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Local unstable entropies of partially hyperbolic diffeomorphisms

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Abstract. Consider a C^1-partially hyperbolic diffeomorphism $f : M \to M$. Following the ideas in establishing the local variational principle for topological dynamical systems, we introduce the notions of local unstable metric entropies (and local unstable topological entropy) relative to a Borel cover $\mathcal{U}$ of M. It is shown that they coincide with the unstable metric entropy (and unstable topological entropy, respectively), when $\mathcal{U}$ is an open cover with small diameter. We also define the unstable tail entropy in the sense of Bowen and the unstable topological conditional entropy in the sense of Misiurewicz, and demonstrate that both of them vanish. Some generalizations of these results to the case of unstable pressure are also investigated.

Key words: partial hyperbolicity, local variational principle, local unstable entropy, local unstable pressure
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1. *Introduction*

Let (X, T) be a topological dynamical system, that is, X is a compact metric space and $T : X \to X$ is a surjective and continuous map. The topological entropy $h_{\text{top}}(T)$ measures the whole complexity of orbits of the dynamical system while the metric entropy $h_\mu(T)$ measures the average complexity of orbits with respect to a T-invariant probability measure μ on X. The well-known variational principle established by Goodwyn [**10**] and Goodman [**9**] states that $h_{\text{top}}(T) = \sup_\mu h_\mu(T)$ where the supremum is taken over all T-invariant probability measures on X. In recent years, local versions of the variational principle have been extensively studied, for example in [**2**, **8**, **13**, **27**]. Given an open cover $\mathcal{U}$ of X, Romagnoli [**27**] proposed two notions of metric entropies relative to $\mathcal{U}$: $h_\mu(T, \mathcal{U})$ and $h^+_\mu(T, \mathcal{U})$ satisfying $h_\mu(T, \mathcal{U}) \le h^+_\mu(T, \mathcal{U})$, and then obtained the following

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local variational principle:

$$h_{\text{top}}(T, \mathcal{U}) = \max\{h_\mu(T, \mathcal{U}) : \mu \text{ is a } T\text{-invariant measure on } X\}.$$

Later, Glasner and Weiss [**8**] proved that if (X, T) is invertible then the local variational principle also holds for $h_\mu^+(T, \mathcal{U})$, i.e.,

$$h_{\text{top}}(T, \mathcal{U}) = \max\{h_\mu^+(T, \mathcal{U}) : \mu \text{ is a } T\text{-invariant measure on } X\}.$$

It is easy to see that the classical variational principle follows from the local ones. Another important result along this line is the relative local variational principle proved by Huang, Ye and Zhang in [**14**]. More precisely, for a factor map ϕ between two topological dynamical systems (X, T) and (Y, S), they introduced two kinds of metric conditional entropies for covers, namely $h_\mu(T, \mathcal{U}|Y)$ and $h_\mu^+(T, \mathcal{U}|Y)$, and then showed that

$$h_{\text{top}}(T, \mathcal{U}|Y) = \max_\mu\{h_\mu(T, \mathcal{U}|Y)\} = \max_\mu\{h_\mu^+(T, \mathcal{U}|Y)\},$$

where the maximum is taken over all T-invariant probability measures on X. This relative local variational principle generalizes the relative variational principle obtained in [**4**, **7**, **17**].

In this paper, we let M be an n-dimensional smooth, connected, and compact Riemannian manifold without boundary and let $f : M \to M$ be a C^1-diffeomorphism. f is said to be *partially hyperbolic* (cf. for example [**25**]) if there exists a non-trivial Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ into so-called stable, center, and unstable distributions, such that all unit vectors $v^\sigma \in E_x^\sigma$ $(\sigma = c, s, u)$ with $x \in M$ satisfy

$$\|T_x f v^s\| < \|T_x f v^c\| < \|T_x f v^u\|,$$

and

$$\|T_x f|_{E_x^s}\| < 1 \quad \text{and} \quad \|T_x f^{-1}|_{E_x^u}\| < 1,$$

for some suitable Riemannian metric on M. The stable distribution E^s and unstable distribution E^u are integrable: there exist so-called stable and unstable foliations W^s and W^u, respectively, such that $TW^s = E^s$ and $TW^u = E^u$.

