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Modeling potential as fiber entropy and pressure as entropy

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Abstract. We first prove that topological fiber entropy potential in a relatively symbolic extension (topological joining with a subshift) of a topological dynamical system (which is a non-negative, non-decreasing, upper semicontinuous and subadditive potential $\mathfrak{H}$) yields topological pressure equal to the topological entropy of the extended system. The terms occurring on the other side of the variational principle for pressure are equal to the extension entropies of the invariant measures. Thus the variational principle for pressure reduces to the usual variational principle (for entropy) applied to the extended system. Next we prove our main theorem saying that every non-negative, upper semicontinuous and subadditive potential $\mathfrak{F}$ (we drop the monotonicity assumption) is 'nearly equal' to the fiber entropy potential $\mathbf{H}$ in some relatively symbolic extension of the system, in the sense that all terms occurring in the variational principle for pressure are the same for both potentials. This gives a new interpretation of all such potentials $\mathfrak{F}$ as a kind of additional information function enhancing the natural information arising from the dynamical system, and provides a new proof of the variational principle for pressure. At the end of the paper we provide examples showing that both assumptions, continuity and additivity, under which so-called lower pressure (defined with the help of spanning sets) equals the pressure, are essential, already in the class of non-negative, upper semicontinuous, subadditive potentials.

1. *Heuristics*

In the theory of dynamical systems, there are two major notions of entropy: measure-theoretic (Kolmogorov–Sinai) and topological (Bowen, Adler–Konheim–McAndrew). Their information-theoretic interpretation is, roughly speaking, the following: every state

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of the system (point in the phase space) carries some natural *information*. This can be imagined to be the information about the point's 'identity', and initially it is fully 'concealed' from an observer. By making a measure-theoretic measurement (applying a measurable partition $\mathcal{P}$) or a topological measurement (distinguishing points that are more than ϵ apart) at time zero, the observer can 'acquire' some part of this information, by learning to which element of the partition (or to which ϵ-ball) the initial state belongs. Repeating the same measurement in every following unit of time, the observer acquires analogous information about the points in the forward orbit of the initial state; hence, by 'backtracking', he/she acquires more and more information about the initial state. In the measure-theoretic setup, the information acquired after n steps, if the initial state is x, equals

$$I_n(x) = -\log \mu(A_x^n),$$

where A_x^n denotes the unique element of the partition $\mathcal{P}^n = \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{P})$ containing x, and μ is the invariant measure considered. In this manner, we have represented the acquired information in the form of an increasing sequence $(I_n)_{n\geq 1}$ of non-negative functions on X. Note that these functions depend on the measure and on the partition. The entropy $h_\mu(T, \mathcal{P})$ is defined as the *average flow of information*, with averaging applied first over the space and then over time:

$$h_\mu(T, \mathcal{P}) = \lim_n \frac{1}{n} \int I_n \, d\mu.$$

Finally, the Kolmogorov–Sinai entropy of μ is the supremum of the above terms over all finite partitions $\mathcal{P}$, i.e. the global (maximal) *flow of information* in the system.

In the topological setup the information function acquired in n steps is constant over the space and equals the logarithm of the number of 'classes of ϵ-indistinguishability' among the n-orbits. We prefer to skip further discussion of indistinguishability and refer to [**4**, §6.1]. Such a constant information function is analogous to the measurable situation in which all classes have the same measure (a measurable partition with atoms of equal measures yields a constant information function equal to the logarithm of the number of atoms; this, however, is just a rough analogy—the topological 'classes of ϵ-indistinguishability' are usually not disjoint). Since the information function is constant, the averaging over the space is not necessary, and the topological ϵ-entropy is defined as the *flow* (average over the time) of this information. Finally, we take supremum over ϵ to get the topological entropy as the *global flow of the topological information* in the system.

We have included the above discussion in order to emphasize one important fact: the *acquired information* has the form of an increasing sequence of non-negative functions defined on the phase space. The measure-theoretic information depends on the measure and partition, while the topological information depends on a topological parameter ϵ (alternatively, on an open cover). Moreover, the topological information need not coincide with the measure-theoretic information of any invariant measure. Nonetheless, the measure-theoretic and topological entropies are strictly related: the variational principle asserts that topological entropy equals the supremum, over all invariant measures, of their Kolmogorov–Sinai entropies.

In this note we will try to extend the above information-theoretic interpretation to the notion of topological pressure. So far, this notion has been mainly attributed to

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so-called thermodynamical formalism, hence we believe our interpretation might shed some new light on it. We would like to interpret a non-negative potential as a kind of *additional information* enhancing the natural information arising from the dynamical system. Moreover, since a potential $\mathfrak{F} = (f_n)_{n\geq 1}$ is just *one* sequence of functions, this additional information must be *universal*, i.e. the same potential should enhance the measure-theoretic information regardless of the invariant measure and partition, as well as the topological information regardless of any topological parameter. Needless to say, we would like to be able to interpret the topological pressure associated with the potential as the global flow of the enhanced topological information, and we would like to be able to interpret the global flow of the enhanced measure-theoretic information as well.

In order to understand whether such an interpretation is possible, we seek standard constructions where such universal additional information occurs naturally. As we shall see, exactly this kind of additional information occurs as so-called *topological fiber entropy* in *relatively symbolic extensions* of topological dynamical systems. In a general topological extension, topological fiber entropy depends on a topological parameter ϵ (or on an open cover), so it has the form of a *family* of sequences of functions. But in a relatively symbolic extension, due to the existence of a relative universal generator in form of a partition into clopen sets (so it relatively generates both topologically as well as measure-theoretically for every invariant measure), one sequence suffices to determine the topological fiber entropy completely. We will call this sequence the *fiber entropy potential* and denote it by $\mathfrak{H} = (\mathbf{H}_n)_{n\geq 1}$. It is non-negative, upper semicontinuous, subadditive and non-decreasing in n.

A heuristic interpretation of the fiber entropy potential as additional information associated to each state is via the fact that the state *splits* in the extension into many (finer) states. Topological fiber entropy at a point equals the logarithm of the number of 'classes of indistinguishability' into which the point splits. So, it corresponds to the additional *topological* information resulting from the splitting. As we shall see, although of topological origin, such information can be successfully applied to enhance the measure-theoretic information as well, by simply adding the potential to the information function (creating, for every invariant measure and partition, the new sequence $(I_n + \mathbf{H}_n)_{n\geq 1}$).

For the fiber entropy potential, we can prove the following facts.

(1) The topological pressure coincides with the topological entropy of the extended system (i.e. with the global flow of the enhanced information, as we desired).
(2) The global flow of the enhanced measure-theoretic information is equal, for each invariant measure μ on the factor system, to the so-called *extension entropy* of μ.
(3) As a result, the variational principle for pressure reduces to the usual variational principle (for entropy) applied to the extended system, which is an evidence for correctness of our interpretation.

Since our interpretation works in the class of fiber entropy potentials, we have attempted to interpret *every* non-negative, non-decreasing, upper semicontinuous, subadditive potential $\mathfrak{F}$ as being in some sense equivalent (nearly equal) to the fiber entropy potential $\mathfrak{H}$ in some relatively symbolic extension, so that the above three facts hold for the potential $\mathfrak{F}$. This attempt was successful; we were even able to drop the assumption that the potential $\mathfrak{F}$ is non-decreasing. So, we have not only proved that our interpretation of the potential

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as the additional information is correct, but also provided a method to model the potential in such a way that this interpretation can be seen in a practical mode. We have shown that the variational principle for pressure, which was a subject of many independent works and has been proved gradually with increasing generality (ultimately it is known for upper semicontinuous subadditive potentials), reduces, for non-negative potentials, to the usual variational principle. In this way our paper provides an alternative proof.

2. *Preliminaries*

We recall some of the notions in the classic theory of topological entropy and topological pressure. Let (X, T) be a topological dynamical system, where X is a compact metric space with a metric d and $T : X \to X$ is a continuous transformation. We denote by d^n the metric

$$d^n(x, y) = \max\{d(T^i x, T^i y) : i = 0, 1, \ldots, n-1\}.$$

Let F be a subset of X. A set $E \subset F$ is (n, ϵ)-*spanning in* F ($n \in \mathbb{N}$, $\epsilon > 0$) if the union of ϵ-balls in the metric d^n around the points in E covers F. If $F = X$ we will just say that E is (n, ϵ)-spanning. A set $E \subset F$ is (n, ϵ)-separated (this definition does not depend on F) if the distances in d^n between any distinct elements of E are at least ϵ. By compactness of X, there exists a finite (n, ϵ)-spanning set, and all (n, ϵ)-separated sets are finite with bounded cardinality. We denote by $\mathsf{r}(n, \epsilon)$ the minimal cardinality of an (n, ϵ)-spanning set and we let $\mathsf{s}(n, \epsilon)$ denote the maximal cardinality of an (n, ϵ)-separated set. It is easily seen that any (n, ϵ)-separated set of maximal cardinality is (n, ϵ)-spanning, so

$$\mathsf{s}(n, \epsilon) \geq \mathsf{r}(n, \epsilon). \tag{2.1}$$

It is also fairly easy to see that if E is an (n, ϵ)-separated set then the $\epsilon/2$-balls in the metric d^n around the elements of E are pairwise disjoint, and every $(n, \frac{\epsilon}{2})$-spanning set must contain at least one element in every such ball, so

$$\mathsf{r}\left(n, \frac{\epsilon}{2}\right) \geq \mathsf{s}(n, \epsilon). \tag{2.2}$$

The *topological entropy* of (X, T) is defined as

$$\mathbf{h}_{\mathrm{top}}(T) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log(\mathsf{r}(n, \epsilon)) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log(\mathsf{s}(n, \epsilon)). \tag{2.3}$$

The proof of the second equality requires the inequalities (2.1) and (2.2) and also (to replace $\liminf_n$ by $\limsup_n$) another equivalent definition of topological entropy, using open covers, which we will skip (see [**4**]). The choice of the base of the logarithm in the above definition is completely arbitrary (it only affects the values of entropies by a multiplicative constant). Most popular choices are e, 2 or 10. Throughout this paper 'log' always stands for $\log_2$.

