(x_j) \in U_j^{\delta}$.

For any $r \in \{1, 2\}^{\ell+1}$, take $s \in \{1, 2\}^{\ell}$ with $s(i) = r(i)$, $i = 1, 2, \ldots, \ell$ and put $r(\ell + 1) = k$. Let $M_r = M_s \cdot p_k$. Then

$$\Pi_{M_r}(x_1) = \Pi_{M_s}(\Pi_{p_k}(x_1)) \in \bigcap_{i=1}^{\ell} \Pi_{g_i^{-1}} U_{s(i)} \quad \text{as } \Pi_{p_k}(x_1) \in B(x_1, \delta).$$

As $\Pi_{g_{\ell+1}} \Pi_{p_k}(x_1) \in W_k^{\delta}$ and G is Abelian, we have

$$\Pi_{g_{\ell+1}} \Pi_{M_r}(x_1) = \Pi_{M_s} \Pi_{g_{\ell+1}} \Pi_{p_k}(x_1) \in U_k.$$

So

$$\Pi_{M_r}(x_1) \in \bigcap_{i=1}^{\ell+1} \Pi_{g_i^{-1}} U_{r(i)},$$

and $\Pi_{M_r}(x_j) = \Pi_{M_s} \Pi_{p_k}(x_j) \in U_j$ as $\Pi_{p_k}(x_j) \in B(x_j, \delta)$ for $j = 1, 2$. This ends the proof of the claim.

Finally, by the claim, $(x_1, x_2) \in SS(X, G)$. □

Remark 3.6. In the above proof, we use the fact $\Pi_{g_{\ell+1}} \Pi_{M_s} = \Pi_{M_s} \Pi_{g_{\ell+1}}$. Hence when G is non-Abelian, our proof does not work.

PROPOSITION 3.7. *Let (X, G) be a minimal G-system and $(x_1, x_2) \in X \times X \setminus \Delta_X$. If (x_1, x_2) is a minimal point of $(X \times X, G)$, then $(x_1, x_2) \in SS(X, G)$ if and only if $(x_1, x_2) \in WM(X, G)$.*

Proof. Let $A = \overline{\{(\Pi_g(x_1), \Pi_g(x_2)) : g \in G\}}$. Clearly A is a minimal set of $(X \times X, G)$. By Lemma 3.5, it remains to show that $\pi_1 : A \to X$ is semi-open.

Let U be a non-empty open set of A. Take a non-empty open set U_1 of A with $\overline{U_1} \subset U$. Then $\bigcup_{g \in G} \Pi_g \times \Pi_g(U_1) = A$ as A is a minimal set of $(X \times X, G)$. Since A is compact, there exist $g_1, g_2, \ldots, g_k \in G$ such that $\bigcup_{i=1}^{k} \Pi_{g_i} \times \Pi_{g_i}(U_1) = A$. Hence $\bigcup_{i=1}^{k} \Pi_{g_i} \pi_1(\overline{U_1}) = X$. As $\Pi_{g_i} \pi_1(\overline{U_1})$, $i = 1, 2, \ldots, k$, are closed sets, $\mathrm{int}(\pi_1(\overline{U_1})) \neq \emptyset$. Thus, $\mathrm{int}(\pi_1(U)) \neq \emptyset$, that is π_1 is semi-open. □

Remark 3.8. In fact, we have shown that a homomorphism between two minimal G-systems is semi-open. This is a well-known fact (see, for example, [**1**]).

Let (X, G) be a G-system with metric d. Two points x_1, x_2 of X are called *proximal* if there exists a sequence $\{g_i\}_{i \in \mathbb{N}}$ of G such that $d(g_i(x_1), g_i(x_2)) \to 0$. x_1, x_2 are called *distal* if they are not proximal. (X, G) is called a *distal system*, if each $x_1 \neq x_2 \in X$, x_1, x_2 are distal.

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COROLLARY 3.9. *Let (X, G) be a minimal distal G-system. Then $SS(X, G) = WM(X, G) = RP(X, G) \setminus \Delta_X$. In particular, (X, G) is tame if and only if it is almost periodic.*

Proof. For any $(x_1, x_2) \in X \times X$, since (X, G) is minimal and distal, (x_1, x_2) is a minimal point of $X \times X$. Now, by Proposition 3.7, $SS(X, G) = WM(X, G) = RP(X, G) \setminus \Delta_X$.

Now if (X, G) is tame, then $SS(X, G) = \emptyset$. Hence $RP(X, G) = \Delta_X$, which shows that (X, G) is equicontinuous. Finally, it is clear that almost periodicity implies tameness. □

Remark 3.10. Glasner in [**11**] proved that for any topological group Γ, a minimal distal Γ-system is tame if and only if it is almost periodic.

We say a G-system (X, G) is *transitive* if there is a point $x \in X$ with a dense orbit, and the point x is called *a transitive point*. We say a G-system (X, G) is *weakly mixing* if $(X \times X, G)$ is transitive. For a transitive G-system we have the following corollary.

PROPOSITION 3.11. *Let (X, G) be a transitive G-system with x_1 a transitive point and p a fixed point, i.e. $Gp = \{p\}$. Then $(x_1, p) \in SS(X, G)$ if and only if $(x_1, p) \in WM(X, G)$.*

Proof. Set $A = \overline{\{(\Pi_g(x_1), \Pi_g(p)) : g \in G\}}$. Then $A = X \times \{p\}$, and hence $\pi_1 : A \to X$ is semi-open. By Lemma 3.5 we get the proposition. □

THEOREM 3.12. *A weakly mixing tame G-system is a trivial one.*

Proof. Let (X, G) be a weakly mixing tame G-system. If (X, G) is not trivial, then there exists $(x_1, x_2) \in X \times X \setminus \Delta_X$ such that (x_1, x_2) is a transitive point of $(X \times X, G)$. Clearly, $(x_1, x_2) \in WM(X, G)$. Let $A = \overline{\{(\Pi_g(x_1), \Pi_g(x_2)) : g \in G\}}$. Then $A = X \times X$ and the projection to the first coordinate $\pi_1 : A \to X$ is semi-open. By Lemma 3.5, $(x_1, x_2) \in SS(X, G)$, which contradicts the assumption that (X, G) is tame (see Theorem 3.2). □

4. *The structure of a tame system*

In this section, we study the structure of a tame system. First let's give some basic definitions in topological dynamical systems.

Let $\pi : (X, G) \to (Y, G)$ be a homomorphism. The extension $\pi : (X, G) \to (Y, G)$ is called *proximal* (respectively *distal*) if $\pi(x_1) = \pi(x_2)$ implies that x_1 and x_2 are proximal (respectively distal). The extension π is called *almost one-to-one* if there exists a dense G_δ set $X_0 \subset X$ such that $\pi^{-1}\pi(x) = \{x\}$ for any $x \in X_0$. The extension π is called *weakly mixing* if (R_π, G) is a transitive subsystem of $(X \times X, G)$, where $R_\pi = \{(x_1, x_2) \in X \times X : \pi(x_1) = \pi(x_2)\}$.

In the following, we will show that a minimal tame system is almost automorphic and uniquely ergodic. First, we recall the notion of almost automorphic.

