Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2021.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

105.E (Yasuo Matsuda)

There are 100 items, ranked from 1st place to 100th place.

Choose these items one by one at random and stop once the best 10 have been chosen. Find the expected number of items that need to be chosen in order to achieve this.

105.F (Chris Starr)

Consider a right tetrahedron $OABC$ with OA, OB and OC mutually perpendicular with lengths a, b and c respectively.

(a) Let a, b and c be successive terms in the Fibonacci sequence. Prove that the area of triangle ABC is rational.

(b) Let a, b be successive terms in the Fibonacci sequence. Show that there are an infinite number of integer values of c such that the area of triangle ABC is rational.

105.G (Dorin Marghidanu)

If a, b, c, x, y, z are positive real numbers, prove that

$$\frac{H_3(a,b,c)}{A_3(x,y,z)} \leqslant \frac{1}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}} + \frac{1}{\frac{y}{a}+\frac{z}{b}+\frac{x}{c}} + \frac{1}{\frac{z}{a}+\frac{x}{b}+\frac{y}{c}} \leqslant \frac{A_3(a,b,c)}{A_3(x,y,z)}$$

where $A_3(m, n, p) = \frac{1}{3}(m + n + p)$, $H_3(m, n, p) = \dfrac{3}{\frac{1}{m} + \frac{1}{n} + \frac{1}{p}}$.

When is there equality?

105.H (Seán Stewart)

(a) Evaluate $\displaystyle\int_0^\infty \frac{(\sin x - x)^2}{x^6}\,dx$.

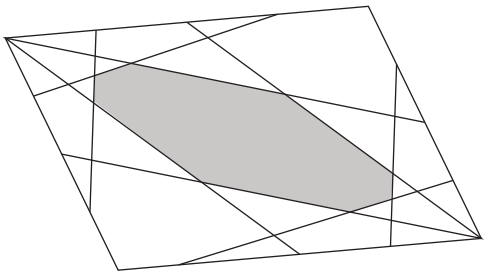
(b) Evaluate $\displaystyle\int_0^\infty \frac{(\sin x - x)^2}{x^5}\,dx$.

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Solutions and comments on **104.I**, **104.J**, **104.K**, **104.L** (November 2020).

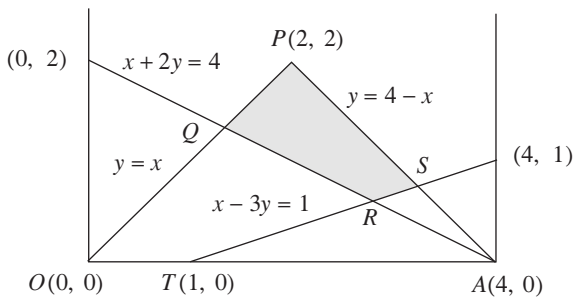
104.I (Arsalan Wares)

The figure shows a parallelogram with points on each side that split the sides into four equal parts. Eight line segments connecting these points in the manner shown enclose the shaded central octagon. Exactly what proportion of the parallelogram is occupied by the shaded octagon?



Answer: $\dfrac{133}{480}$

By far the most common solution to this popular problem was to apply a ratio-preserving affine transformation to map the parallelogram to a square, with most choosing a side length of 4. The central octagon is then composed of four congruent quadrilaterals, one of which is shown shaded in the figure.



The points of intersection of the lines are readily calculated as $Q\left(\frac{4}{3}, \frac{4}{3}\right)$, $R\left(\frac{14}{5}, \frac{3}{5}\right)$, $S\left(\frac{13}{4}, \frac{3}{4}\right)$. The shaded area $PQRS$ is given by

"area OPA − area OQA − area TSA + area TRA"

which evaluates as $\dfrac{133}{120}$, so the required ratio is $\dfrac{\text{area } PQRS}{\text{area } OPA} = \dfrac{133}{480}$.

Correct solutions were received from: S. Abbott, M. V. Channakeshava, N. Curwen, C. Davis, S. Dolan, M. G. Elliott, G. Howlett, P. F. Johnson, J. A. Mundie, C. Starr, G. Strickland, L. Wimmer and the proposer Arsalan Wares.