One of the most important characterizations of topological entropy is the following theorem.

THEOREM 2.1. (Variational principle)

$$\mathbf{h}_{\mathrm{top}}(T) = \sup\{h_\mu(T) : \mu \textit{ is an invariant measure on } X\}.$$

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We omit the preliminaries for the Kolmogorov–Sinai entropy $h_\mu(T)$. See [**4**] or any textbook on ergodic theory.

For cosmetic reasons, from now on we will denote our dynamical system by (Y, S). A *potential* on (Y, S) is a sequence $\mathfrak{F} = (f_n)_{n\geq 1}$ of functions $f_n : Y \to \mathbb{R}$. Sometimes a potential is understood as just one function $f : Y \to \mathbb{R}$. This corresponds to the sequence $(f_n)_{n\geq 1}$, where $f_n = f + f \circ S + f \circ S^2 + \cdots + f \circ S^{n-1}$. Potentials determined by one function are characterized as *additive*, i.e. satisfying, for any $n, m \in \mathbb{N}$ the equality $f_{n+m} = f_n + f_m \circ S^n$. Other potentials are usually assumed to be only *subadditive*, i.e. to satisfy $f_{n+m} \leq f_n + f_m \circ S^n$ or (rarely) *superadditive* (fulfilling the reverse inequality). Classically all functions f_n are assumed continuous, but we will consider more general *upper semicontinuous* potentials, i.e. such that the sets $\{y : f_n(y) \geq t\}$ are closed for every $n \in \mathbb{N}$, $t \in \mathbb{R}$. We will be mainly concerned with *non-negative potentials*, i.e. such that $f_n \geq 0$ for each $n \geq 1$.

For a potential $\mathfrak{F} = (f_n)_{n\geq 1}$ we make the following definition.

Definition 2.2. We let

$$\mathsf{S}_{n,\epsilon}(\mathfrak{F}) = \sup\Big\{\sum_{y\in E} 2^{f_n(y)} : E \text{ is an } (n, \epsilon)\text{-separated set in } Y\Big\},$$

and then we define *topological pressure* associated with the potential as

$$\mathbf{P}(\mathfrak{F}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathsf{S}_{n,\epsilon}(\mathfrak{F}).$$

It is clear that the limit over $\epsilon \to 0$ exists by monotonicity. We will also briefly address a variant of the notion with $\liminf_n$ in place of $\limsup_n$ which will be denoted as $\underline{\mathbf{P}}(\mathfrak{F})$. It is immediately seen that the topological pressure of the zero potential equals $\mathbf{h}_{\text{top}}(S)$ (the same is true for $\underline{\mathbf{P}}(\mathfrak{F})$).

By analogy with the alternative definition of topological entropy, it is natural to consider the following alternative definition of pressure.

Definition 2.3. Let

$$\mathsf{R}_{n,\epsilon}(\mathfrak{F}) = \inf\Big\{\sum_{y\in E} 2^{f_n(y)} : E \text{ is an } (n, \epsilon)\text{-spanning set in } Y\Big\},$$

and define the *lower topological pressure* of the potential $\mathfrak{F} = (f_n)_{n\geq 1}$ as

$$\mathbf{Q}(\mathfrak{F}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \mathsf{R}_{n,\epsilon}(\mathfrak{F})$$

(with the convention that $\log 0 = -\infty$).

Again, we will also briefly address a variant with $\liminf_n$, denoted by $\underline{\mathbf{Q}}(\mathfrak{F})$. It is clear that $\mathsf{S}_{n,\epsilon}(\mathfrak{F}) \geq \mathsf{R}_{n,\epsilon}(\mathfrak{F})$ because in the supremum defining $\mathsf{S}_{n,\epsilon}(\mathfrak{F})$ it suffices to take maximal (n, ϵ)-separated sets (the expression $2^{f_n(y)}$ is always positive), and such sets are (n, ϵ)-spanning. As a result, the pressure always dominates lower pressure. Equality between lower pressure and pressure holds with rather strong assumptions: continuity and additivity of the potential (see [**13**]). We will give examples showing that neither assumption can be dropped.

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One of the most important facts known for upper semicontinuous subadditive potentials is the following theorem, which we provide in the most recent formulation from [**1**] and [**2**] (see also [**3, 8, 11, 14**] for the local versions).

THEOREM 2.4. (Variational principle for pressure) *If $\mathfrak{F} = (f_n)_{n\geq 1}$ is a subadditive potential consisting of upper semicontinuous functions, then the equality*

$$\mathbf{P}(\mathfrak{F}) = \sup\left\{h_\nu(S) + \lim_n \frac{1}{n}\int f_n\, d\nu : \nu \text{ is an invariant measure on } Y\right\}$$

holds (where, by convention, $\infty - \infty = -\infty$).

For non-negative potentials we will provide a new proof based on a new interpretation of the notions involved. For convenience, the expression $\lim_n (1/n)\int f_n\, d\nu$ will be called the *limit integral* of the potential with respect to the measure ν. The existence of such limits for subadditive potentials and invariant measures follows from the trivial observation that the sequence $(\int f_n\, d\nu)_{n\geq 1}$ is subadditive (and from basic properties of subadditive sequences).

Here we will prove a very simple 'replacement theorem' allowing any subadditive upper semicontinuous potential to be replaced by a continuous one without changing the values of the parameters of interest to us.

THEOREM 2.5. *Let (Y, S) be a topological dynamical system, and let $\mathfrak{F} = (f_n)_{n\geq 1}$ be a subadditive, upper semicontinuous potential on Y. Then there exists a subadditive* continuous *potential $\mathfrak{G} = (g_n)_{n\geq 1}$ with $g_n \geq f_n$ for all n, which has the same topological pressure, lower topological pressure, and, for every invariant measure, the same limit integral, as $\mathfrak{F}$, i.e.*

$$\mathbf{P}(\mathfrak{G}) = \mathbf{P}(\mathfrak{F}) \quad \text{and} \quad \underline{\mathbf{P}}(\mathfrak{G}) = \underline{\mathbf{P}}(\mathfrak{F}); \tag{2.4}$$

$$\mathbf{Q}(\mathfrak{G}) = \mathbf{Q}(\mathfrak{F}) \quad \text{and} \quad \underline{\mathbf{Q}}(\mathfrak{G}) = \underline{\mathbf{Q}}(\mathfrak{F}); \tag{2.5}$$

$$\lim_n \frac{1}{n}\int g_n\, d\nu = \lim_n \frac{1}{n}\int f_n\, d\nu \text{ for every invariant measure } \nu \text{ on } Y. \tag{2.6}$$

If the space Y is zero-dimensional and the potential is non-negative then we can additionally require that each g_n assumes as values only logarithms of integers.

Proof. Each function f_n, being upper semicontinuous, is a (pointwise) decreasing limit of a sequence of continuous functions $(g_{n,m})_{m\geq 1}$. Fix a sequence of positive numbers ϵ_k decreasing to zero. Let $\mathbf{E}_{n,k}$ denote the collection of all (n, ϵ_k)-separated sets. This can be viewed as a compact subset of the formal disjoint union of finitely many finite Cartesian powers of Y:

$$\mathbf{E}_{n,k} \subset \mathbf{Y}_{n,k} = \bigcup_{i=1}^{\mathsf{S}(n,\epsilon_k)} Y^i.$$

On $\mathbf{Y}_{n,k}$ we consider the (positive) function $\phi_n(y_1, \ldots, y_i) = \sum_{j=1}^{i} 2^{f_n(y_j)}$, and we know that on the set $\mathbf{E}_{n,k}$ it does not exceed the constant $\mathsf{S}_{n,\epsilon_k}(\mathfrak{F})$. Since the continuous functions $\psi_{n,m}(y_1, \ldots, y_i) = \sum_{j=1}^{i} 2^{g_{n,m}(y_j)}$ decrease (with m) to the upper semicontinuous function ϕ_n, there exists an index m_k such that $\psi_{n,m_k} < 2\mathsf{S}_{n,\epsilon_k}(\mathfrak{F})$ on the entire compact set $\mathbf{E}_{n,k}$. We let $m(n)$ be the maximum of the indices m_k obtained as above

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for all $k \le n$. We now redefine the sequence $(g_{n,m})_{m\ge 1}$ by starting it from the index $m(n)$ and shifting the enumeration so it runs from 1 again. It is clear that now every potential whose nth function is located between f_n and $g_{n,1}$ has the same topological pressure as $\mathfrak{F}$.

Next we take care of the lower topological pressure. For each n and k there exists a finite (n, ϵ_k)-spanning set $\{y_1, \dots, y_i\}$ such that $\phi_n(y_1, \dots, y_i) < 2\mathsf{R}_{n,\epsilon_k}(\mathfrak{F})$ (or $< 2^{-n^2}$ if $\mathsf{R}_{n,\epsilon_k}(\mathfrak{F}) = 0$). As before, by starting the sequence $(g_{n,m})_{m\ge 1}$ from a large enough index $m(n)$ we can arrange that $\psi_{n,1}(y_1, \dots, y_i) < 2\mathsf{R}_{n,\epsilon_k}(\mathfrak{F})$ (or $< 2^{-n^2}$) for every $k \le n$. With this arrangement, every potential whose nth function is located between f_n and $g_{n,1}$ has (in addition to topological pressure) also the same lower topological pressure as $\mathfrak{F}$.