Definition 4.1. Let (X, G) be a G-system. $x \in X$ is an almost automorphic point if any sequence $\{g'_1, g'_2, \dots\}$ in G has a subsequence $\{g_1, g_2, \dots\}$ such that $\lim_{n\to\infty} \Pi_{g_n}(x) = y$ for some $y \in X$ and $\lim_{n\to\infty} \Pi_{g_n^{-1}}(y) = x$. We say a G-system (X, G) is almost automorphic if there is an almost automorphic point with a dense orbit in X.

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PROPOSITION 4.2. [**29**] *A G-system (X, G) is almost automorphic if and only if it is an almost one-to-one extension of a minimal equicontinuous G-system (Y, G).*

Remark 4.3. It is easy to see that an almost automorphic G-system is an almost one-to-one extension of its maximal equicontinuous factor.

Let X, Y be two compact metric spaces and $\pi : X \to Y$ be a continuous map. We call π an *open map* if for every open subset U in X, $\pi(U)$ is open in Y. In particular, an extension $\pi : (X, G) \to (Y, G)$ is called *open* if $\pi : X \to Y$ is an open map. The following lemma is a result of Glasner [**10**, Corollary 6.4].

LEMMA 4.4. *Let (X, G) be a minimal G-system and $\pi : (X, G) \to (Y, G)$ be a proximal open extension, then π is a weakly mixing extension.*

Let $\pi : (X, G) \to (Y, G)$ be an extension of minimal G-systems. Consider $\pi^{-1} : Y \to 2^X$ with $y \mapsto \pi^{-1}(y)$. Then π^{-1} is a upper-semicontinuous map, hence its continuous point forms a dense G_δ subset of Y, denoted by Y_0.

Let $Y^* = \overline{\{\pi^{-1}(y) : y \in Y_0\}}$. Then Y^* is a G-invariant closed subset 2^X. Hence (Y^*, G) is a subsystem of $(2^X, G)$. It is easy to see that for any $y^* \in Y^*$, $\pi(y^*)$ is a singleton set. So we can well define $\tau : Y^* \to Y$ such that $\tau(y^*) = \pi(y)$ for $y \in y^*$, and it is easy to see that $\tau : (Y^*, G) \to (Y, G)$ is an extension.

Now let $X^* = \{(x, y^*) \in X \times Y^* : x \in y^*\}$; then X^* is a closed G-invariant subset of $(X \times Y^*, G)$. Moreover, (X^*, G) is a subsystem of $(X \times Y^*, G)$. Let $\tau' : X^* \to X$ and $\phi : X^* \to Y^*$ be the naturally coordinate projection map. The following is a classical result (see, for example, [**1**, Lemma 14.41]).

LEMMA 4.5. *(X^*, G) be a minimal G-system, and in the following commutative diagram of homomorphisms of minimal G-systems*

$$\begin{array}{ccc} (X, G) & \xleftarrow{\tau'} & (X^*, G) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\phi} \\ (Y, G) & \xleftarrow{\tau} & (Y^*, G) \end{array}$$

ϕ is an open extension and τ, τ' are almost one-to-one extensions.

PROPOSITION 4.6. *Let $\pi : (X, G) \to (Y, G)$ be an extension of minimal G-systems. If π is proximal and $SS(X, G) = \emptyset$, then π is almost one-to-one.*

Proof. Let Y^*, X^* and ϕ, τ, τ' be defined as above. By Lemma 4.5, one knows that ϕ is an open extension and τ, τ' are almost one-to-one extensions.

For any $y^* \in Y^*$ and $x, x' \in y^*$, since $\pi(x) = \pi(x') = \tau(y^*)$ and π is proximal, (x, x') is proximal. Moreover, $((x, y^*), (x', y^*))$ is proximal. This shows that ϕ is a proximal open extension. Now, using Lemma 4.4, one knows that (R_ϕ, G) is transitive.

If π is not an almost one-to-one extension, then ϕ is not an isomorphism as τ, τ' are almost one-to-one extensions. Then $R_\phi \setminus \Delta_{X^*} \neq \emptyset$. Let (x_1^*, x_2^*) be a transitive point of (R_ϕ, G); then $(x_1^*, x_2^*) \in WM(X^*, G)$. Let $\phi_1 : R_\phi \to X^*$ be the projection map onto the first coordinate.

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Now for any open subsets U, V of X^*, $\phi_1((U \times V) \cap R_\phi) = \phi^{-1}(\phi(U) \cap \phi(V)) \cap U$ is an open subset of X^* as ϕ is an open extension. This shows ϕ_1 is an open extension. Hence $(x_1^*, x_2^*) \in SS(X^*, G)$ by Lemma 3.5. Let $x_i^* = (x_i, \phi(x_i^*)), i = 1, 2$. Since $(x_1^*, x_2^*) \in R_\phi \setminus \Delta_{X^*}$, $\phi(x_1^*) = \phi(x_2^*)$ and $x_1 \neq x_2$. It is easy to see that $(x_1, x_2) \in SS(X, G)$ as $(x_1^*, x_2^*) \in SS(X^*, G)$ and $\tau'(x_i^*) = x_i$, a contradiction. Hence π is an almost one-to-one extension. □

Now we are ready to show the following.

THEOREM 4.7. *A minimal tame G-system is an almost one-to-one extension of its maximal equicontinuous factor, that is almost automorphic.*

Proof. Let $\pi : (X, G) \to (Y, G)$ be the extension between a minimal tame system (X, G) and its maximal equicontinuous factor (Y, G). Since (X, G) is tame, $SS(X, G) = \emptyset$ by Theorem 3.2. Now, by Proposition 4.6, it remains to show that π is proximal.

Recall that for a minimal G-system the regionally proximal relation is an equivalence relation and it induces the maximal equicontinuous factor (see for example [**1**]). Let $R_\pi = \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$; then $R_\pi = RP(X, G) = WM(X, G) \cup \Delta_X$ by Proposition 3.4. If π is not proximal, then there exists a distal pair $(x_1, x_2) \in R_\pi \setminus \Delta_X$.

Let $A = \overline{\{(\Pi_g(x_1), \Pi_g(x_2)) : g \in G\}}$; then $A \subset WM(X, G)$ and $A \cap \Delta_X = \emptyset$ as (x_1, x_2) is a distal pair. Since A is closed G-invariant, there exists a minimal point (x_1', x_2') in A. Since $(x_1', x_2') \in WM(X, G)$, one has $(x_1', x_2') \in SS(X, G)$ by Proposition 3.7. This contradicts $SS(X, G) = \emptyset$. □

Let (X, G) be a G-system and $\mathcal{M}(X)$ be the set of all Borel probability measures. For $\mu \in \mathcal{M}(X)$, we say μ is *G-invariant* if $g\mu = \mu$ for each $g \in G$, where $g\mu(A) = \mu(g^{-1}A)$ for any Borel subset A of X. Denote the set of all G-invariant Borel probability measures by $\mathcal{M}(X, G)$. Since G is Abelian, $\mathcal{M}(X, G) \neq \emptyset$. In the weak*-topology, $\mathcal{M}(X, G)$ is a compact convex metric subspace of the local convex space $C(X)^*$. For $\mu \in \mathcal{M}(X, G)$, if for any Borel set A with $g^{-1}(A) = A (\mathrm{mod}\, \mu)$, one has $\mu(A) = 0$ or $\mu(X \setminus A) = 0$, then we say μ is *ergodic*. Denote the set of all G-invariant ergodic Borel probability measures by $\mathcal{M}^e(X, G)$. Similar to [**32**, Theorem 6.10 (3)], we easily prove that a measure $\mu \in \mathcal{M}(X, G)$ is an extreme of $\mathcal{M}(X, G)$ if and only if μ is ergodic. Hence $\mathcal{M}^e(X, G)$ is a G_δ subset of $\mathcal{M}(X, G)$ and for $\mu \in \mathcal{M}(X, G)$ there exists a unique Borel measure m on $\mathcal{M}^e(X, G)$ such that $\mu = \int_{\mathcal{M}^e(X,G)} \theta \, dm(\theta)$(see for example, [**24**]); it is called the *ergodic decomposition* of μ. We say (X, G) is *uniquely ergodic* if $\mathcal{M}(X, G)$ has one element.