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104.J (Isaac Sofair)

Prove that

$$2\sum_{k=0}^{\infty}\frac{(-1)^k\binom{n+k}{k}}{(2n+2k+1)\binom{2n+2k}{2k}} = \binom{2n}{n}\sum_{k=0}^{\infty}\frac{2^k}{(2n+2k+1)\binom{2n+2k}{n+k}}.$$

This summation problem attracted a fascinating range of solutions and I apologise if this short summary does not do them all justice.

One approach is to examine the recurrence relations satisfied by each side of the identity.

Let $F_1(n,k) = \dfrac{2(-1)^k\binom{n+k}{k}}{(2n+2k+1)\binom{2n+2k}{2k}}$, so that $S_1(n) = \sum_{k=0}^{\infty} F_1(n,k)$ denotes the left-hand side of the identity. Then, either by manipulating binomial coefficients, as in Graham Howlett's solution, or be employing Zeilberger's algorithm, as in Fabio Velandia's solution, we obtain the recurrence relation

$$2(n+1)F_1(n+1,k) - 2(2n+1)F_1(n,k) = (2n+1)\big[F_1(n,k+1) - F_1(n,k)\big].$$

For fixed n, summing over $k \geqslant 0$ and using $F_1(n,0) = \dfrac{2}{2n+1}$ gives the recurrence relation

$$2(n+1)S_1(n+1) - 2(2n+1)S_1(n) = -(2n+1)F_1(n,0) = -2$$

with initial condition $S_1(0) = 2\sum_{k=0}^{\infty}\dfrac{(-1)^k}{2k+1} = \dfrac{\pi}{2}$ (from the Gregory-Leibniz series).

Next, let $F_2(n,k) = \binom{2n}{n}\dfrac{2^k}{(2n+2k+1)\binom{2n+2k}{n+k}}$ so that $S_2(n) = \sum_{k=0}^{\infty} F_2(n,k)$ denotes the right-hand side of the identity. Then $F_2(n,k)$ satisfies the recurrence relation

$$(n+1)F_2(n+1,k) - (2n+1)F_2(n,k) = (2n+1)\big[F_2(n,k+1) - F_2(n,k)\big]$$

which sums to give

$$(n+1)S_2(n+1) - (2n+1)S_2(n) = -(2n+1)F_2(n,0) = -1$$

with initial condition $S_2(0) = \sum_{k=0}^{\infty}\dfrac{2^k}{(2k+1)\binom{2k}{k}} = \dfrac{\pi}{2}$. (This is a less well-known series, but its value can be found by substituting $x = \dfrac{1}{\sqrt{2}}$ into

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the Maclaurin expansion of $\dfrac{\sin^{-1} x}{\sqrt{1 - x^2}} = \displaystyle\sum_{k=0}^{\infty} \frac{2^{2k}}{(2k+1)\binom{2k}{k}} \cdot x^{2k+1}$.)

Since $S_1(n)$ and $S_2(n)$ satisfy the same recurrence relation and same initial conditions, we have $S_1(n) = S_2(n)$ for all n as required.

M. G. Elliott and the proposer, Isaac Sofair, manipulated a partial fraction decomposition of the left-hand side to show equality.

Seán Stewart used Euler's transformation for alternating series. This states that $\displaystyle\sum_{k=0}^{\infty} (-1)^k a_k(n) = \sum_{k=0}^{\infty} \frac{\Delta^k a_0(n)}{2^{k+1}}$ where Δ^k denotes the k-fold composition of the forward difference operator $\Delta a_0(n) = a_{k+1}(n) - a_k(n)$. Applying this with $a_k(n) = \dfrac{\binom{n+k}{k}}{(2n+2k+1)\binom{2n+2k}{2k}}$ where, by induction, $\Delta^k a_0(n) = \binom{2n}{n} \dfrac{2^{2k}}{(2n+2k+1)\binom{2n+2k}{n+k}}$ establishes the result.