In the next step we will take care of subadditivity and the limit integrals. Define $\mathfrak{G} = (g_n)_{n\ge 1}$ by setting, for each $n \ge 1$,

$$g_n = \inf\{g_{n_1,n} + g_{n_2,n} \circ S^{n_1} + g_{n_3,n} \circ S^{n_1+n_2} + \cdots + g_{n_k,n} \circ S^{n_1+\cdots+n_{k-1}}\}, \tag{2.7}$$

where the infimum ranges over all tuples $n_1, n_2, \dots, n_k \ge 1$ (of arbitrary lengths k between 1 and n) such that $n_1 + n_2 + \cdots + n_k = n$. Subadditivity of this new potential is obvious, and, by subadditivity of $\mathfrak{F}$, we have also $f_n \le g_n$. Clearly $g_n \le g_{n,n} \le g_{n,1}$, so that $\mathfrak{G}$ has the same pressure and lower pressure as $\mathfrak{F}$. Fix an invariant measure ν on Y. For every $k > l \ge 0$ we have

$$\frac{1}{2^k}\int g_{2^k}\,d\nu \le \frac{1}{2^k}\int g_{2^l,2^k} + g_{2^l,2^k} \circ S^{2^l} + \cdots + g_{2^l,2^k} \circ S^{2^k-2^l}\,d\nu = \frac{1}{2^l}\int g_{2^l,2^k}\,d\nu.$$

By subadditivity of $\mathfrak{G}$ on the one hand, and by the monotone Lebesgue theorem on the other, by passing to the limit as k grows to infinity we obtain

$$\lim_n \frac{1}{n}\int g_n\,d\nu \le \frac{1}{2^l}\int f_{2^l}\,d\nu.$$

Since this is true for every $l \ge 0$, by subadditivity of $\mathfrak{F}$ we can replace the right-hand side by $\lim_n (1/n)\int f_n\,d\nu$. The reverse inequality is obvious.

The last statement (concerning zero-dimensional spaces and non-negative potentials) is proved by introducing a two-step modification of the functions g_n. By an elementary argument involving a base of the topology consisting of closed-and-open sets, for each n we can find an integer-valued continuous function h_n with $2^{g_n} \le h_n \le 2^{g_n} + 2$. The potential $\mathfrak{G}' = (g'_n)_{n\ge 1}$, where $g'_n = \log h_n$, clearly has the same pressure, lower pressure and all limit integrals, as $\mathfrak{G}$ (hence as $\mathfrak{F}$). To make sure the new potential is subadditive we redefine the potential $\mathfrak{G}$ as follows:

$$g_n = \inf\{g'_{n_1} + g'_{n_2} \circ S^{n_1} + g'_{n_3} \circ S^{n_1+n_2} + \cdots + g'_{n_k} \circ S^{n_1+\cdots+n_{k-1}}\},$$

with the infimum ranging over the tuples as in (2.7) (note that the sum of logarithms of integers is again a logarithm of an integer). This new version of the potential $\mathfrak{G}$ fulfills all our requirements. □

We continue with more preliminaries. Let (Y, S) and (Z, σ) be two topological dynamical systems. By a topological joining of these systems we will understand any system (X, T) where $X \subset Y \times Z$ with the product action $T = S \times \sigma$ (for that X must be *T-invariant*, i.e. $T(X) \subset X$), such that X maps by the natural projections *onto* Y and

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onto Z. We will use the symbol $Y \vee Z$ to denote the space X (and sometimes also the system (X, T)) in a topological joining of (Y, S) with (Z, σ).

A system (X, T) is a *topological extension* of (Y, S) (equivalently, (Y, S) is a *topological factor* of (X, T)) if there exists a continuous surjection $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$ (π is called the *factor map*). We will say that the systems are *topologically conjugate* if the factor map π is a homeomorphism. For example, if (X, T) is a topological joining of (Y, S) and (Z, σ) then it is a topological extension of both (Y, S) and (Z, σ); the factor maps are provided by the natural projections onto the first and second coordinate, respectively. In fact every topological extension has this form; an extension (X, T) of (Y, S) is topologically conjugate to the joining $Y \vee X$ given by the relation $(y, x) \in Y \vee X$ whenever $y = \pi(x)$.

A topological factor map π from (X, T) to (Y, S) sends (by preimage) the Borel sigma-algebra $\mathfrak{B}$ in Y to a subinvariant sub-sigma-algebra $\pi^{-1}(\mathfrak{B})$ of the Borel sigma-algebra $\mathfrak{A}$ in X. We will often identify $\mathfrak{B}$ with $\pi^{-1}(\mathfrak{B})$ (and write $\mathfrak{B}$ for either one). The map π also transforms every invariant measure μ on X (i.e. on $\mathfrak{A}$) to an invariant measure ν on Y (i.e. on $\mathfrak{B}$) by the formula $\nu(B) = \mu(\pi^{-1}(B))$ ($B \in \mathfrak{B}$). We will write $\nu = \pi(\mu)$. Conversely, it is known that every invariant measure ν on Y lifts to at least one invariant measure μ on X (i.e. the map π is surjective on invariant measures). We will be using the *conditional entropy* of μ *given* the factor (Y, S) which is denoted, depending on the context, as $h_\mu(T|S)$, $h(\mu|\pi(\mu))$ or $h(\mathfrak{A}|\mathfrak{B})$ (when the measure is understood). Basically, $h_\mu(T|S) = h_\mu(T) - h_{\pi(\mu)}(S)$; however, this does not apply when the latter entropy is infinite (in which case the notion $h_\mu(T|S)$ still has a well-defined, finite or infinite, value). We will not go into more details (see [**4**, §4.1], [**12**], or [**13**]).

In the same context we define the topological fiber entropy on Y in the following steps (we replicate the notation from [**4**]). For an open cover $\mathcal{U}$ of X and a set $K \subset X$ we denote

$$\mathbf{N}(\mathcal{U}|K) := \min\left\{\#\mathcal{V} : \mathcal{V} \subset \mathcal{U}, \bigcup \mathcal{V} \supset K\right\}$$

(if $K = X$ we just write $\mathbf{N}(\mathcal{U})$). Next, we define

$$\mathbf{N}(\mathcal{U}|y) = \mathbf{N}(\mathcal{U}|\pi^{-1}(\{y\})),$$

$$\mathbf{H}(\mathcal{U}|y) = \log \mathbf{N}(\mathcal{U}|y),$$

$$\mathbf{H}(\mathcal{U}|\nu) = \int \mathbf{H}(\mathcal{U}|y)\, d\nu \quad \text{where } \nu \text{ is an invariant measure on } Y,$$

$$\mathbf{h}(\mathcal{U}|\nu) = \lim_{n\to\infty} \frac{1}{n}\mathbf{H}(\mathcal{U}^n|\nu) \quad \text{where } \mathcal{U}^n = \bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}),$$

$$\mathbf{h}(T|\nu) = \sup\{\mathbf{h}(\mathcal{U}|\nu) : \mathcal{U} \text{ is an open cover of } X\}.$$

(The limit above exists by subadditivity of the sequence $\mathbf{H}(\mathcal{U}^n|\nu)$.) The term $\mathbf{H}(\mathcal{U}|y)$ is called the *(static) fiber entropy of* $\mathcal{U}$ *given* y, $\mathbf{h}(\mathcal{U}|\nu)$ is the *(dynamic) fiber entropy of* $\mathcal{U}$ *given* ν and $\mathbf{h}(T|\nu)$ is the *(dynamic) fiber entropy of* T *given* ν. One of the most important theorems concerning the fiber entropy is as follows (see [**10**] and also [**7**]).

THEOREM 2.6. (Inner variational principle)

$$\mathbf{h}(T|\nu) = \sup\{h(\mu|\nu) : \mu \in \pi^{-1}(\nu)\}.$$

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We will also need the notion provided below.

Definition 2.7. In the context as above, the *extension entropy* of ν is defined as

$$h^{\pi}(\nu) = \sup\{h_{\mu}(T) : \ \mu \in \pi^{-1}(\nu)\}.$$

The inner variational principle and the formula $h_{\mu}(T) = h_{\nu}(S) + h(\mu|\nu)$ provide a strict connection between fiber entropy and extension entropy:

$$h^{\pi}(\nu) = h_{\nu}(S) + \mathbf{h}(T|\nu). \tag{2.8}$$

3. *Fiber entropy as a potential*

In this section we will address fiber entropy in a *relatively symbolic* extension, i.e. an extension (X, T) of (Y, S) conjugate to a topological joining $Y \vee Z$, where (Z, σ) is a *subshift*. By a subshift we mean a shift-invariant subset $Z \subset \Lambda^{\mathbb{S}}$, where Λ is a finite set (called the *alphabet*) and $\mathbb{S} = \mathbb{Z}$ (then we have *bilateral* subshift) or $\mathbb{S} = \mathbb{N} \cup \{0\}$ (then we have *unilateral* subshift). In subshifts, the transformation σ always denotes the shift map given by $\sigma(z)(n) = z(n+1)$ ($z = (z(n))_{n\in\mathbb{S}} \in \Lambda^{\mathbb{S}}$). By default, the metric in the symbolic space is $d(z, z') = 1/(1 + \min\{|n| : z(n) \neq z'(n)\})$. In the space $X = Y \vee Z$ we will fix a specific open cover $\mathcal{U}$ (which is at the same time a partition into closed sets) corresponding to the *zero-coordinate partition* of Z, given by the rule

$$\mathcal{U} = \{U_a : a \in \Lambda\} \quad \text{where } U_a = \{(y, z) : z(0) = a\}.$$

In this case $\mathcal{U}^n$ is the partition into *cylinder sets* corresponding to blocks of length n. We will identify blocks with their cylinder sets and write

$$B \in \Lambda^n \quad \text{as well as } B = \{(y, z) : z[0, n-1] = B\} \in \mathcal{U}^n.$$

In this setup we will prove the following (not very surprising, but with surprisingly complicated proof) fact.

THEOREM 3.1. *Let $(X, T) = Y \vee Z$ be a relatively symbolic extension of a system (Y, S). Let $\mathcal{U}$ denote the zero-coordinate partition (and cover) of X corresponding to the symbols in Z. Then the sequence $\mathfrak{H} = (\mathbf{H}_n)_{n\geq 1}$ of fiber entropy functions $\mathbf{H}_n(y) = \mathbf{H}(\mathcal{U}^n|y)$ is an upper semicontinuous, non-negative, non-decreasing, subadditive potential on Y, and the associated topological pressure $\mathbf{P}(\mathfrak{H})$ equals the topological entropy $\mathbf{h}_{\text{top}}(T)$ of the system (X, T). The same is true for $\underline{\mathbf{P}}(\mathfrak{H})$.*

The proof uses one direction (usually referred to as the 'easy direction') of the variational principle for pressure. It holds for a large class of potentials. We now present a short proof of this inequality.