Now we show that a minimal tame G-system is uniquely ergodic.

THEOREM 4.8. *Let (X, G) be a minimal tame G-system. Then (X, G) is uniquely ergodic.*

Proof. Let (Y, G) be the maximal equicontinuous factor of (X, G) and $\pi : (X, G) \to (Y, G)$ be the extension (see Theorem 4.7) and ν be the unique ergodic measure on (Y, G).

Assume μ is a G-invariant measure on X and consider the disintegration of μ over ν. That is, for almost every $y \in Y$ we have a measure μ_y on X such that $\mathrm{supp}(\mu_y) \subset \pi^{-1}(y)$ (see [**26**] or [**7**]) and $\mu = \int_Y \mu_y \, d\nu(y)$. We claim that for almost every $y \in Y$, μ_y is concentrated on one point.

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Proof of the claim. Let $R_\pi = \{(x, y) \in X \times X : \pi(x) = \pi(y)\}$. So $\mathrm{supp}(\mu_y \times \mu_y) \subset \pi^{-1}(y) \times \pi^{-1}(y) \subset R_\pi$, for ν-almost every $y \in Y$. Now we define a measure $\mu \times_Y \mu$ on Borel sets of $X \times X$ such that if A is a Borel set of $X \times X$, then

$$\mu \times_Y \mu(A) = \int_Y \mu_y \times \mu_y(A)\, d\nu(y).$$

So $\mu \times_Y \mu(R_\pi) = 1$. In the following we show

$$\mu \times_Y \mu(\Delta_X) = 1. \tag{4.1}$$

If (4.1) is not true, then $\mu \times_Y \mu(R_\pi \setminus \Delta_X) > 0$. Consider the ergodic decomposition of $\mu \times_Y \mu$. It is not hard to see that there is an ergodic measure θ on (R_π, G) such that $\theta(R_\pi \setminus \Delta_X) > 0$. As each point in R_π is a proximal pair in X and $\theta(R_\pi \setminus \Delta_X) > 0$, $(\mathrm{supp}(\theta), G)$ is a transitive G-system with Δ_X the only minimal set and $\mathrm{supp}(\theta) \setminus \Delta_X \neq \emptyset$.

Collapsing the G-invariant closed subset Δ_X of $\mathrm{supp}(\theta)$ to a point p, we get a new non-trivial transitive G-system (Z, G) having a fixed point p. Let $\widetilde{\theta}$ be the image of θ under the collapsing. Clearly, $\mathrm{supp}(\widetilde{\theta}) = Z$. Fix a transitive point $z_1 \in Z$. As (X, G) is tame, (Z, G) is also tame by Proposition 2.4. Hence $SS(Z, G) = \emptyset$ and $(z_1, p) \notin WM(Z, G)$ by Theorem 3.2 and Proposition 3.11. Therefore, we can find an open neighborhood U_1 of z_1 and an open neighborhood U_2 of p such that $N(U_1, U_1) \cap N(U_1, U_2) = \emptyset$. Put $N(z_1, U_2) = \{g \in G : gz_1 \in U_2\}$; then we have

$$N(U_1, U_1) \cap N(z_1, U_2) = \emptyset. \tag{4.2}$$

Since z_1 is a transitive point of Z and p is a fixed point, $N(z_1, U_2)$ is thick, that is, for any finite subset K of G there is $g \in G$ such that $Kg \subseteq N(z_1, U_2)$. Now we are to find inductively $\{g_n\} \subset G$ such that $g_j g_i^{-1} \in N(z_1, U_2)$ for each $1 \leq i < j$. First, we take $g_1 \in N(z_1, U_2)$. Let $K_1 = \{g_1^{-1}\}$; then there exists $g_2 \in G$ such that $K_1 g_2 \subseteq N(z_1, U_2)$, that is, $g_2 g_1^{-1} \in N(z_1, U_2)$. Assume that for $k = m$ one has find $g_1, g_2, \ldots, g_m$ such that $g_j g_i^{-1} \in N(z_1, U_2)$ for each $1 \leq i < j \leq m$. For $k = m + 1$, let $K_m = \{g_1^{-1}, \ldots, g_m^{-1}\}$; then there exists $g_{m+1} \in G$ such that $K_m g_{m+1} \subseteq N(z_1, U_2)$, that is, $g_{m+1} g_i^{-1} \in N(z_1, U_2)$ for $1 \leq i \leq m$. This ends the induction.

Since $g_j g_i^{-1} \in N(z_1, U_2)$ for each $1 \leq i < j$, using (4.2) one has $g_j g_i^{-1} \notin N(U_1, U_1)$, that is, $g_i(U_1) \cap g_j(U_1) = \emptyset$ for each $1 \leq i < j$. Note that $\widetilde{\theta}(g_i(U_1)) = \widetilde{\theta}(U_1) > 0$ for each $i \in \mathbb{N}$, one has $\widetilde{\theta}\big(\bigcup_{i\in\mathbb{N}} g_i(U_1)\big) = \sum_{i\in\mathbb{N}} \widetilde{\theta}(g_i(U_1)) = +\infty$, a contradiction. Hence (4.1) is true, that is $\mu \times_Y \mu(\Delta_X) = 1$. Since

$$\int_Y \mu_y \times \mu_y(\Delta_X)\, d\nu(y) = \mu \times_Y \mu(\Delta_X) = 1,$$

we have $\mu_y \times \mu_y(\Delta_X) = 1$ for ν-almost every $y \in Y$. A simple calculation shows that μ_y is concentrated on one point, say $c(y)$ for ν-almost every $y \in Y$. This finishes the proof of the claim. □

We now show that (X, G) is uniquely ergodic. Assume that μ^1 and μ^2 are two G-invariant measures on X. Let $\mu = \frac{1}{2}(\mu^1 + \mu^2)$. By the claim, μ_y^i is concentrated on $c^i(y)$ for ν-almost every $y \in Y$. Thus μ_y is concentrated on $c^1(y)$ and $c^2(y)$ for ν-almost every $y \in Y$. By the claim we have $c^1(y) = c^2(y)$ for ν-almost every $y \in Y$. Thus $\mu^1 = \mu^2$. This proves that (X, G) is uniquely ergodic. □

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LEMMA 4.9. *Let* (X, G) *be a tame* G*-system and* $\mu \in \mathcal{M}^{\mathrm{e}}(X, G)$. *Then* $(\mathrm{supp}(\mu), G)$ *is almost automorphic.*

Proof. First, $(\mathrm{supp}(\mu), G)$ is transitive as μ is ergodic. If $(\mathrm{supp}(\mu), G)$ is not minimal, then there exists a G-invariant closed A of $(\mathrm{supp}(\mu), G)$ such that $A \neq \mathrm{supp}(\mu)$. Collapsing A to a point p, we get a new G-system (Z, G) such that p is a fixed point. Let $\widetilde{\mu}$ be the image of μ under the collapsing. Clearly $\mathrm{supp}(\widetilde{\mu}) = Z$.