In the seminal papers [**18**, **19**], the metric entropy along a hierarchy of local unstable manifolds corresponding to positive Lyapunov exponents is introduced for any C^2-diffeomorphism. It is used to characterize the invariant measures satisfying Pesin's entropy formula. Furthermore, the well-known Ledrappier–Young formula is established, relating positive Lyapunov exponents, the metric entropy, and dimensions of invariant measures. If we assume f is partially hyperbolic, the metric entropy along the unstable foliation is defined as the conditional entropy $h_\mu(f, \xi) := H_\mu(f^{-1}\xi|\xi)$, where ξ is an increasing measurable partition subordinate to unstable manifolds. Let $\mathcal{Q}^u$ denote the set of all such increasing partitions. See [**18**, **19**] and §2.3 for more details. In [**11**], the authors give a new definition for the unstable metric entropy $h_\mu^u(f)$, by using measurable partitions subordinate to unstable manifolds that can be obtained by refining a finite Borel partition into pieces of unstable leaves. We will denote the set of such measurable partitions by $\mathcal{P}^u$ (see §2.2). It is proved that the unstable metric entropy $h_\mu^u(f)$ coincides with $h_\mu(f, \xi)$,

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where $\xi \in \mathcal{Q}^u$. The following variational principle relating the unstable topological entropy $h^u_{\text{top}}(f)$ and unstable metric entropy $h^u_\mu(f)$ is established in [**11**]:

$$h^u_{\text{top}}(f) = \sup_{\mu \in \mathcal{M}_f(M)} h^u_\mu(f) = \sup_{\mu \in \mathcal{M}^e_f(M)} h^u_\mu(f),$$

where $\mathcal{M}_f(M)$ ($\mathcal{M}^e_f(M)$) denotes the set of all f-invariant (f-ergodic, respectively) Borel probability measures on M. Since the unstable metric entropy provides the complexity caused by the unstable direction of partially hyperbolic diffeomorphisms, it has been used to study Gibbs u-states [**12**, **29**] and measures of high entropy [**28**] in recent works.

It is a natural question whether there are local variational principles for unstable entropies relative to a given open cover $\mathcal{U}$ of M. To study this question, we follow the ideas of Romagnoli [**27**]. Given a Borel cover $\mathcal{U}$ of M, we introduce two notions of *local unstable metric entropies* $h_\mu(f, \mathcal{U}|\zeta)$ and $h^+_\mu(f, \mathcal{U}|\zeta)$, where ζ is a measurable partition from $\mathcal{P}^u$ or $\mathcal{Q}^u$ (see §3.1), and a notion of *local unstable topological entropy* $h^u_{\text{top}}(f, \mathcal{U})$ (see §3.2). The following interesting result is then obtained.

THEOREM 1.1. *Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism and $\mathcal{U}$ an open cover of M with sufficiently small diameter. Then, for any $\mu \in \mathcal{M}_f(M)$ and $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$,*

$$h_\mu(f, \mathcal{U}|\zeta) = h^+_\mu(f, \mathcal{U}|\zeta) = h_\mu(f|\zeta) = h^u_\mu(f).$$

Furthermore,

$$h^u_{\text{top}}(f, \mathcal{U}) = h^u_{\text{top}}(f).$$

In fact, we can prove $h_\mu(f, \mathcal{U}|\zeta) = h^+_\mu(f, \mathcal{U}|\zeta) = h_\mu(f|\zeta) = h^u_\mu(f)$ and $h^u_{\text{top}}(f) \le h^u_{\text{top}}(f, \mathcal{U})$ for any Borel cover $\mathcal{U}$ of M. If $\mathcal{U}$ is an open cover, then $h^u_{\text{top}}(f, \mathcal{U}) \le h^u_{\text{top}}(f)$ by definition, which might be false for some Borel cover $\mathcal{U}$ (see [**14**, Proposition A.4]).