LEMMA 3.2. *Let $\mathfrak{F} = (f_n)_{n\geq 1}$ be a subadditive potential on (Y, S) consisting of Borel-measurable functions, with f_1 bounded from above. Then for any invariant measure ν we have*

$$\underline{\mathbf{P}}(\mathfrak{F}) \geq h_{\nu}(S) + \lim_n \frac{1}{n} \int f_n \, d\nu$$

(with the convention that $\infty - \infty = -\infty$). Obviously, the same holds for $\mathbf{P}(\mathfrak{F})$.

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Proof. Of course the limit on the right exists by subadditivity. Let us denote it by L. If $L = -\infty$ then the inequality is trivial. In the remaining case it suffices to consider an ergodic measure ν. By the subadditive ergodic theorem (see [**9**, Theorem 5.3]), the functions $(1/n) f_n$ converge ν-almost surely to the (finite) constant L. So, given $\delta > 0$, there exist a set Y_0 of positive measure and an n_0 such that, for every $y \in Y_0$ and $n \geq n_0$, we have $f_n(y) > n(L - \delta)$. Then, for any $\epsilon > 0$ and $n \geq n_0$,

$$\begin{aligned} \mathsf{S}_{n,\epsilon}(\mathfrak{F}) &\geq \sup\left\{\sum_{y\in E} 2^{f_n(y)} : E \text{ is an } (n, \epsilon)\text{-separated set in } Y_0\right\} \\ &\geq 2^{n(L-\delta)} \sup\{\#E : E \text{ is an } (n, \epsilon)\text{-separated set in } Y_0\}. \end{aligned}$$

Taking logarithms, dividing by n and passing to $\liminf_n$, we obtain

$$\underline{\mathbf{P}}(\mathfrak{F}) \geq (L - \delta) + \liminf_{n\to\infty} \frac{1}{n} \log \sup\{\#E : E \text{ is an } (n, \epsilon)\text{-separated set in } Y_0\}.$$

Since $\nu(Y_0) > 0$, the $\liminf_n$ on the right is, for sufficiently small ϵ, not smaller than $h_\nu(S) - \delta$ (if $h_\nu(S) < \infty$, or larger than any positive constant, if $h_\nu(S) = \infty$); this follows from a result of Katok (see [**13**, Theorem 8.19]). Since δ is arbitrary, the lemma is proved. □

We now return to the relatively symbolic extension (X, T) of (Y, S) and the fiber entropy potential $\mathfrak{H}$. We will need the following easy fact.

LEMMA 3.3. *For every positive $\epsilon < 1$ and every $n \geq 1$, we have*

$$\mathsf{s}(n, \epsilon) \geq \mathsf{S}_{n,\epsilon}(\mathfrak{H}),$$

where $\mathsf{s}(n, \epsilon)$ refers to (X, T) $(= Y \vee Z)$ with the 'maximum metric' $d((y, z), (y', z')) = \max\{d(y, y'), d(z, z')\}$.

Proof. Because $\mathcal{U}^n$ is a partition, $\mathbf{H}(\mathcal{U}^n | y)$ equals the logarithm of the number of cylinders of length n in Z which have non-empty intersection with the fiber of y in the joining X. So, the term $2^{\mathbf{H}_n(y)}$ in the definition of $\mathsf{S}_{n,\epsilon}(\mathfrak{H})$ is this number of cylinders. Consider an (n, ϵ)-separated set E in Y. Now lift the set E to a subset of X by choosing one point in every cylinder intersecting the fiber of $y \in E$. Since $\epsilon < 1$, single symbols in Z are more than ϵ apart, hence we will obtain an (n, ϵ)-separated set in X (perhaps not maximal) of cardinality $\sum_{y\in E} 2^{\mathbf{H}_n(y)}$, implying $\mathsf{s}(n, \epsilon) \geq \sum_{y\in E} 2^{\mathbf{H}_n(y)}$. We conclude the proof by taking the supremum over all sets E. □

As an immediate consequence of Lemma 3.3 we obtain the following result.

COROLLARY 3.4. *We have*

$$\mathbf{h}_{\text{top}}(T) \geq \mathbf{P}(\mathfrak{H}).$$

Obviously, the same holds for $\underline{\mathbf{P}}(\mathfrak{H})$.

We can now prove Theorem 3.1.

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Proof of Theorem 3.1. It is obvious that $\mathfrak{H}$ is non-negative and non-decreasing. Subadditivity and upper semicontinuity of such a potential is shown in [**7**] (see also [**4**, Fact 6.7.3]). Further, for $\mathbf{H}_n(y) = \mathbf{H}(\mathcal{U}^n|y)$, the limit $\lim_n (1/n) \int \mathbf{H}_n \, d\nu$ equals the (dynamic) fiber entropy $\mathbf{h}(\mathcal{U}|\nu)$ (see the definition provided in §2). Moreover, the number $\mathbf{H}(\mathcal{U}^n|y)$ will not change if we join $\mathcal{U}$ with any open cover $\mathcal{V}$ lifted from Y, and such joined covers 'generate' in X, i.e. every open cover of X is refined by a cover of the form $(\mathcal{U} \vee \mathcal{V})^n$. Thus $\mathbf{h}(\mathcal{U}|\nu) = \mathbf{h}(T|\nu)$ and so

$$h_\nu(S) + \lim_n \frac{1}{n} \int \mathbf{H}_n \, d\nu = h_\nu(S) + \mathbf{h}(\mathcal{U}|\nu) = h_\nu(S) + \mathbf{h}(T|\nu) = h^\pi(\nu) \tag{3.1}$$

(we have used formula (2.8)). So, eventually

$$\begin{aligned} \mathbf{h}_{\mathrm{top}}(T) &\geq \mathbf{P}(\mathfrak{H}) \\ &\geq \sup\left\{ h_\nu(S) + \lim_n \frac{1}{n} \int \mathbf{H}_n \, d\nu : \nu \text{ is an invariant measure on } Y \right\} \\ &= \sup\{h^\pi(\nu) : \nu \text{ is an invariant measure on } Y\} \\ &= \sup\{h_\mu(T) : \mu \text{ is an invariant measure on } X\}. \end{aligned} \tag{3.2}$$

The variational principle (for entropy) closes the chain, concluding the proof. □

Notice that, for the fiber entropy potential, we have given an alternative proof of the variational principle for pressure. We have also shown that in this case

$$\underline{\mathbf{P}}(\mathfrak{H}) = \mathbf{P}(\mathfrak{H}).$$

We end this section with two observations. The first one is rather trivial: the fiber entropy potential assumes only countably many values, so it can be continuous only on disconnected spaces. The second remark is an interpretation of the case where the fiber entropy potential is additive. It is not hard to see that this corresponds precisely to the case when the relatively symbolic extension is (topologically) *relatively independent*, i.e. once the single symbols allowed for each $y \in Y$ in the joining are determined, then for any $n \geq 1$ the allowed n-cylinders for y are *all* blocks obtained by concatenating symbols allowed for $y, Sy, \ldots, S^{n-1}y$.

4. *Potential as a fiber entropy*

In this section we will prove a perhaps more interesting reverse statement: every potential (in a suitable class) is 'equivalent' to the fiber entropy potential in some relatively symbolic extension. Notice that we do not assume $\mathfrak{F}$ to be non-decreasing.

THEOREM 4.1. *Let (Y, S) be a topological dynamical system, and let $\mathfrak{F} = (f_n)_{n\geq 1}$ be a non-negative, subadditive and upper semicontinuous potential on Y. Then there exists a relatively symbolic extension $(X, T) = Y \vee Z$ such that the fiber entropy potential $\mathfrak{H} = (\mathbf{H}_n)_{n\geq 1}$, where $\mathbf{H}_n(y) = \mathbf{H}(\mathcal{U}^n|y)$ and $\mathcal{U}$ denotes the zero-coordinate partition (and cover) of X corresponding to the symbols in Z, satisfies*

$$\lim_n \frac{1}{n} \int \mathbf{H}_n \, d\nu = \lim_n \frac{1}{n} \int f_n \, d\nu \text{ for every invariant measure } \nu \text{ on } Y; \tag{4.1}$$

$$\mathbf{P}(\mathfrak{F}) = \mathbf{P}(\mathfrak{H}). \tag{4.2}$$

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In order to prove this theorem we must introduce a number of technical tools. To begin, we will associate with an arbitrary potential $\mathfrak{F}$ two sequences of integers, (n_k) and (p_k), the first of which is established in the following, quite trivial, lemma.