Fix a transitive point $z_1 \in Z$. By Proposition 2.4, one knows that (Z, G) is tame. Hence $SS(Z, G) = \emptyset$ and $(z_1, p) \notin WM(Z, G)$ by Theorem 3.2 and Proposition 3.11. Therefore, we can find an open neighborhood U_1 of z_1 and an open neighborhood U_2 of p such that $N(U_1, U_1) \cap N(U_1, U_2) = \emptyset$. Similar to the proof of the claim of Theorem 4.8, we can find inductively $\{g_n\} \subset G$ such that $g_j g_i^{-1} \in N(z, U_2)$ for each $1 \leq i < j$.

Since $g_j g_i^{-1} \in N(z_1, U_2)$ for each $1 \leq i < j$, one has $g_j g_i^{-1} \notin N(U_1, U_1)$, that is, $g_i(U_1) \cap g_j(U_1) = \emptyset$ for each $1 \leq i < j$. Noting that $\widetilde{\mu}(g_i(U_1)) = \widetilde{\mu}(U_1) > 0$ for each $i \in \mathbb{N}$, one has $\widetilde{\mu}\big(\bigcup_{i\in\mathbb{N}} g_i(U_1)\big) = \sum_{i\in\mathbb{N}} \widetilde{\mu}(g_i(U_1)) = +\infty$, a contradiction. Hence $(\mathrm{supp}(\mu), G)$ is minimal. Now, by Theorem 4.7, one has $(\mathrm{supp}(\mu), G)$ is almost automorphic. □

THEOREM 4.10. *Let* (X, G) *be a tame* G*-system. If* (X, G) *is transitive and there exists* $\mu \in \mathcal{M}(X, G)$ *with* $\mathrm{supp}(\mu) = X$, *then* (X, G) *is minimal. Hence it is almost automorphic and uniquely ergodic.*

Proof. It is completely similar to the proof of Lemma 4.9. □

5. *Tame and null system for* $\mathbb{Z}$*-actions*

In this section, we will prove that for $\mathbb{Z}$-actions a tame system must be a measure-null system for any invariant measure. At the same time, we will give examples to show that tameness is strictly weaker than nullness for $\mathbb{Z}$-actions.

By a *cascade* we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism from X to X. The set of T-invariant probability measures defined on Borel sets of X, $\mathcal{B}(X)$, is denoted by $\mathcal{M}(X, T)$. In this context measurability will be always related to $\mathcal{B}(X)$. A probability measure $\mu \in \mathcal{M}(X, T)$ induces a *measure-theoretical dynamical system* (m.t.d.s.) $(X, \mathcal{B}(X), \mu, T)$ (or just (X, μ, T)), that is, $T : X \to X$ is a $\mathcal{B}(X) \mapsto \mathcal{B}(X)$ measurable bijection and $T\mu = \mu$.

Let $\mathcal{A} = \{0 \leq t_1 < t_2 < \cdots\} \subseteq \mathbb{N}$ be an increasing sequence of natural numbers and $\mathcal{U}$ be a finite cover of X. The *topological sequence entropy* of $\mathcal{U}$ with respect to (X, T) along $\mathcal{A}$ is defined by

$$h_{\mathrm{top}}^{\mathcal{A}}(T, \mathcal{U}) = \limsup_{n\to+\infty} \frac{1}{n} \log N\Big(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}\Big),$$

where $N\big(\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}\big)$ is the minimal cardinality among all cardinalities of sub-covers of $\bigvee_{i=1}^{n} T^{-t_i}\mathcal{U}$. The *topological sequence entropy* of (X, T) along $\mathcal{A}$ is

$$h_{\mathrm{top}}^{\mathcal{A}}(T) = \sup_{\mathcal{U}} h_{\mathrm{top}}^{\mathcal{A}}(T, \mathcal{U}),$$

where supremum is taken over all finite open covers of X (that is, made of open sets). If $\mathcal{A} = \mathbb{N}$ we recover standard topological entropy. In this case we omit the superscript $\mathbb{N}$.

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Analogously, given $\mu \in \mathcal{M}(X,T)$ and α a finite measurable partition of X we define the sequence entropy of α with respect to (X,μ,T) along $\mathcal{A}$ by

$$h_\mu^{\mathcal{A}}(T,\alpha) = \limsup_{n\to+\infty} -\frac{1}{n} \sum_{A\in\bigvee_{i=1}^n T^{-t_i}\alpha} \mu(A)\log\mu(A).$$

The *sequence entropy* of (X,T,μ) along $\mathcal{A}$ is

$$h_\mu^{\mathcal{A}}(T) = \sup_\alpha h_\mu^{\mathcal{A}}(T,\alpha),$$

where supremum is taken over all finite measurable partitions. As in the topological case, when $\mathcal{A} = \mathbb{N}$ we recover entropy of T with respect to μ. In this case we omit the superscript $\mathbb{N}$. For the classical theory of measure-theoretical entropy see [**9, 23**], and classical theory of topological entropy can be found in [**3, 9**]. For sequence entropy see [**15, 22**].

Let (X,T) be a cascade and $\mu \in \mathcal{M}(X,T)$. An *eigenfunction* for T is some non-zero function $f \in L^2(X,\mathcal{B}_X,\mu) = L^2(\mu)$ such that $Uf = f\circ T = \lambda f$ for some $\lambda \in \mathbb{C}$. λ is called the *eigenvalue corresponding to* f. It is easy to see every eigenvalue has norm one, that is $|\lambda| = 1$. If $f \in L^2(\mu)$ is an eigenfunction, then $\mathrm{cl}\{U^n f : n \in \mathbb{Z}\}$ is a compact subset of $L^2(\mu)$. Generally, we say f is *almost periodic* if $\mathrm{cl}\{U^n f : n \in \mathbb{Z}\}$ is compact in $L^2(\mu)$. It is well known that the set of all bounded almost periodic functions forms a U-invariant and conjugated-invariant algebra of $L^2(\mu)$ (denoted by A_c). The set of all almost periodic functions is just the closure of A_c (denoted by H_c), and is also spanned by the set of eigenfunctions. T is said to be *compact* or to have *discrete spectrum* if $L^2(\mu)$ can be spanned by the set of eigenfunctions, that is $H_c = L^2(X,\mathcal{B},\mu)$. The following theorem is a classic result of Kushnirenko [**22**].