Finally, Phil Kitchenside and Stan Dolan utilised properties of hypergeometric functions. By defintion, ${}_2F_1(a, b; c; z) = \displaystyle\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!}$ where $(a)_k = a(a+1)\ldots(a+k-1)$. Then, manipulating factorials,

$$S_1(n) = \frac{2}{2n+1}\, {}_2F_1\left(\tfrac{1}{2}, 1; n + \tfrac{3}{2}; -1\right)$$

$$\text{and } S_2(n) = \frac{1}{2n+1}\, {}_2F_1\left(1, n+1; n + \tfrac{3}{2}; \tfrac{1}{2}\right).$$

Equality of $S_1(n)$ and $S_2(n)$ now follows from the Pfaff transformation

$${}_2F_1(a, b; c; z) = (1 - z)^{-b}\, {}_2F_1\left(b, c - a; c; \frac{z}{z-1}\right).$$

Correct solutions were received from: S. Dolan, M. G. Elliott, G. Howlett, P. Kitchenside, S. M. Stewart, F. A. Velandia and the proposer Isaac Sofair.

104.K (Jacob Siehler)

Two unit squares are drawn in the plane with their edges parallel to the coordinate axes and their centres chosen randomly and independently in the region $-1 \leqslant x, y \leqslant 1$. Determine the expected value for the area of their intersection.

Answer: $\dfrac{25}{144}$.

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Most solvers either calculated the expected value via an appropriate double integral or via density functions as in the solution which follows.

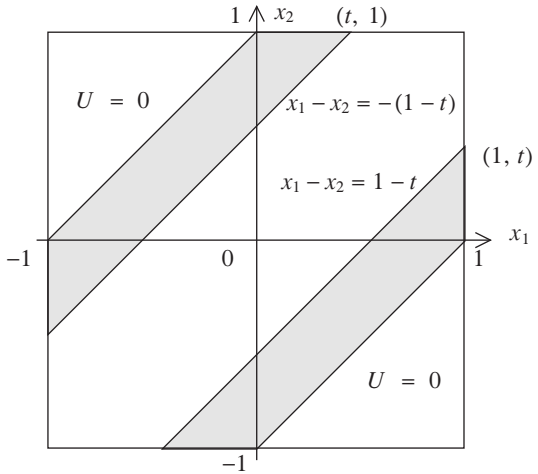
Let (X_1, Y_1), (X_2, Y_2) denote the centres of the two unit squares with X_i, Y_i all uniformly distributed on $[-1, 1]$. The area of their overlap is given by UV where

$$U = \begin{cases} 1 - |X_1 - X_2| & \text{if } |X_1 - X_2| \leqslant 1 \\ 0 & \text{otherwise} \end{cases},$$

$$V = \begin{cases} 1 - |Y_1 - Y_2| & \text{if } |Y_1 - Y_2| \leqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

Since U and V are independent with the same distribution,

$$\mathrm{E}(UV) = \mathrm{E}(U)\,\mathrm{E}(V) = \mathrm{E}(U)^2.$$



From the diagram, $\mathrm{P}(U = 0) = \frac{1}{4}$ and, for $0 < t \leqslant 1$, $\mathrm{P}(0 < U \leqslant t)$ is the relative area of the shaded region,

$$\mathrm{P}(0 < U \leqslant t) = \frac{1}{4} \times 2 \times \frac{1}{2}\left[(1 + t)^2 - 1\right] = \frac{1}{4}\left(t^2 + 2t\right).$$

The probability density function for U is thus

$$f(t) = \begin{cases} \frac{1}{4} & \text{if } t = 0 \\ \frac{1}{2}(1 + t) & \text{if } 0 < t \leqslant 1 \end{cases}$$

from which $\mathrm{E}(U) = \int_0^1 t \times \frac{1}{2}(1 + t)\,dt = \frac{5}{12}$ and $\mathrm{E}(UV) = \mathrm{E}(U)^2 = \frac{25}{144}$.

As Peter Johnson commented, having the density function enables further calculations. For example,

$$\mathrm{E}\left(U^2\right) = \int_0^1 t^2 \times \tfrac{1}{2}(1 + t)\,dt = \frac{7}{24}$$

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$$\text{so that } \operatorname{var}(U) = \frac{7}{24} - \left(\frac{5}{12}\right)^2 = \frac{17}{144}$$

$$\text{and } \operatorname{var}(UV) = \mathrm{E}\left(U^2V^2\right) - [\mathrm{E}(UV)]^2$$

$$= \left(\frac{7}{24}\right)^2 - \left(\frac{25}{144}\right)^2 = \frac{1139}{20736}.$$