To distinguish the two different types of measurable partitions, we make a convention that η is always a partition from $\mathcal{P}^u$, ξ from $\mathcal{Q}^u$, and ζ from either $\mathcal{P}^u$ or $\mathcal{Q}^u$. In Theorem 1.1, we have given a unified statement for both partitions from $\mathcal{P}^u$ and $\mathcal{Q}^u$. We will give a direct proof of Theorem 1.1 for the case $\eta \in \mathcal{P}^u$ in §3 following classical techniques in proving the (relative) local variational principle. For the case $\xi \in \mathcal{P}^u$, we do not know a direct proof, so its proof is postponed until in §4 after proving Theorem 1.3. From the proof, one will see some advantage of partition $\eta \in \mathcal{P}^u$ compared with $\xi \in \mathcal{Q}^u$, for example, $H_\mu(\eta|\zeta) < \infty$ but we do not know if $H_\mu(\xi|\zeta) < \infty$ (See Lemma 3.3). Note that an increasing partition $\xi \in \mathcal{Q}^u$ plays an important role in proving the Ledrappier–Young formula (cf. [**18**, **19**]), while $\eta \in \mathcal{P}^u$ finds its novelty in proving the unstable variational principle in [**11**]. In this paper, we try to put our results in a more unified way, using both of the above measurable partitions.

The following local variational principle for unstable entropies follows from the above theorem and the variational principle for unstable entropies recalled above (cf. [**11**, Theorem D]).

COROLLARY 1.2. (Local variational principle) *Let $f : M \to M$ be a C^1-partially hyperbolic diffeomorphism and $\mathcal{U}$ an open cover of M with sufficiently small diameter.*

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Then, for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$,

$$h^u_{\text{top}}(f, \mathcal{U}) = \sup_{\mu \in \mathcal{M}_f(M)} h_\mu(f, \mathcal{U}|\zeta) = \sup_{\mu \in \mathcal{M}_f(M)} h^+_\mu(f, \mathcal{U}|\zeta).$$

Moreover,

$$h^u_{\text{top}}(f, \mathcal{U}) = \sup_{\mu \in \mathcal{M}^e_f(M)} h_\mu(f, \mathcal{U}|\zeta) = \sup_{\mu \in \mathcal{M}^e_f(M)} h^+_\mu(f, \mathcal{U}|\zeta).$$

We may compare Theorem 1.1 with the relative local variational principle in [**14**] mentioned above. A major difference between them is the following: for a factor map π between (X, T) and (Y, S), $h_\mu(T, \mathcal{U}|Y)$ and $h^+_\mu(T, \mathcal{U}|Y)$ are conditional entropies with respect to a T-*invariant* partition $\{\pi^{-1}(y) : y \in Y\}$; however, unstable metric entropies $h_\mu(f, \mathcal{U}|\zeta)$ and $h^+_\mu(f, \mathcal{U}|\zeta)$ are conditional entropies with respect to a partition ζ subordinate to unstable manifolds. The expansion of unstable manifolds under f plays an important role here. To illustrate this and interpret Theorem 1.1 from another point of view, we introduce the notions of *unstable tail entropy* following Bowen [**3**] and *unstable topological conditional entropy* following Misiurewicz [**22**]. We show that both of them vanish due to the expansion of unstable manifolds under f.

THEOREM 1.3. *Let* $f : M \to M$ *be a* C^1*-partially hyperbolic diffeomorphism. The unstable topological conditional entropy of* f *in the sense of Misiurewicz* $h^{*u}(f) = 0$*; moreover, the unstable tail topological entropy in the sense of Bowen* $h^{*u}(f, \epsilon) = 0$ *for sufficiently small* ϵ.

For a given Borel cover $\mathcal{U}$, $h_\mu(f|\xi) - h_\mu(f, \mathcal{U}|\xi)$ is bounded from above by the unstable topological conditional entropy relative to $\mathcal{U}$ (see Proposition 4.12). This allows us to complete the proof of Theorem 1.1 for the case $\xi \in \mathcal{Q}^u$ based on Theorem 1.3.

At last, we make some generalizations of the above results to the case of unstable pressure. A variational principle for unstable pressure has been obtained in [**12**]. Combining it with Theorem 1.1, we prove the following theorem.