THEOREM 5.1. *Let (X,T) be a cascade and $\mu \in \mathcal{M}(X,T)$. Then $(X,\mathcal{B}_X,\mu,T)$ has discrete spectrum if and only if $h_\mu^{\mathcal{A}}(T) = 0$ for any increasing sequence $\mathcal{A}$ of natural numbers.*

Let (X,T) be a cascade. We say (X,T) is *M-null* if for any $\mu \in \mathcal{M}(X,T)$ and any increasing sequence $\mathcal{A}$ of natural numbers, $h_\mu^{\mathcal{A}}(T) = 0$ (see [**17**]).

THEOREM 5.2. *Let (X,T) be a tame cascade. Then $(X,\mathcal{B}_X,\mu,T)$ has discrete spectrum for each $\mu \in \mathcal{M}(X,T)$. Moreover, (X,T) is M-null.*

Proof. Let $\mu \in \mathcal{M}(X,T)$. To show that $(X,\mathcal{B}_X,\mu,T)$ has discrete spectrum, it is sufficient to show that for any $f \in L^\infty(X,\mathcal{B}_X,\mu)$, $\mathrm{cl}\{U^n f : n \in \mathbb{Z}\}$ is compact in $(L^2(\mu), \|\cdot\|_{L^2(\mu)})$, where $\|g\|_{L^2(\mu)} = (\int_X |g|^2\,d\mu)^{1/2}$ for $g \in L^2(\mu)$.

Let $f \in L^\infty(X,\mathcal{B}_X,\mu)$. For any fixed sequence $\{h_i\}_{i\in\mathbb{N}} \subset \mathrm{cl}\{U^n f : n \in \mathbb{Z}\}$, take $n_i \in \mathbb{Z}$ such that $\|U^{n_i} f - h_i\|_{L^2(\mu)} < 1/2^i$ for each $i \in \mathbb{N}$. Since (X,T) is tame, there exist $p \in E(X,T)$ and subsequence $i_1 < i_2 < \cdots$ of natural numbers such that $\lim_{k\to+\infty} T^{n_{i_k}}(x) = p(x)$ for each $x \in X$. Clearly, p is a Borel map from X to X. Let $g = f\circ p$. Then, $g \in L^\infty(X,\mathcal{B}_X,\mu)$. Since $\lim_{k\to+\infty} f(T^{n_{i_k}}x) = g(x)$ for μ-almost every x and $f, g \in L^\infty(X,\mathcal{B}_X,\mu)$, it is not hard to see that $\lim_{k\to+\infty} U^{n_{i_k}} f = g$

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in $(L^2(\mu), \|\cdot\|_{L^2(\mu)})$ and $g \in \mathrm{cl}\{U^n f : n \in \mathbb{Z}\}$. Moreover,

$$\lim_{k\to+\infty} \|h_{i_k} - g\|_{L^2(\mu)} \le \lim_{k\to+\infty} (\|h_{i_k} - U^{n_{i_k}} f\|_{L^2(\mu)} + \|U^{n_{i_k}} f - g\|_{L^2(\mu)})$$
$$\le \lim_{k\to+\infty} \frac{1}{2^{i_k}} + \lim_{k\to+\infty} \|U^{n_{i_k}} f - g\|_{L^2(\mu)} = 0.$$

That is, $\lim_{k\to+\infty} h_{i_k} = g$ in $(L^2(\mu), \|\cdot\|_{L^2(\mu)})$. This implies that the complete metric space $(\mathrm{cl}\{U^n f : n \in \mathbb{N}\}, \|\cdot\|_{L^2(\mu)})$ is sequential compact. Hence $(\mathrm{cl}\{U^n f : n \in \mathbb{N}\}, \|\cdot\|_{L^2(\mu)})$ is compact since, for a metric space, sequential compactness is equivalent to compactness. □

A cascade (X, T) is *null* if $h^{\mathcal{A}}_{\mathrm{top}}(T) = 0$ for any sequence $\mathcal{A} \subseteq \mathbb{N}$. Kerr and Li in [**20**] proved that a null cascade is tame. In [**15**] Goodman obtained a minimal non-equicontinuous cascade which is null. In the following, we continue the discussion concerning tameness and nullness for a cascade. Finally, we will construct a tame cascade which is not null.

Definition 5.3. Let (X, T) be a cascade. A pair $(x_1, x_2) \in X \times X$ is called weakly scrambled if $x_1 \neq x_2$ and there exists a closed neighborhood U_i of x_i, $i = 1, 2$ with $U_1 \cap U_2 = \emptyset$ such that for any $k \in \mathbb{N}$ there exist $\{n_1^k < n_2^k < \cdots < n_k^k\} \subseteq \mathbb{N}$ satisfying $\bigcap_{i=1}^k T^{-n_i^k} U_{s(i)} \neq \emptyset$ for any $s = (s(1), s(2), \ldots, s(k)) \in \{1, 2\}^k$.

Denote by $WS(X, T)$ the set of all weakly scrambled pairs of (X, T). Clearly, $SS(X, T) \subseteq S(X, T) \subseteq WS(X, T)$.

The following combinatorial lemma appeared in [**27**] (see also [**33**, Proposition 8.2]).

LEMMA 5.4. *For every $h > 0$, there exist $b(h) > 0$ and $M_h \in \mathbb{N}$ such that if $m \ge M_h$ and $B \subset \{1, 2\}^m$ satisfies $|B| \ge 2^{mh}$, then we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \ge b(h)m$ and $B|_{J_m} = \{1, 2\}^{J_m}$, that is, for any $s \in \{1, 2\}^{J_m}$ there is $b \in B$ with $b(j) = s(j)$ for $j \in J_m$.*

PROPOSITION 5.5. *Let (X, T) be a cascade. If (X, T) is null, then $WS(X, T) = \emptyset$. Moreover, if in addition (X, T) has zero dimension, then (X, T) is null if and only if $WS(X, T) = \emptyset$.*

Proof. Let (X, T) be null. If $WS(X, T) \neq \emptyset$, then there exist two closed subsets U_1, U_2 of X with $U_1 \cap U_2 = \emptyset$ such that for any $k \in \mathbb{N}$ there exist $\{n_1^k < n_2^k < \cdots < n_k^k\} \subseteq \mathbb{N}$ satisfying $\bigcap_{i=1}^k T^{-n_i^k} U_{s(i)} \neq \emptyset$ for any $s = (s(1), s(2), \ldots, s(k)) \in \{1, 2\}^k$. Without loss of generality, we may require $n_k^k < n_1^{k+1}$ for each $k \in \mathbb{N}$.

Let $C_k = \{n_1^k < n_2^k < \cdots < n_k^k\}$ for $k \in \mathbb{N}$ and $m_j = (j + 1)! - j!$ for $j = 1, 2, \ldots$. Let $\mathcal{A} = \bigcup_{j=1}^\infty C_{m_j}$ and $A_\ell = \bigcup_{j=1}^\ell C_{m_j}$ for $\ell \in \mathbb{N}$. Let $\mathcal{U} = \{U_1^c, U_2^c\}$. Then $\mathcal{U}$ is a finite open cover of X and

$$h^{\mathcal{A}}_{\mathrm{top}}(T, \mathcal{U}) \ge \liminf_{\ell\to\infty} \frac{\log N\big(\bigvee_{i\in A_\ell} T^{-i}\mathcal{U}\big)}{\sum_{j=1}^\ell m_j} \ge \liminf_{\ell\to\infty} \frac{\log N\big(\bigvee_{j\in C_{m_\ell}} T^{-j}\mathcal{U}\big)}{\sum_{i=1}^\ell m_i}$$
$$= \liminf_{\ell\to\infty} \frac{\log 2^{m_\ell}}{\sum_{i=1}^\ell m_i} = \lim_{\ell\to\infty} \frac{m_\ell}{\sum_{i=1}^\ell m_i} \log 2 = \log 2.$$

This implies that (X, T) is not null, a contradiction.