LEMMA 4.2. *Let $\mathfrak{F}$ be an arbitrary potential on Y. There exists a sequence (n_k), with $n_1 = 1$ and $n_{k+1} > 2n_k$ for every $k \geq 1$, and $n_k/n_{k+1} \to 0$, such that*

$$\mathbf{P}(\mathfrak{F}) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{n_k} \log \mathsf{S}_{n_k,\epsilon}(\mathfrak{F}).$$

Proof. Assume that the pressure is finite. Since both limits in $\epsilon \to 0$ (the one above and the one in the definition of $\mathbf{P}(\mathfrak{F})$) are non-decreasing, these limits can be replaced by limits over a fixed sequence ϵ_j decreasing to zero. Consider the following sequence of pairs:

$$(\epsilon_1, \delta_1), (\epsilon_1, \delta_2), (\epsilon_2, \delta_2), (\epsilon_1, \delta_3), (\epsilon_2, \delta_3),$$
$$(\epsilon_3, \delta_3), (\epsilon_1, \delta_4), (\epsilon_2, \delta_4), (\epsilon_3, \delta_4), (\epsilon_4, \delta_4), \ldots,$$

where the δ_i tend to zero. We let $n_1 = 1$ and then, for the kth consecutive pair ($k = 2, 3, \ldots$) of the form (ϵ_j, δ_i), we choose n_k so that

$$\frac{1}{n_k} \log \mathsf{S}_{n_k,\epsilon_j}(\mathfrak{F}) > \limsup_{n \to \infty} \frac{1}{n} \log \mathsf{S}_{n,\epsilon_j}(\mathfrak{F}) - \delta_i \quad \text{and} \quad n_k > 2n_{k-1}^2.$$

We skip the similar proof in the case of infinite pressure (as we shall not use it). □

Once (n_k) is established, we let $p_1 = 1$, and for $k \geq 1$ we let p_{k+1} be the largest multiple of p_k not exceeding n_{k+1}. Since $n_{k+1} \geq 2n_k \geq 2p_k$, we have $p_{k+1} \geq 2p_k$, i.e. the sequence (p_k) is a base of some odometer (G, τ) (see below for an explanation). We also note two easy convergences:

$$\frac{n_k}{p_k} \to 1 \quad \text{and} \quad \frac{p_k}{p_{k+1}} \to 0.$$

Our desire is to replace the sequence (n_k) in the formulation of Lemma 4.2 by (p_k).

LEMMA 4.3. *If the potential $\mathfrak{F}$ is subadditive with f_1 bounded from above then*

$$\mathbf{P}(\mathfrak{F}) = \lim_{\epsilon \to 0} \limsup_{k \to \infty} \frac{1}{p_k} \log \mathsf{S}_{p_k,\epsilon}(\mathfrak{F}).$$

The proof is preceded by another lemma.

LEMMA 4.4. *Assume that the potential $\mathfrak{F}$ on a dynamical system (Y, S) is subadditive with f_1 bounded from above. Then for every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that, for every $n \geq 1$,*

$$\mathsf{S}_{n+1,\epsilon}(\mathfrak{F}) \leq C_\epsilon \cdot \mathsf{S}_{n,\epsilon}(\mathfrak{F}).$$

Proof. Fix a finite cover $\{B_1, B_2, \ldots, B_{N_\epsilon}\}$ of Y by $\epsilon/2$-balls and consider an $(n+1, \epsilon)$-separated set E. Denote $E_i = E \cap T^{-n}(B_i)$ ($i = 1, 2, \ldots, N_\epsilon$). The points in E_i are not ϵ-separated in the nth iterate, thus they must be (n, ϵ)-separated. In this manner, E is represented as the (not necessarily disjoint) union of N_ϵ (n, ϵ)-separated sets E_i. Thus, for at least one index i we must have

$$\sum_{y \in E} 2^{f_{n+1}(y)} \leq N_\epsilon \sum_{y \in E_i} 2^{f_{n+1}(y)} \leq 2^l N_\epsilon \sum_{y \in E_i} 2^{f_n(y)} \leq C_\epsilon \cdot \mathsf{S}_{n,\epsilon}(\mathfrak{F}),$$

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where l is an upper bound for f_1 and C_ϵ is defined as $2^l N_\epsilon$. The application of the supremum over all $(n+1, \epsilon)$-separated sets E yields the assertion. □

Proof of Lemma 4.3. For a subadditive potential with f_1 bounded from above we have, by Lemma 4.4, for every $k \geq 1$,

$$\frac{1}{n_k} \log \mathsf{S}_{n_k,\epsilon}(\mathfrak{F}) \leq \frac{(n_k - p_k)}{n_k} \log(C_\epsilon) + \frac{1}{n_k} \log \mathsf{S}_{p_k,\epsilon}(\mathfrak{F}),$$

which has the same limit superior as $(1/p_k) \log \mathsf{S}_{p_k,\epsilon}(\mathfrak{F})$. □

Since we will work with systems having odometer factors, we introduce a few definitions and lemmas concerning such systems.

Definition 4.5. Let (X, T) be a dynamical system and choose $p \in \mathbb{N}$. A set $F \subset X$ is called a *p-marker* if it is closed and open (we will say *clopen*) and $\{F, T^{-1}(F), T^{-2}(F), \ldots, T^{-p+1}(F)\}$ is a partition of X. By a *system of markers* we will mean a sequence of sets $(F_k)_{k\geq 1}$ such that F_k is a p_k-marker, $p_1 = 1$, and $F_{k+1} \subsetneq F_k$, for each $k \geq 1$.

It follows from the above definition that a p-marker F is always T^p-forward invariant (i.e. $T^p(F) \subset F$); a 1-marker equals X. Moreover, if a system of markers exists then p_{k+1} is an essential multiple of p_k (i.e. $p_{k+1} \geq 2p_k$), for every k.

If (X, T) admits a system of markers such that the associated partitions separate points, then it is called the *odometer to base* (p_k). For a fixed base (p_k), the odometer is unique up to topological conjugacy and it will be denoted by (G, τ). It is well known that (G, τ) is minimal and equicontinuous, hence, for an appropriate invariant metric, τ is an isometry. The marker sets F_k form a base of the topology at zero. From now on, we will always regard odometers as equipped with such an invariant metric. We refer to [**5**] for more details concerning odometers.

For a system (X, T), the condition of admitting a system of markers (without satisfying the separation condition) is equivalent to having the corresponding odometer as a topological factor.

LEMMA 4.6. *Suppose that (X, T) has an odometer factor with base (p_k) and marker sets F_k. Define $\mathsf{r}'(p_k, \epsilon)$ and $\mathsf{s}'(p_k, \epsilon)$ as the minimal cardinality of a (p_k, ϵ)-spanning set in F_k and the maximal cardinality of a (p_k, ϵ)-separated set in F_k, respectively. Then*

$$\lim_{\epsilon\to 0} \liminf_{k\to\infty} \frac{1}{p_k} \log(\mathsf{r}'(p_k, \epsilon)) = \lim_{\epsilon\to 0} \limsup_{k\to\infty} \frac{1}{p_k} \log(\mathsf{s}'(p_k, \epsilon)) = \mathbf{h}_{\mathrm{top}}(T).$$

Proof. Clearly, $\mathsf{r}'(p_k, \epsilon) \leq \mathsf{s}'(p_k, \epsilon) \leq \mathsf{s}(p_k, \epsilon)$, which implies both inequalities '$\leq$'. It remains to prove that the left-hand side dominates the entropy. Denote by $\mathcal{U}_{(p_k,\epsilon)}$ the cover by the (p_k, ϵ)-balls. Notice that this is a subcover of $\mathcal{U}^{p_k}_{(1,\epsilon)} = \bigvee_{i=0}^{p_k-1} T^{-i}(\mathcal{U}_{(1,\epsilon)})$. Thus,

$$\mathsf{r}'(p_k, \epsilon) = \mathbf{N}(\mathcal{U}_{(p_k,\epsilon)}|F_k) \geq \mathbf{N}(\mathcal{U}^{p_k}_{(1,\epsilon)}|F_k)$$

(compare [**4**, (6.1.10)]). Because F_k is T^{p_k}-invariant, we can consider the system (F_k, T^{p_k}). It is well known (and fairly obvious) that for any topological dynamical system (Y, S) and any open cover $\mathcal{V}$ of Y, the number $\log(\mathbf{N}(\mathcal{V}))$ estimates from above the

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topological entropy $\mathbf{h}(S, \mathcal{V})$ (which is defined as $\lim_n (1/n) \log \mathbf{N}(\mathcal{V}^n)$; see [**4**, (6.1.6)], then use properties of subadditive sequences). So,

$$\log(\mathsf{r}'(p_k, \epsilon)) \geq \mathbf{h}(T^{p_k}, \mathcal{V}) = \mathbf{h}(T^{p_k}, \mathcal{V}^2),$$

where $\mathcal{V} = \mathcal{U}^{p_k}_{(1,\epsilon)}|_{F_k}$ and $\mathcal{V}^2 = \mathcal{U}^{2p_k}_{(1,\epsilon)}|_{F_k}$ (the 'power' p_k is computed with respect to T, while the 'power' 2 is computed with respect to T^{p_k}). The second equality is [**4**, Fact 6.2.4]. For each $i = 1, 2, \ldots, p_k - 1$ observe the system $(T^{-i}(F_k), T^{p_k})$ and the cover $\mathcal{U}^{p_k}_{(1,\epsilon)}|_{T^{-i}(F_k)}$. Note that by applying $T^{-(p_k - i)}$ this cover is transformed to a cover of $T^{-p_k}(F_k) = F_k$ which is refined by $\mathcal{V}^2$. This implies that

$$\mathbf{h}(T^{p_k}, \mathcal{U}^{p_k}_{(1,\epsilon)}|_{T^{-i}(F_k)}) \leq \mathbf{h}(T^{p_k}, \mathcal{V}^2),$$

for each $i = 0, 1, \ldots, p_k - 1$, and hence

$$\log(\mathsf{r}'(p_k, \epsilon)) \geq \mathbf{h}(T^{p_k}, \mathcal{U}^{p_k}_{(1,\epsilon)}),$$

where now both the map T^{p_k} and the cover $\mathcal{U}^{p_k}_{(1,\epsilon)}$ are considered on the entire space X (the right-hand side equals the maximum, over $i = 0, 1, \ldots, p_k - 1$, of the entropies restricted to the sets $T^{-i}(F_k)$). Now we can apply the 'power rule' (see the proof of Fact 6.2.3 in [**4**]) to obtain that

$$\mathbf{h}(T^{p_k}, \mathcal{U}^{p_k}_{(1,\epsilon)}) = p_k \, \mathbf{h}(T, \mathcal{U}_{(1,\epsilon)}).$$

The desired inequality follows directly by dividing by p_k, passing to the limit inferior over k and passing with ϵ to zero. □