THEOREM 1.4. *Let* $f : M \to M$ *be a* C^1*-partially hyperbolic diffeomorphism and* $\mathcal{U}$ *an open cover of* M *with sufficiently small diameter. Then, for any* $\phi \in C(M, \mathbb{R})$,

$$P^u(f, \phi, \mathcal{U}) = P^u(f, \phi).$$

Here $C(M, \mathbb{R})$ denotes the Banach space of continuous functions $\phi : M \to \mathbb{R}$. For topological dynamical systems, the local variational principle of pressure and its relative versions were established in [**13**, **15**, **20**, **21**, **30**], etc. Like the entropy case, the local unstable variational principle of pressure in our setting follows easily from Theorem 1.4, Theorem 1.1, and the unstable variational principle of pressure in [**12**].

COROLLARY 1.5. (Local variational principle for unstable pressure) *Let* $f : M \to M$ *be a* C^1*-partially hyperbolic diffeomorphism and* $\mathcal{U}$ *an open cover of* M *with sufficiently small diameter. Then, for any* $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$,

$$\begin{aligned} P^u(f, \phi, \mathcal{U}) &= \sup_{\mu \in \mathcal{M}_f(M)} \left\{ h_\mu(f, \mathcal{U}|\zeta) + \int_M \phi \, d\mu \right\} \\ &= \sup_{\mu \in \mathcal{M}_f(M)} \left\{ h^+_\mu(f, \mathcal{U}|\zeta) + \int_M \phi \, d\mu \right\}. \end{aligned}$$

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Moreover,

$$P^u(f,\phi,\mathcal{U}) = \sup_{\mu\in\mathcal{M}_f^e(M)}\left\{h_\mu(f,\mathcal{U}|\zeta)+\int_M \phi\,d\mu\right\}$$
$$= \sup_{\mu\in\mathcal{M}_f^e(M)}\left\{h_\mu^+(f,\mathcal{U}|\zeta)+\int_M \phi\,d\mu\right\}.$$

In the main results above, we only need to assume that the partially hyperbolic diffeomorphism f is C^1, for which the unstable manifolds always exist. Both measurable partitions $\eta\in\mathcal{P}^u$ and $\xi\in\mathcal{Q}^u$ can be constructed under the C^1 regularity of f. For an arbitrary diffeomorphism, f is usually assumed to be $C^{1+\alpha}$ so that there exists a hierarchy of local unstable manifolds. This assumption is necessary in many important results such as the Ledrappier–Young formula (cf. [**18**, **19**]).

Furthermore, for a larger class of diffeomorphisms with a uniformly expanding f-invariant foliation, we can also define the entropy along this foliation, and analogs of the above main results still hold. In the current paper, we only focus on the partially hyperbolic diffeomorphism defined as above, which has a center distribution. In this setting, one can see clearly how the unstable metric entropy is related to the metric entropy of the whole system. In particular, if f is a $C^{1+\alpha}$-partially hyperbolic diffeomorphism without positive Lyapunov exponents in the center direction for any invariant measure, then the unstable metric entropy is equal to the metric entropy of f. See [**18**, Corollary 5.3] and also [**11**, Corollary A.1].

The paper is organized as follows. In §2, we recall some necessary preliminaries of the unstable entropies, including two types of measurable partitions in $\mathcal{P}^u$ and $\mathcal{Q}^u$, and the unstable variational principle. Two notions of local unstable metric entropies and a notion of local unstable topological entropy are introduced in §3. Then Theorem 1.1 for the case $\eta\in\mathcal{P}^u$ is proved there. In §4, we define the notions of unstable topological conditional entropy and unstable tail entropy and then prove Theorem 1.3. Their connection to local unstable entropies is also discussed there, completing the proof of Theorem 1.1. Theorem 1.4 is proved in §5. At the end, we give a counterexample to show that an *inner* variational principle for unstable pressure (in a form similar to Ledrappier–Walters (cf. [**7**, **17**])) does not hold, which illustrates a significant difference between the unstable entropy of a partially hyperbolic diffeomorphism and the relative entropy of a factor map.