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Now we assume that (X, T) is zero-dimensional. Let $WS(X, T) = \emptyset$. If (X, T) is not null, then there exist two clopen (closed and open) subsets U_1, U_2 of X and an increasing sequence $\mathcal{A} = \{t_1 < t_2 < \cdots\}$ of natural numbers such that $U_1 \cap U_2 = \emptyset$, $U_1 \cup U_2 = X$ and $h^{\mathcal{A}}_{\text{top}}(T, \{U_1, U_2\}) > 0$. Let $h := (h^{\mathcal{A}}_{\text{top}}(T, \{U_1, U_2\}))/2$. By Lemma 5.4, there exist $b(h) > 0$ and $M_h \in \mathbb{N}$ such that if $m \geq M_h$ and $B \subset \{1, 2\}^m$ satisfies $|B| \geq 2^{mh}$, then we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \geq b(h)m$ and $B|_{J_m} = \{1, 2\}^{J_m}$, that is, for any $s \in \{1, 2\}^{J_m}$ there is $b \in B$ with $b(j) = s(j)$ for $j \in J_m$.

For each $k \in \mathbb{N}$ take $m \geq \max\{k/b(h), M_h\}$ with $N\big(\bigvee_{i=1}^m T^{-t_i}\{U_1, U_2\}\big) > 2^{mh}$. Let $B = \{s \in \{1, 2\}^m : \bigcap_{i=1}^m T^{-t_i} U_{s(i)} \neq \emptyset\}$. Then $|B| = N\big(\bigvee_{i=1}^m T^{-t_i}\{U_1, U_2\}\big) > 2^{mh}$. Hence we can find $J_m \subset \{1, 2, \ldots, m\}$ with $|J_m| \geq b(h)m \geq k$ and $B|_{J_m} = \{1, 2\}^{J_m}$, that is, for any $s \in \{1, 2\}^{J_m}$, $\bigcap_{i \in J_m} T^{-t_i} U_{s(i)} \neq \emptyset$. Let $J_m = \{j^k_1 < \cdots < j^k_k < j^k_{k+1} < \cdots < j^k_{|J_m|}\}$ and $n^k_i = t_{j^k_i}$ for $1 \leq i \leq |J_m|$. Then $\{n^k_1 < n^k_2 < \cdots < n^k_k\} \subseteq \mathbb{N}$ satisfies $\bigcap_{i=1}^k T^{-n^k_i} U_{s(i)} \neq \emptyset$ for any $s = (s(1), s(2), \ldots, s(k)) \in \{1, 2\}^k$. Since k is arbitrary, this shows that $\emptyset \neq U_1 \times U_2 \subseteq WS(X, T)$, a contradiction. □

Let S be a subset of $\mathbb{Z}$. The *upper Banach density* of S is

$$BD^*(S) = \limsup_{|I| \to +\infty} \frac{|S \cap I|}{|I|},$$

where I ranges over intervals of $\mathbb{Z}$. We say the *lower Banach density* of S is one if for each $a < 1$ there is N such that for any subinterval I of $\mathbb{Z}$ with $|I| \geq N$ we have $|S \cap I| \geq a|I|$. Let (X, T) be a cascade. We shall say that (X, T) is *Banach-transitive* if for any non-empty open subsets U, V of X, $N(U, V)$ has positive upper Banach density. For $x \in X$ and $U \subset X$ let $N(x, U) = \{n \in \mathbb{Z} : T^n(x) \in U\}$. A cascade is an *E-system* if it is transitive and there is an invariant measure with full support. It is known [**13**] that if (X, T) is an E-system, then $N(U, V)$ is syndetic, i.e. having bounded gap, for any non-empty open subsets U, V of X, and thus is Banach-transitive. Note that an infinite subset A of $\mathbb{Z}$ is *syndetic* if it has bounded gaps, and it is *thick* if it contains arbitrary long intervals of integers. It is easy to see that the intersection of a syndetic set and a thick set is not empty. Now we can prove the following.

THEOREM 5.6. *Let (X, T) be a cascade. If (X, T) is non-minimal and Banach-transitive, then (X, T) is not tame.*

Proof. Without loss of generality, we assume that (X, T) has a fixed point p (using Proposition 2.4). Let x_1 be a transitive point and U_1 and U_2 be open neighborhoods of x_1 and p respectively. There are two cases.

Case 1. (X, T) is an E-system. As (X, T) is an E-system, $N(U_1, U_1)$ is syndetic [**13**]. As p is a fixed point, $N(x_1, U_2)$ is thick. Thus $N(U_1, U_2) = N(x_1, U_2) - N(x_1, U_1)$ is a thick set. Combining the above facts, one has $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$. Hence $(x_1, p) \in WM(X, T)$. By Proposition 3.11, we know that $(x_1, p) \in SS(X, T)$. So (X, T) is not tame by Theorem 3.2.

Case 2. (X, T) is not an E-system. Without loss of generality, we assume that (X, T) is uniquely ergodic and the unique ergodic measure is δ_p (collapsing the closure of the union of the supports of all invariant measures and using Proposition 2.4). We have the following.

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CLAIM. *The lower Banach density of* $N(x_1, U_2)$ *is* 1.

Proof of claim. If this is not the case, then $N(x_1, U_2^c)$ has positive upper Banach density. Thus, there exist $k \in \mathbb{N}$ and $a_k < b_k$ such that

$$\lim_{k\to+\infty} \frac{1}{b_k - a_k} |N(x_1, U_2^c) \cap \{a_k, a_k+1, \ldots, b_k-1\}| > 0 \quad \text{and} \quad \lim_{k\to+\infty} (b_k - a_k) = +\infty.$$

Now, set

$$\mu_k = \frac{1}{b_k - a_k} \sum_{i=a_k}^{b_k-1} \delta_{T^i x_1}.$$

Let $\mu = \lim_{i\to+\infty} \mu_{k_i}$ be a limit point of $\{\mu_k\}_{k=1}^{+\infty}$ in the weak*-topology. Clearly, μ is an invariant measure of (X, T) and

$$\begin{aligned}\mu(U_2^c) &\geq \lim_{i\to+\infty} \mu_{k_i}(U_2^c) = \lim_{i\to+\infty} \frac{1}{b_{k_i} - a_{k_i}} \sum_{i=a_{k_i}}^{b_{k_i}-1} \delta_{T^i x_1}(U_2^c) \\ &= \lim_{i\to+\infty} \frac{|N(x_1, U_2^c) \cap \{a_{k_i}, a_{k_i}+1, \ldots, b_{k_i}-1\}|}{b_{k_i} - a_{k_i}} > 0.\end{aligned}$$

Thus $\mu \neq \delta_p$, which contradicts the unique ergodicity of (X, T). This finishes the proof of the claim. □

As (X, T) is Banach-transitive, $N(U_1, U_1)$ has positive upper Banach density. By the claim, the lower Banach density of $N(x_1, U_2)$ is 1. This implies that the lower Banach density of $N(U_1, U_2) = N(x_1, U_2) - N(x_1, U_1)$ is 1. Thus, $N(U_1, U_1) \cap N(U_1, U_2) \neq \emptyset$. Hence $(x_1, p) \in WM(X, T)$. By Proposition 3.11, we know that $(x_1, p) \in SS(X, T)$. So (X, T) is not tame, by Theorem 3.2. □

A cascade (X, T) is *topologically ergodic* if for each pair of non-empty open sets U and V, $N(U, V)$ is syndetic. A cascade is *extremely scattering* if its product with any topologically ergodic system is transitive. It is proved in [**18**] that a weakly mixing cascade is extremely scattering and an extremely scattering cascade is Banach-transitive. In [**19**], the author gives an example to show that for a cascade, extreme scattering is strictly weaker than weak mixing. Thus we have the following.