LEMMA 4.7. *For a non-negative, subadditive potential $\mathfrak{F} = (f_n)_{n\geq 1}$ with f_1 bounded from above, on a system having an odometer factor (with base (p_k) and marker sets F_k) we have, for every invariant measure ν,*

$$\lim_n \frac{1}{n} \int f_n \, d\nu = \lim_k \int_{F_k} f_{p_k} \, d\nu.$$

Proof. For n much larger than p_k the n-orbit $(y, Sy, \ldots, S^{n-1}y)$ of any $y \in Y$ breaks as

$$\begin{aligned}(y, Sy, \ldots, S^{p-1}y), (y_1, Sy_1, \ldots, S^{p_k-1}y_1), (y_2, Sy_2, \ldots, S^{p_k-1}y_2), \ldots \\ \ldots, (y_q, Sy_q, \ldots, S^{p_k-1}y_q), (y', Sy', \ldots, S^{r-1}y'),\end{aligned} \tag{4.3}$$

where both y' and each y_i belong to F_k $(i = 1, 2, \ldots, q)$, $q = \lfloor n/p_k \rfloor$ or $q = \lfloor n/p_k \rfloor - 1$, $p, r \in \{0, \ldots, p_k - 1\}$ (if $p = 0$ or $r = 0$, the corresponding part of the orbit is empty). In fact p is identified as the smallest non-negative integer such that $y \in S^{-p}(F_k)$. Thus, by subadditivity,

$$f_n(y) \leq f_p(y) + \sum_{i=1}^{q} f_{p_k}(y_i) + f_r(y').$$

The sum of the first and last terms is bounded from above by $2p_k l$ where l is an upper bound for f_1. We integrate over $y \in S^{-p}(F_k)$. Each y_i ranges over F_k, hence, by invariance of the measure, we will get

$$\int_{S^{-p}(F_k)} f_n \, d\nu \leq q \int_{F_k} f_{p_k} \, d\nu + 2l \tag{4.4}$$

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(the term p_k in the estimate of the extreme terms cancels with the measure $1/p_k$ of $S^{-p}(F_k)$). Now we can divide both sides by n and sum over all sets $S^{-p}(F_k)$ as $p = 0, \ldots, p_k - 1$ to get

$$\frac{1}{n}\int f_n \, d\nu \le \frac{qp_k}{n}\int_{F_k} f_{p_k} \, d\nu + \frac{2p_k l}{n}.$$

Since, for large n, qp_k/n is close to 1 and the last term is small, we obtain

$$\lim_n \frac{1}{n}\int f_n \, d\nu \le \int_{F_k} f_{p_k} \, d\nu. \tag{4.5}$$

Thus we have obtained 'half' of the assertion of the lemma:

$$\lim_n \frac{1}{n}\int f_n \, d\nu \le \liminf_k \int_{F_k} f_{p_k} \, d\nu.$$

On the other hand, for k' much larger than k, $y \in F_{k'}$ and $p \in \{0, \ldots, p_k - 1\}$ we can break the $p_{k'}$-orbit of y as

$$\begin{aligned}&(y, Sy, \ldots, S^{p_k-p-1}y), (y_1, Sy_1, \ldots, S^{p_k-1}y_1), (y_2, Sy_2, \ldots, S^{p_k-1}y_2), \ldots\\ &\quad \ldots, (y_q, Sy_q, \ldots, S^{p_k-1}y_q), (y', Sy', \ldots, S^{p-1}y'),\end{aligned} \tag{4.6}$$

where now both y' and each y_i belong to $S^{-p}(F_k)$. Thus

$$f_{p_{k'}}(y) \le f_{p_k-p}(y) + \sum_{i=1}^{q} f_{p_k}(y_i) + f_p(y').$$

The extreme terms together are estimated from above by $p_k l$. As we integrate over $y \in F_{k'}$, each y_i (for fixed i) ranges over $S^{ip_k-p}(F_{k'}) \subset S^{-p}(F_k)$, and it is not hard to notice that these sets (with $i = 1, \ldots, q$) are disjoint (because points from $F_{k'}$ return to $F_{k'}$ only after $p_{k'} > qp_k$ steps). Thus,

$$\sum_{i=1}^{q}\int_{F_{k'}} f_{p_k}(y_i) \, d\nu(y) \le \int_{S^{-p}(F_k)} f_{p_k} \, d\nu,$$

hence we obtain

$$\int_{F_{k'}} f_{p_{k'}} \, d\nu \le \int_{S^{-p}(F_k)} f_{p_k} \, d\nu + \frac{p_k l}{p_{k'}}. \tag{4.7}$$

Now we can average over $p = 0, 1, \ldots, p_k - 1$ to get

$$\int_{F_{k'}} f_{p_{k'}} \, d\nu \le \frac{1}{p_k}\int f_{p_k} \, d\nu + \frac{p_k l}{p_{k'}}. \tag{4.8}$$

Applying lim sup as k' goes to infinity, and then the limit over k, we get

$$\limsup_{k'} \int_{F_{k'}} f_{p_{k'}} \, d\nu \le \lim_k \frac{1}{p_k}\int f_{p_k} \, d\nu = \lim_n \frac{1}{n}\int f_n \, d\nu. \tag{4.9}$$

This completes the proof of the lemma. □

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LEMMA 4.8. *Let (Y, S) be a topological dynamical system and let $\mathfrak{F} = (f_n)_{n\geq 1}$ be a non-negative and subadditive potential. Let (Y', S') be the direct product of (Y, S) with the odometer (G, τ) (i.e. $Y' = Y \times G$ with the standard 'maximum metric' and $S'(y, g) = (Sy, \tau g)$), where the odometer has the base (p_k) established after Lemma 4.2. On Y' define the potential $\mathfrak{F}' = (f'_n)_{n\geq 1}$ by lifting: $f'_n(y, g) = f_n(y)$. Then this new potential is obviously non-negative and subadditive. Let F_k denote the p_k-marker in G. The set $F'_k = Y \times F_k$ is a p_k-marker in Y'. Set*

$$\mathcal{S}'_{p_k,\epsilon}(\mathfrak{F}') = \sup\left\{\sum_{y'\in E'} 2^{f'_{p_k}(y')} : E' \text{ is a } (p_k, \epsilon)\text{-separated set in } F'_k\right\}$$

and

$$\mathbf{P}'(\mathfrak{F}') = \lim_{\epsilon\to 0} \limsup_{k\to\infty} \frac{1}{p_k} \log \mathcal{S}'_{p_k,\epsilon}(\mathfrak{F}').$$

Then

$$\mathbf{P}'(\mathfrak{F}') = \mathbf{P}(\mathfrak{F}).$$

Proof. Note that for fixed $\epsilon > 0$, if k is sufficiently large then the set F_k has diameter smaller than ϵ in G. Together with the fact that τ is an isometry this easily implies, for every sufficiently large k, that a set $E' \subset F'_k$ is (p_k, ϵ)-separated if and only if its projection E onto Y is (p_k, ϵ)-separated, and in this case the projection is a bijection from E' to E, and preserves the values of the potential. Thus, we obtain $\mathsf{S}'_{p_k,\epsilon}(\mathfrak{F}') = \mathsf{S}_{p_k,\epsilon}(\mathfrak{F})$ and the assertion now follows from Lemma 4.3. □

An extension $(\tilde{Y}, \tilde{S})$ of (Y, S) (via a map $\tilde{\pi} : \tilde{Y} \to Y$) is called *principal* if $h_{\tilde{\nu}}(\tilde{S}|S) = 0$ for every invariant measure $\tilde{\nu}$ on $\tilde{Y}$. In particular, $h_{\tilde{\nu}}(\tilde{S}) = h_\nu(S)$ where $\nu = \tilde{\pi}(\tilde{\nu})$. It is elementary that a composition of principal extensions is a principal extension. In our main proof the following fact will be exploited (see [**6**]).

THEOREM 4.9. *Every topological dynamical system (Y, S) has a zero-dimensional principal extension $(\tilde{Y}, \tilde{S})$.*

We are now in a position to deliver the main proof.

Proof of Theorem 4.1. Initially, we will build a relatively symbolic extension (X', T') not of (Y, S) (equipped with the upper semicontinuous potential $\mathfrak{F}$) but only of its extension (Y', S') and a continuous potential $\mathfrak{G}'$ obtained as follows. First we replace (Y, S) by its principal zero-dimensional extension $(\tilde{Y}, \tilde{S})$, on which we consider the potential $\tilde{\mathfrak{F}} = (\tilde{f}_n)_{n\geq 1}$ *lifted* from $\mathfrak{F}$:

$$\tilde{f}_n(\tilde{y}) = f_n(y),$$

where $y = \tilde{\pi}(\tilde{y})$ and $\tilde{\pi}$ is the factor map from $\tilde{Y}$ to Y. It is clear that this passage preserves the limit integrals, i.e. if $\nu = \tilde{\pi}(\tilde{\nu})$ then

$$\lim_n \frac{1}{n} \int \tilde{f}_n \, d\tilde{\nu} = \lim_n \frac{1}{n} \int f_n \, d\nu.$$

It is an easy general fact that the lifted potential has at least the same pressure†: $\mathbf{P}(\tilde{\mathfrak{F}}) \geq \mathbf{P}(\mathfrak{F})$. Then, according to the last statement of Theorem 2.5 (concerning zero-dimensional

† It is true that the pressure is preserved, but this does not follow directly without using the 'hard' direction of the variational principle for pressure, and we will not use it.

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spaces and non-negative potentials), we replace the upper semicontinuous potential $\tilde{\mathfrak{F}}$ by a slightly larger one, say $\tilde{\mathfrak{G}} = (\tilde{g}_n)_{n\geq 1}$, which is continuous and assumes as values only logarithms of integers, maintaining the pressure and the limit integrals for all invariant measures. Next, we let (Y', S') be the direct product of $(\tilde{Y}, \tilde{S})$ with the odometer (G, τ) to the base (p_k) satisfying the assertion of Lemma 4.3 applied to $(\tilde{Y}, \tilde{S})$ and the potential $\tilde{\mathfrak{G}}$. Notice that since the odometer has topological entropy zero, (Y', S') is also a principal extension of (Y, S). On Y' we regard the potential $\mathfrak{G}' = (g'_n)_{n\geq 1}$ lifted from $\tilde{\mathfrak{G}}$. Obviously, the functions g'_n are continuous, assume as values logarithms of integer numbers, and the limit integrals of $\mathfrak{G}'$ with respect to invariant measures ν' on Y' are the same as the limit integrals of $\mathfrak{F}$ for the measures $\nu = \pi'(\nu')$, where π' is the (composed) factor map from Y' onto Y. For the time being we make no statements about the pressure of $\mathfrak{G}'$, but in due course we will use the assertion of Lemma 4.8 (applied to $(\tilde{Y}, \tilde{S})$ and the potential $\tilde{\mathfrak{G}}$).