2. *Unstable entropies and unstable variational principle*

This section is devoted to collecting some useful results on unstable metric entropy and unstable topological entropy, and the unstable variational principle relating them. Unstable metric entropy is defined as a conditional entropy, via two different types of measurable partitions. The Ledrappier–Young formula (cf. [**19**]) is established by using increasing partitions subordinate to unstable foliations, which are constructed in [**16**], and also in [**24**] for partially hyperbolic attractors. In [**11**], the authors considered measurable partitions consisting of local unstable leaves that can be obtained by refining a finite partition into pieces of unstable leaves. Both of these two types of measurable partitions are considered in this paper.

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2.1. *Conditional entropy.* For a partition ζ of a set X, let $\zeta(x)$ be the element of ζ containing x. If ζ_1 and ζ_2 are two partitions such that $\zeta_1(x) \subset \zeta_2(x)$, we then write $\zeta_1 \succeq \zeta_2$ or $\zeta_2 \preceq \zeta_1$. $\zeta_1 \vee \zeta_2 := \{A \cap B : A \in \zeta_1, B \in \zeta_2\}$ is called the refinement of ζ_1 and ζ_2. Let $(X, \mathcal{A}, \nu)$ be a standard probability space. A partition ζ of X is called *measurable* if there exists a countable set $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\zeta)$ such that for almost every pair $C_1, C_2 \in \zeta$, we can find some A_n, which separates them in the sense that $C_1 \subset A_n$, $C_2 \subset X - A_n$, where $\mathcal{B}(\zeta)$ is the sub-σ-algebra of elements of $\mathcal{A}$, which are unions of elements of ζ. The *canonical system of conditional measures for ν and ζ* is a family of probability measures $\{\nu_x^\zeta : x \in X\}$ with $\nu_x^\zeta\big(\zeta(x)\big) = 1$, such that for every measurable set $B \subset X$, $x \mapsto \nu_x^\zeta(B)$ is $\mathcal{B}(\zeta)$-measurable and

$$\nu(B) = \int_X \nu_x^\zeta(B)\, d\nu(x).$$

The classical result of Rokhlin (cf. [**26**]) says that if ζ is a measurable partition, then there exists a system of conditional measures relative to ζ. It is essentially unique in the way that two such systems coincide in a set of full ν-measure.

Suppose that $f : M \to M$ is a C^1-partially hyperbolic diffeomorphism. Let d denote the Riemannian metric on M and d^u the metric induced by the Riemannian structure on any unstable manifold. Define $W^u(x, \varepsilon) = \{y \in W^u(x) : d^u(y, x) < \varepsilon\}$, where $W^u(x)$ is the global unstable manifold through x. Since W^u is a continuous foliation with smooth leaves and M is compact, we know that there exist $\varepsilon_1 > 0$ small enough and $C_0 > 1$ such that $d(\cdot\,,\cdot) \le d^u(\cdot\,,\cdot) \le C_0 d(\cdot\,,\cdot)$ on any local unstable manifold $W^u(x, \varepsilon_1)$.

Let $\mathcal{C}_M$ denote the set of Borel covers of M and $\mathcal{C}_M^o \subset \mathcal{C}_M$ the set of open covers of M. Let $\mathcal{P}_M$ denote the set of all finite Borel partitions of M. Note that $\mathcal{P}_M \subset \mathcal{C}_M$.

Definition 2.1. Suppose that μ is a Borel probability measure on M. We define the *information function of $\alpha \in \mathcal{P}_M$* as

$$I_\mu(\alpha)(x) := -\log \mu(\alpha(x)),$$

and the *entropy of partition α* as

$$H_\mu(\alpha) := \int_M I_\mu(\alpha)(x)\, d\mu(x) = -\int_M \log \mu(\alpha(x))\, d\mu(x).$$

The *conditional information function of $\alpha \in \mathcal{P}_M$ with respect to a measurable partition ζ of M* is defined as

$$I_\mu(\alpha|\zeta)(x) := -\log \mu_x^\zeta(\alpha(x)).$$

Then the *conditional entropy of α with respect to ζ* is defined as

$$H_\mu(\alpha|\zeta) := \int_M I_\mu(\alpha|\zeta)(x)\, d\mu(x) = -\int_M \log \mu_x^\zeta(\alpha(x))\, d\mu(x).$$

Remark 2.2. For any two measurable partitions ζ_1 and ζ_2, $I_\mu(\zeta_1|\zeta_2)$ and $H_\mu(\zeta_1|\zeta_2)$ are defined as above possibly being infinity. See the discussion in [**5**, §2.4], and [**18**, **19**].