COROLLARY 5.7. *An extremely scattering tame cascade is trivial.*

Proof. Let (X, T) be an extremely scattering tame system. By Theorem 5.6, (X, T) is minimal. Since a minimal cascade is topologically ergodic, (X, T) is weakly mixing. So (X, T) is trivial by Theorem 3.12. □

LEMMA 5.8. *Let* $\pi : (X, T) \to (Y, S)$ *be a homomorphism between cascades,* $X_0 = \{x \in X : \#\pi^{-1}(\pi(x)) = 1\}$ *and* (Y, S) *be an equicontinuous system. If* $X \setminus X_0$ *is a countable set, then* (X, T) *is tame.*

Proof. By Theorem 2.1, it remains to show that the cardinality $\mathrm{card}(E(X, T)) \leq 2^{\aleph_0}$. Since (Y, S) is equicontinuous, $\mathrm{card}(E(Y, S)) \leq 2^{\aleph_0}$. For $p \in E(X, T)$, let $\pi_*(p) \in E(Y, S)$ with $\pi_*(p)(y) = \pi(p(x))$ for each $y = \pi(x)$. Then $\pi_* : E(X, T) \to E(Y, S)$ is surjective.

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Fix $q \in E(Y, S)$. Since (Y, S) is equicontinuous, q is a homeomorphism from Y to Y. Let $Y_0 = \{y \in Y : \#\pi^{-1}(y) = 1\}$. Then $\pi^{-1}(Y_0) = X_0$, so $Y \setminus Y_0$ is a countable set as $X \setminus X_0$ is countable. Let $X_q = \pi^{-1}(q^{-1}(Y \setminus Y_0))$. Then X_q is a countable set. For $x \in X \setminus X_q$, as $q(\pi(x)) \in Y_0$, $\#\pi^{-1}(q(\pi(x))) = 1$. Thus there exists a unique $x_q \in X$ with $\{x_q\} = \pi^{-1}(q(\pi(x)))$. Now for any $p \in \pi_*^{-1}(q)$, it is clear that $p(x) = x_q$ for $x \in X \setminus X_q$. Note that

- $p|_{X_q} \in X^{X_q}$ for $p \in \pi_*^{-1}(q)$ and
- for any $p, p' \in \pi_*^{-1}(q)$, $p \neq p'$ if and only if $p|_{X_q} \neq p'|_{X_q}$.

We have that the map $\phi_q : \pi_*^{-1}(q) \ni p \mapsto p|_{X_q} \in X^{X_q}$ is one-to-one. Since X_q is countable, there exists a one-to-one map $\varphi_q : X^{X_q} \to X^{\mathbb{N}}$. Moreover, the composition map $\varphi_q \circ \phi_q : \pi_*^{-1}(q) \to X^{\mathbb{N}}$ is also one-to-one.

Finally, we consider the map

$$\Phi : E(X, T) \to E(Y, S) \times X^{\mathbb{N}} \quad \text{with } \Phi(p) = (\pi_*(p), \varphi_{\pi_*(p)} \circ \phi_{\pi_*(p)}(p)).$$

Clearly, Φ is a one-to-one map. Hence $\operatorname{card}(E(X, T)) \leq \operatorname{card}(E(Y, S) \times X^{\mathbb{N}}) \leq 2^{\aleph_0}$ since $\operatorname{card}(X) \leq 2^{\aleph_0}$ and $\operatorname{card}(E(Y, S)) \leq 2^{\aleph_0}$. This finishes the proof. □

Example 5.9. Let α be an irrational number in $[0, 1]$. Let $Y = S^1$ denote the unit circle in the complex plane and S the rotation $S(z) = e^{2\pi i\alpha} z$ for $z \in Y$. Then (Y, S) is a minimal equicontinuous cascade.

Define $A_0 = \{e^{2\pi i\theta} : 0 \leq \theta \leq 1 - \alpha\}$, $A_1 = \{e^{2\pi i\theta} : 1 - \alpha \leq \theta \leq 1\}$ and

$$X = \left\{x \in \{0, 1\}^{\mathbb{Z}} : \bigcap_{n=-\infty}^{+\infty} S^{-n} A_{x_n} \neq \emptyset\right\}.$$

Then X is a σ-invariant closed subset of $\{0, 1\}^{\mathbb{Z}}$, where $(\sigma x)_n = x_{n+1}$ for $x \in \{0, 1\}^{\mathbb{Z}}$. So (X, σ) is a cascade and it is not hard to check that (X, σ) is minimal but not equicontinuous.

Define a homomorphism π from (X, σ) to (Y, S) by putting $\pi(x) \in \bigcap_{n=-\infty}^{+\infty} S^{-i} A_{x_n}$. Then $\#(\pi^{-1}(y)) = 1$ for $y \in Y \setminus \{S^i(1) : i \in \mathbb{Z}\}$ and $\#(\pi^{-1}(y)) = 2$ for $y \in \{S^i(1) : i \in \mathbb{Z}\}$. Hence $X \setminus X_0$ is a countable set, where $X_0 = \{x \in X : \#(\pi^{-1}(\pi(x))) = 1\}$. By Lemma 5.8, (X, σ) is tame. Hence (X, σ) is minimal tame but not equicontinuous. It is needed to mention that Goodman has shown that (X, σ) is null in [**15**, Example 6.5].

Let $\pi : (X, T) \to (Y, S)$ be a homomorphism between cascades. The extension π is called *asymptotic* if $\pi(x_1) = \pi(x_2)$ implies that $\lim_{|n| \to +\infty} d(T^n x_1, T^n x_2) = 0$.

LEMMA 5.10. *Let $\pi : (X, T) \to (Y, S)$ be an asymptotic homomorphism between cascades, $Y_0 = \{y \in X : \#\pi^{-1}(y) = 1\}$ and (Y, S) be an equicontinuous system. If $Y \setminus Y_0$ is a countable set, then (X, T) is tame.*

Proof. By Theorem 2.1, it remains to show that the cardinality $\operatorname{card}(E(X, T)) \leq 2^{\aleph_0}$. Since (Y, S) is equicontinuous, $\operatorname{card}(E(Y, S)) \leq 2^{\aleph_0}$. For $p \in E(X, T)$, let $\pi_*(p) \in E(Y, S)$ with $\pi_*(p)(y) = \pi(p(x))$ for each $y = \pi(x)$. Then $\pi_* : E(X, T) \to E(Y, S)$ is surjective.