The relatively symbolic extension (X', T') of (Y', S') (the construction of which will occupy most of the proof) will determine a fiber entropy potential $\mathfrak{H}' = (\mathbf{H}'_n)_{n\geq 1}$ on Y'. In order to avoid an abundance of 'primes', from now until specified otherwise, we will write (Y, S), y, F_k, g_n and $\mathbf{H}_n$ in place of (Y', S'), y', F'_k, g'_n and $\mathbf{H}'_n$, respectively.

In fact, we will build a decreasing (with respect to inclusion) sequence of subshifts Z_k (over a common alphabet Λ) and decreasing joinings $Y \vee Z_k$, so that in the intersection we will obtain a joining of Y with $Z = \bigcap Z_k$.

Because each of the extensions $Y \vee Z_k$ comes with its own sequence of fiber entropies (and the dependence on k is not shown in the notation), we write

$$\mathbf{H}_n^{(k)}(y) = \mathbf{H}(\mathcal{U}^n | y) \quad \text{in the extension } Y \vee Z_k$$

(where $\mathcal{U}$ denotes the zero-coordinate partition corresponding to the symbols in Λ). The joining $Y \vee Z_k$ will be defined by assigning to every $y \in F_k$ a family $\mathcal{F}_k(y)$ of blocks of length p_k (over a specified earlier alphabet Λ), which will be called *k-blocks allowed for y*. Then we include in the joining all pairs (y, z), where z is a symbolic sequence satisfying the following condition:

$$\text{whenever } S^j(y) \in F_k \text{ then } z[j,\ j + p_k - 1] \text{ is a } k\text{-block allowed for } S^j y. \tag{4.10}$$

(This rule may be insufficient to determine z at coordinates left from the first such j, but only for non-invertible actions and only for $k > 1$; in such a case we will admit here all configurations allowed in step $k - 1$). The thus determined collection of pairs (y, z) is shift-invariant (which is clear in the invertible case, while in the non-invertible case forward shift invariance will follow by some consistency rules), and later we will verify that our procedure makes it closed (in the product topology). It is also clear that for $y \in F_k$ the fiber entropy $\mathbf{H}_{p_k}^{(k)}(y)$ (in the extension $Y \vee Z_k$) equals the logarithm of the cardinality of the family $\mathcal{F}_k(y)$.

We are ready to build the joinings. We denote $l = \max\{2^{g_1}(y) : y \in Y\}$ and let $\Lambda = \{1, 2, \ldots, l\}$. This will be the alphabet in all our symbolic systems Z_k.

In step 1, for each $y \in F_1 = Y$ we should assign a family of p_1-blocks, i.e. of single symbols (recall that $p_1 = 1$). We set

$$\mathcal{F}_1(y) = \{1, 2, 3, \ldots, 2^{g_1(y)}\}.$$

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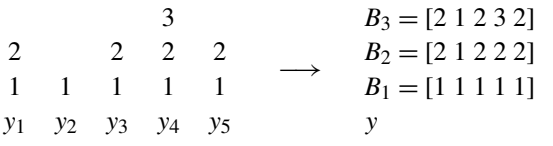


FIGURE 1. On the left, above each $y_i = S^{i-1}(y)$ we show a list of allowed 1-blocks. On the right, above $y \in F_2$ we show the resulting first $l_1(y)$ allowed 2-blocks.

Note that since 2^{g_1} is continuous and assumes values in a discrete set, it is locally constant, which implies that the same family of symbols is assigned on a closed-and-open set. So, the set of pairs (y, z) resulting from the rule (4.10) is closed. In this manner, we have defined $Y \vee Z_1$ so that $\mathbf{H}_1^{(1)}(y) = g_1(y)$ for all $y \in F_1 = Y$. In step 2 we should assign 2-blocks (of lengths p_2) allowed for $y \in F_2$. After step 1 there are

$$2^{g_1(y_1)+g_1(y_2)+\cdots+g_1(y_{q_1})}$$

(we are using the notation of (4.3) with $p = r = 0$ and $q_1 = p_2/p_1$) blocks of length p_2 allowed for y (we do not call them 2-blocks yet; only some of these blocks will receive this name). The complication that follows now is necessary to assure that in the future steps we will not 'lose' any of the allowed 1-blocks. In the above collection of blocks of length p_2 we introduce a special order: we denote $l_1(y) = \max\{2^{g_1(y_i)} : i = 1, \ldots, q_1\}$ and then the first $l_1(y)$ blocks B_j $(j = 1, 2, \ldots, l_1(y))$ are selected by the rule

$$B_j(i) = \min\{j, 2^{g_1(y_i)}\} \quad (i = 1, 2, \ldots, q_1).$$

That is, B_j consists of the symbols j, except at positions i where j is not available—the largest available symbol appears there (see Figure 1).

The rest of the blocks are ordered lexicographically. By subadditivity of the potential $\mathfrak{G}$, the number $2^{g_1(y_1)+g_1(y_2)+\cdots+g_1(y_{q_1})}$ of available blocks is not less than $2^{g_{p_2}(y)}$ (and, of course, than $l_1(y)$). We now select the initial $\max\{2^{g_{p_2}(y)}, l_1(y)\}$ of them. This is our family $\mathcal{F}_2(y)$ of 2-blocks allowed for y. Again, the function $\max\{2^{g_{p_2}(y)}, l_1(y)\}$ is locally constant and all the functions (of y) $2^{g_1(y)}, \ldots, 2^{g_1(S^{p_2-1}y)}$ are locally constant, hence the same family of 2-blocks is allowed over a closed-and-open set of ys. So, the resulting set of pairs (y, z) is closed†. We have obtained an extension $Y \vee Z_2 \subset Y \vee Z_1$ such that, for $y \in F_2$,

$$g_{p_2}(y) \le \mathbf{H}_{p_2}^{(2)}(y) = \log(\max\{2^{g_{p_2}(y)}, l_1(y)\}) \le \max\{g_{p_2}(y), \log l\}.$$

Notice that, by including the initial blocks $B_1, \ldots, B_{l_1(y)}$ in $\mathcal{F}_2(y)$, we have assured that for every $y_i = S^{i-1}(y)$ $(i = 1, \ldots, q_1)$ all formerly allowed 1-blocks (i.e. symbols) are used at least once, hence we have $\mathbf{H}_1^{(2)}(y) = \mathbf{H}_1^{(1)}(y)$ for all $y \in F_1 = Y$.

† It is in this very place that we need the potential $\mathfrak{G}$ to be continuous, and since it assumes values in a discrete set, this also requires the space Y to be zero-dimensional. Otherwise we would be dealing with an upper semicontinuous potential and unable to arrange the sets Z_k (for $k > 1$) to be closed, maintaining the cardinality of $\mathcal{F}_k(y)$ close enough to $g_{p_k}(y)$.

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After step k, for each $y \in F_{k+1}$ there are

$$2^{\mathbf{H}^{(k)}_{p_k}(y_1)+\mathbf{H}^{(k)}_{p_k}(y_2)+\cdots+\mathbf{H}^{(k)}_{p_k}(y_{q_k})}$$

(we are using again the notation of (4.3) with $p = r = 0$ and $q_k = p_{k+1}/p_k$) blocks of length p_{k+1} allowed for y. We order them analogously as in step 1 (with the reference to the order established in step k), so that the first

$$l_k(y) = \max\{\mathbf{H}^{(k)}_{p_k}(y_i) : i = 1, \ldots, q_k\}$$

of them guarantee that all allowed k-blocks appear (at the appropriate positions) at least once. Then we select the initial $\max\{2^{g_{p_{k+1}}(y)}, l_k(y)\}$ (which is not more than we have) of them to create $\mathcal{F}_{k+1}(y)$. In this manner we build an extension $Y \vee Z_{k+1} \subset Y \vee Z_k$ such that

$$\text{for } y \in F_{k+1}, \quad g_{p_{k+1}}(y) \le \mathbf{H}^{(k+1)}_{p_{k+1}}(y) \le \max\{g_{p_{k+1}}(y), \log(l^{p_k})\} \le g_{p_{k+1}}(y) + p_k \log l \tag{4.11}$$

and maintaining formerly established values of $\mathbf{H}^{(j)}_{p_j}(y)$ for $j \le k$ and $y \in F_j$. In the end, the final extension $Y \vee Z$ (obtained by intersecting of $Y \vee Z_k$ over k) satisfies, for every $k \ge 1$,

$$\text{for } y \in F_{k+1}, \quad g_{p_{k+1}}(y) \le \mathbf{H}_{p_{k+1}}(y) \le g_{p_{k+1}}(y) + p_k \log l. \tag{4.12}$$

This concludes the construction of the relatively symbolic extension $Y \vee Z$.

We recall that we have been working with (Y', S'), and we now restore the proper notation (with primes). The inequalities (4.12) take on the form

$$\text{for } y' \in F'_{k+1}, \quad g'_{p_{k+1}}(y') \le \mathbf{H}'_{p_{k+1}}(y') \le g'_{p_{k+1}}(y') + p_k \log l. \tag{4.13}$$

By Lemma 4.7, the inequalities (4.13), the choice of $\mathfrak{G}'$ and since the fractions p_k/p_{k+1} tend to zero with k, we get a version of (4.1):

$$\lim_n \frac{1}{n} \int \mathbf{H}'_n \, d\nu' = \lim_n \frac{1}{n} \int g'_n \, d\nu' = \lim_n \frac{1}{n} \int f_n \, d\nu, \tag{4.14}$$

for every invariant measure ν' on Y' and its projection ν on Y.