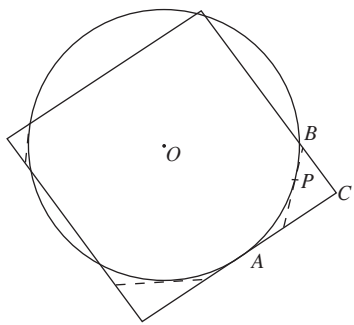
And Samuele Riccarelli computed the probability density function $g(t)$ for the area, UV,

$$g(t) = \begin{cases} \frac{7}{16} & \text{if } t = 0 \\ \frac{1}{4}[2 - 2t - (t+1)\ln t] & 0 < t \leqslant 1 \end{cases}.$$

Correct solutions were received from: N. Curwen, C. Davis, S. Dolan, M. G. Elliott, G. Howlett, P. F. Johnson, S. Riccarelli and the proposer Jacob Siehler.

104.L (Paul Stephenson)

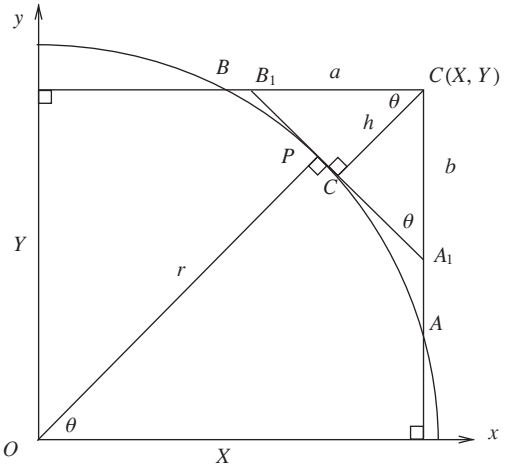
A stiff heavy rectangular cloth is thrown carelessly across a round table, centre O, as shown in the figure. The friction between the cloth and the table is such that it stays where thrown but single flaps like ABC, where A and B are the points where adjacent edges of the flap cut the table, fall vertically from folds tangential to the table edge. If the fold falls in such a way that OPC is a straight line, then prove that the vertical portion of the flap is isosceles in shape.



The intention in this problem was that, given points A, B, the fold would position itself so as to minimise the potential energy of the vertical portion of the flap. M. G. Elliott, whose solution is summarised below, showed first that OPG is necessarily a straight line where G is the centre of mass of the vertical portion of the flap.

The figure below shows a rectangular portion of the cloth with flap ABC and fold line A_1B_1. The potential energy of the hanging portion of the flap, A_1B_1C, is $-\rho g \frac{1}{2}ab\frac{1}{3}h$ (where ρ is the density of the cloth and g the acceleration due to gravity), so it suffices to maximise $V = abh$.

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The equation of A_1B_1 is $x\cos\theta + y\sin\theta = r$ so that

$$h = X\cos\theta + Y\sin\theta - r, \qquad a = \frac{h}{\cos\theta},\ b = \frac{h}{\sin\theta}$$

$$\text{and } V = \frac{(X\cos\theta + Y\sin\theta - r)^3}{\cos\theta\sin\theta}.$$

For stationary points, $\dfrac{dV}{d\theta} = 0$ whence

$$3\cos\theta\sin\theta\left(-X\sin\theta + Y\cos\theta\right) - \left(X\cos\theta + Y\sin\theta - r\right)\left(\cos^2\theta - \sin^2\theta\right) = 0.$$

Substituting $\cos\theta = \dfrac{b}{c}$, $\sin\theta = \dfrac{a}{c}$ and (by areas)

$$X\cos\theta + Y\sin\theta - r = h = \frac{ab}{c}$$

gives

$$3ab\left(-aX + bY\right) - ab\left(b^2 - a^2\right) = 0$$

so that

$$\frac{a}{b} = \frac{Y - \frac{b}{3}}{X - \frac{a}{3}} = \tan\theta \qquad (*)$$

meaning that, since G has coordinates $\left(X - \frac{1}{3}a, Y - \frac{1}{3}b\right)$, O, P, G lie in a straight line. (The stationary value maximises V because both ab and h are smaller when P is near A or B.)

In the problem, we are further given that OPC is a straight line so that $\tan\theta = \dfrac{Y}{X}$. Coupled with $(*)$, this forces $X = Y$, $\theta = 45°$ and $a = b$, as required.

Correct solutions were received from: S. Dolan, M. G. Elliott, C. Starr and the proposer Paul Stephenson.

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