First, it is not hard to check that for any $p \in E(X, T) \setminus \{T^n : n \in \mathbb{Z}\}$, $p(\pi^{-1}(y))$ is a singleton for each $y \in Y$. Fix $q \in E(Y, S)$. Since (Y, S) is equicontinuous, q is a

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homeomorphism from Y to Y. Let $Y_q = q^{-1}(Y \setminus Y_0)$ and $X_q = \pi^{-1}(Y_q)$. Then Y_q is a countable set. For $x \in X \setminus X_q$, as $q(\pi(x)) \in Y_0$, $\#\pi^{-1}(q(\pi(x))) = 1$. Thus there exists a unique $x_q \in X$ with $\{x_q\} = \pi^{-1}(q(\pi(x)))$. Now for any $p \in \pi_*^{-1}(q) \setminus \{T^n : n \in \mathbb{N}\}$, it is clear that $p(x) = x_q$ for $x \in X \setminus X_q$. For $y \in Y_q$, we define $T_p(y) = p(x)$ for some $x \in \pi^{-1}(y)$. Since $p(\pi^{-1}(y))$ is a singleton, $T_p(y)$ is well defined and is independent of the choice of $x \in \pi^{-1}(y)$. Note that

- $T_p \in X^{Y_q}$ for $p \in \pi_*^{-1}(q) \setminus \{T^n : n \in \mathbb{N}\}$ and
- for any $p, p' \in \pi_*^{-1}(q) \setminus \{T^n : n \in \mathbb{N}\}$, $p \neq p'$ if and only if $T_p \neq T_{p'}$.

We have that the map $\phi_q : \pi_*^{-1}(q) \setminus \{T^n : n \in \mathbb{Z}\} \ni p \mapsto T_p \in X^{Y_q}$ is one-to-one. Since Y_q is countable, there exists a one-to-one map $\varphi_q : X^{Y_q} \to X^{\mathbb{N}}$. Moreover, the composition map $\varphi_q \circ \phi_q : \pi_*^{-1}(q) \setminus \{T^n : n \in \mathbb{Z}\} \to X^{\mathbb{N}}$ is also one-to-one.

Finally, we consider the map

$$\Phi : E(X, T) \setminus \{T^n : n \in \mathbb{Z}\} \to E(Y, S) \times X^{\mathbb{N}} \quad \text{with } \Phi(p) = (\pi_*(p), \varphi_{\pi_*(p)} \circ \phi_{\pi_*(p)}(p)).$$

Clearly, Φ is a one-to-one map. Hence $\operatorname{card}(E(X, T)) \leq \operatorname{card}(E(Y, S) \times X^{\mathbb{N}} \cup \{T^n : n \in \mathbb{N}\}) \leq 2^{\aleph_0}$ since $\operatorname{card}(X) \leq 2^{\aleph_0}$ and $\operatorname{card}(E(Y, S)) \leq 2^{\aleph_0}$. This finishes the proof. □

Example 5.11. Let $K = \mathbb{R}/\mathbb{Z}$ denote the circle and K^n the n-torus. Let $S : K^n \to K^n$ $(n \geq 2)$ be an arbitrary minimal equicontinuous homeomorphism with $S(x_1, x_2, \ldots, x_n) = (x_1 + \alpha_1, \ldots, x_n + \alpha_n)$. Let $z_1, \ldots, z_m \in K^n$ be such that $S^r(z_i) \neq z_j$ for any $r \in \mathbb{Z}$, if $i \neq j$. Then by [**25**, §4.2] there exist a minimal cascade (K^n, T) and a homomorphism $\pi : (K^n, T) \to (K^n, S)$ such that $\pi^{-1}(z)$ is a singleton except for $z = S^r(z_i)$, where $i \in \{1, 2, \ldots, m\}$ and $r \in \mathbb{Z}$, in which case $\pi^{-1}(z)$ is homeomorphic to an $(n-1)$-cube. Moreover, for each $i \in \{1, 2, \ldots, m\}$, $\operatorname{diam}(\pi^{-1}(S^n(z_i))) \to 0$ as $|n| \to +\infty$.

It is clear that (K^n, T) is not equicontinuous, π is an asymptotic homomorphism and $K^n \setminus K_0^n = \bigcup_{i=1}^m \{S^r z_i : r \in \mathbb{Z}\}$ is a countable set, where $K_0^n = \{z \in K^n : \#\pi^{-1}(z) = 1\}$. By Lemma 5.10, (K^n, T) is tame but not equicontinuous.

Example 5.12. Let $(\{0, 1\}^{\mathbb{Z}}, \sigma)$ be the full shift with two symbols, where $\sigma(x)_i = x_{i+1}$ for $i \in \mathbb{Z}$. Set $S_1 = \{2\}$ and $q_1 = 2$, and for $n \geq 1$, set $S_{n+1} = \{2^{q_n+i} : i = 1, 2, \ldots, n+1\}$ and $q_{n+1} = 2^{q_n+n+1}$. Let $W^n = \{x \in \{0, 1\}^{\mathbb{Z}} : \{i \in \mathbb{Z} : x(i) = 1\} \subseteq S_n\}$ for $n = 1, 2, \ldots$ and $X = cl(\bigcup_{i \in \mathbb{Z}} \sigma^i(\bigcup_{n=1}^{\infty} W^n))$. Then (X, σ) is a subsystem of $(\{0, 1\}^{\mathbb{Z}}, \sigma)$. Now we claim that (X, σ) is tame but not null.

First, it is not hard to compute that $X = \bigcup_{i \in \mathbb{Z}} \sigma^i \big(\bigcup_{n=1}^{\infty} W^n\big) \cup \bigcup_{j \in \mathbb{Z}} \sigma^j w^1 \cup \{w^0\}$, where $w^0 \in \{0, 1\}^{\mathbb{Z}}$ with $w^0(i) \equiv 0$ for $i \in \mathbb{Z}$ and $w^1 \in \{0, 1\}^{\mathbb{Z}}$ with $w^1(i) = 1$ if and only if $i = 0$. Hence X is countable. Thus, (X, T) is tame by Lemma 5.8.

Let $U_0 = \{x \in X : x(0) = 0\}$ and $U_1 = \{x \in X : x(1) = 0\}$. Clearly, U_i is a closed neighborhood of w^i, $i = 0, 1$. For any $k \in \mathbb{N}$ and $s \in \{0, 1\}^{S_k}$, $\bigcap_{j \in S_k} \sigma^{-j} U_{s(j)} \neq \emptyset$. This implies $(w^0, w^1) \in WS(X, \sigma)$. By Proposition 5.5, (X, σ) is not null.

Finally, let (Y, S) be an irrational rotation on a circle or an adding machine. Then the product $(X \times Y, T \times S)$ is still tame but not null. It is clear that $X \times Y$ is a perfect space, that is, without isolated points. Particularly, when (Y, S) is an adding machine, the product space $X \times Y$ is a Cantor set.

To end the paper let us ask the following questions:

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Question 1. Is a minimal tame cascade null?

Question 2. Is a cascade (X, T) null if and only if $WS(X, T) = \emptyset$?

Question 3. Is a G-system (X, G) tame if and only if $S(X, G) = \emptyset$?

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