So, using consequently the variational principle for the fiber entropy potential (Theorem 3.1—more precisely, the equality in (3.2)), the equality (4.14), the 'easy direction' of the variational principle for pressure (Lemma 3.2), Lemma 4.8 and the estimates (4.13), we can write

$$\begin{aligned}
\mathbf{P}(\mathfrak{H}') &= \sup\left\{h_{\nu'}(S') + \lim_n \frac{1}{n} \int \mathbf{H}'_n \, d\nu' : \nu' \text{ is an invariant measure on } Y'\right\} \\
&= \sup\left\{h_\nu(S) + \lim_n \frac{1}{n} \int f_n \, d\nu : \nu \text{ is an invariant measure on } Y\right\} \\
&\le \mathbf{P}(\mathfrak{F}) \le \mathbf{P}(\tilde{\mathfrak{F}}) = \mathbf{P}(\tilde{\mathfrak{G}}) = \mathbf{P}'(\mathfrak{G}') = \mathbf{P}'(\mathfrak{H}')
\end{aligned}$$

(we have also used the equality $h_{\nu'}(S') = h_\nu(S)$, which is true because the odometer has entropy zero, so (Y', S') is a principal extension of (Y, S)). Now the extreme right term

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is obviously dominated by the extreme left term†, so we have equalities, which yields a version of (4.2):

$$\mathbf{P}(\mathfrak{F}) = \mathbf{P}(\mathfrak{H}'). \tag{4.15}$$

We have 'almost' proved the theorem, as we have built a relatively symbolic extension of (Y', S'), while we need one of (Y, S) (although (X', S') is an extension of (Y, S), it need not be relatively symbolic). But now the creation of the desired extension is easy. Since (X', T') is a joining of (Y', S') and (Z, σ), and (Y', S') is an extension of (Y, S) (via a map π'), the map $(y', z) \mapsto (\pi'(y'), z)$ sends X' to a joining $Y \vee Z$, which we declare to be (X, T). By definition, (X, T) is a relatively symbolic extension of (Y, S). We denote by $\mathbf{H}_n(y)$ the potential corresponding to the fiber entropy $\mathbf{H}(\mathcal{U}^n|y)$ in this extension (where $\mathcal{U}$ is, as usual, the partition into cylinders corresponding to single symbols in Z). We need to replace $\mathfrak{H}'$ by $\mathfrak{H}$ in (4.14) and (4.15). For this we will need to recall a standard fact in entropy theory.

LEMMA 4.10. *Because Y' is a principal extension of Y, $X' = Y' \vee Z$ is a principal extension of $X = Y \vee Z$.*

Proof. Let $\mathfrak{A}'$, $\mathfrak{A}$, $\mathfrak{B}'$, $\mathfrak{B}$ and $\mathfrak{C}$ denote the Borel sigma-algebras in X', X, Y', Y and Z, respectively. By lifting them against the appropriate factor maps to X', we can consider all of them as sub-sigma-algebras of $\mathfrak{A}'$. We have $\mathfrak{A}' = \mathfrak{B}' \vee \mathfrak{C}$ and $\mathfrak{A} = \mathfrak{B} \vee \mathfrak{C}$. We need to show that $h(\mathfrak{A}'|\mathfrak{A}) = 0$, where the entropy is computed with respect to any invariant measure on X'. Indeed, by the subadditivity rules [**4**, Fact 4.1.7],

$$\begin{aligned} h(\mathfrak{A}'|\mathfrak{A}) &= h(\mathfrak{B}' \vee \mathfrak{C}|\mathfrak{B} \vee \mathfrak{C}) \le h(\mathfrak{B}'|\mathfrak{B} \vee \mathfrak{C}) + h(\mathfrak{C}|\mathfrak{B} \vee \mathfrak{C}) \\ &\le h(\mathfrak{B}'|\mathfrak{B}) + h(\mathfrak{C}|\mathfrak{C}) = 0. \end{aligned}$$

□

Now the derivation of (4.2) from (4.15) is immediate; we know from Theorem 3.1 that $\mathbf{P}(\mathfrak{H}') = \mathbf{h}_{\text{top}}(T')$ and analogously $\mathbf{P}(\mathfrak{H}) = \mathbf{h}_{\text{top}}(T)$. But the system (X', T'), being a principal extension, has the same topological entropy as (X, T). Thus $\mathbf{P}(\mathfrak{H}) = \mathbf{P}(\mathfrak{H}') = \mathbf{P}(\mathfrak{F})$.

Let us denote the factor maps: $\phi' : X' \to Y'$ and $\phi : X \to Y$. Also let $\psi' : X' \to X$ (this one is principal). Recall that we have already denoted the map $\pi' : Y' \to Y$ (also principal). Since both $\mathfrak{H}'$ and $\mathfrak{H}$ are fiber entropy potentials (for ϕ' and ϕ, respectively), we have, as mentioned above equality (3.1),

$$\lim_n \frac{1}{n} \int \mathbf{H}'_n \, d\nu' = \mathbf{h}(T'|\nu') = \sup\{h(\mu'|\nu') : \mu' \in \phi'^{-1}(\nu')\}$$

and an analogous formula holds for the terms with the 'primes' removed (for $\nu = \pi'(\nu')$). By (4.14), the right-hand side above does not depend on $\nu' \in \pi'^{-1}(\nu)$, thus we can freely apply the supremum over such measures. We intend to prove that, for any invariant measure ν on Y,

$$\sup\{h(\mu'|\nu') : \nu' \in \pi'^{-1}(\nu),\ \mu' \in \phi'^{-1}(\nu')\} = \sup\{h(\mu|\nu) : \mu \in \phi^{-1}(\nu)\}. \tag{4.16}$$

† We cannot use Lemma 4.8 to write $\mathbf{P}'(\mathfrak{H}') = \mathbf{P}(\mathfrak{H}')$ directly because the potential $\mathfrak{H}'$ is not lifted against (constant on fibers of) the projection from Y' to Y.

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For μ' (and $\nu' = \phi'(\mu')$) which nearly achieves the supremum on the left, we take $\mu = \psi'(\mu')$. Then, by commutation of the factor maps, $\mu \in \phi^{-1}(\nu)$ and we apply the 'triangle inequality' for conditional entropies (see [**4**, (4.1.12)]; in the central part below all entropies refer to μ'):

$$h(\mu'|\nu') = h(\mathfrak{A}'|\mathfrak{B}') \le h(\mathfrak{A}'|\mathfrak{A}) + h(\mathfrak{A}|\mathfrak{B}) + h(\mathfrak{B}|\mathfrak{B}') = 0 + h(\mu|\nu) + 0.$$

This proves the inequality '$\le$' in (4.16). If μ nearly achieves the supremum on the right then we take any $\mu' \in {\psi'}^{-1}(\mu)$ and $\nu' = \phi'(\mu')$ (this measure belongs to ${\pi'}^{-1}(\nu)$). Now we write

$$h(\mu|\nu) = h(\mathfrak{A}|\mathfrak{B}) \le h(\mathfrak{A}|\mathfrak{A}') + h(\mathfrak{A}'|\mathfrak{B}') + h(\mathfrak{B}'|\mathfrak{B}) = 0 + h(\mu'|\nu') + 0.$$

We have shown that $\lim_n (1/n) \int \mathbf{H}'_n \, d\nu' = \lim_n (1/n) \int \mathbf{H}_n \, d\nu$, which combined with (4.14) yields (4.1), concluding the proof. □

Remark 4.11. An identical proof (requiring symmetric versions of Lemmas 4.2, 4.3 and 4.8) yields $\underline{\mathbf{P}}(\mathfrak{F}) = \underline{\mathbf{P}}(\mathfrak{H})$, and, since we have already shown that $\underline{\mathbf{P}}(\mathfrak{H}) = \mathbf{P}(\mathfrak{H})$, we obtain

$$\underline{\mathbf{P}}(\mathfrak{F}) = \mathbf{P}(\mathfrak{F}).$$

5. *Lower pressure—examples*

As mentioned earlier, it has long been known that for additive continuous potentials the pressure and lower pressure coincide (as do the versions with $\liminf_n$); see [**13**]. It came to the authors' attention that it is not known whether any of the assumptions can be relaxed. We will give negative examples for both additivity and continuity. We start with a very simple example, where both assumptions are violated.

Example 5.1. Let (X, T) be a relatively symbolic almost one–one extension of (Y, S) which has larger topological entropy than (Y, S). For example, (X, T) can be a *Toeplitz system* of positive entropy and (Y, S) can be its maximal equicontinuous factor (which is an odometer and has topological entropy zero). We refer to [**5**] for an extensive exposition on almost one–one extensions and Toeplitz systems. By Theorem 3.1, the pressure of the fiber entropy potential equals the entropy of (X, T). On the other hand, since the set of points in Y which have singleton fibers in X is dense, any (n, ϵ)-spanning set in Y can be insignificantly modified so that it consists of such points. For any such point y we have $\mathbf{H}_n(y) = 0$, which implies that the lower pressure (and its variant) is equal to the topological entropy of Y, hence is strictly smaller than the pressure.

Example 5.2. It is known that there exist relatively symbolic almost one–one extensions which are relatively independent and increase the entropy. For example, there exist Toeplitz systems with positive entropy which are relatively independent over the odometer. In such a case we obtain an example with the lower pressure strictly smaller than the pressure for an *additive* (but only upper semicontinuous) potential.

Example 5.3. In order to obtain an example with a *continuous* (but only subadditive) potential it suffices to apply Theorem 2.5 to either of the preceding examples. Notice that even if we apply this to an *additive* potential $\mathfrak{H}$ taken from Example 5.2, the resulting continuous potential $\mathfrak{G}$ is no longer additive, hence there is no conflict with the theorem.

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