The properties in the following lemma are well known (cf. for example [**26**]).

LEMMA 2.3. *Let α, β, and γ be measurable partitions with $H_\mu(\alpha|\gamma)$, $H_\mu(\beta|\gamma) < \infty$.*
(i) *If $\alpha \le \beta$, then $I_\mu(\alpha|\gamma)(x) \le I_\mu(\beta|\gamma)(x)$ and $H_\mu(\alpha|\gamma) \le H_\mu(\beta|\gamma)$.*

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(ii) $I_\mu(\alpha \vee \beta|\gamma)(x) = I_\mu(\alpha|\gamma)(x) + I_\mu(\beta|\alpha \vee \gamma)(x)$ *and* $H_\mu(\alpha \vee \beta|\gamma) = H_\mu(\alpha|\gamma) + H_\mu(\beta|\alpha \vee \gamma)$.
(iii) $H_\mu(\alpha \vee \beta|\gamma) \le H_\mu(\alpha|\gamma) + H_\mu(\beta|\gamma)$.
(iv) $H_\mu(\beta|\gamma) \le H_\mu(\alpha|\gamma) + H_\mu(\beta|\alpha)$.
(v) *If* $\beta \le \gamma$, *then* $H_\mu(\alpha|\beta) \ge H_\mu(\alpha|\gamma)$.

2.2. *Partitions and unstable metric entropy.* For a partition $\alpha \in \mathcal{P}_M$, denote $\operatorname{diam}\alpha = \sup\{\operatorname{diam} A : A \in \alpha\}$. Choose and fix $\lambda > \|T_x f|_{E^u_x}\| > 1$ for any $x \in M$. Fix $L > 0$, and choose a sufficiently small number $0 < \varepsilon_0 \ll \min\{\varepsilon_1, L\}$ such that:
(1) $W^u(x, L) \cap B(x, \varepsilon_0)$ has only one connected component for any $x \in M$;
(2) $\lambda C_0 \varepsilon_0 \ll L$.
Let $\mathcal{P} \subset \mathcal{P}_M$ denote the set of all finite Borel partitions α of M with $\operatorname{diam}\alpha < \varepsilon_0$. For a partition $\alpha \in \mathcal{P}$, let $\alpha^u > \alpha$ be the partition that satisfies $\alpha^u(x) = \alpha(x) \cap W^u(x, \bar{\varepsilon}_0)$ for any $x \in M$, where $C_0\varepsilon_0 < \bar{\varepsilon}_0 < L$ (thus $W^u(x, \bar{\varepsilon}_0)$ contains the connected component of $W^u(x) \cap B(x, \varepsilon_0)$ that contains x, and does not intersect other connected components). As W^u is a continuous foliation, α^u is a measurable partition with respect to any Borel probability measure μ if $\alpha \in \mathcal{P}$. Recall that a partition ζ of M is said to be *subordinate to unstable manifolds* of f with respect to a measure μ if for μ-almost every x, $\zeta(x) \subset W^u(x)$ and $\zeta(x)$ contains an open neighborhood of x in $W^u(x)$. It is clear that if $\alpha \in \mathcal{P}$ such that $\mu(\partial\alpha) = 0$, where $\partial\alpha := \bigcup_{A\in\alpha} \partial A$, then the corresponding α^u is a partition subordinate to unstable manifolds of f. Let $\mathcal{P}^u$ be the set of all partitions α^u subordinate to unstable manifolds obtained in this way, where $\alpha \in \mathcal{P}$.

When $\alpha \in \mathcal{C}_M$ or α is a measurable partition, we write $\alpha_m^n := \bigvee_{i=m}^n f^{-i}\alpha$ for any $m \le n$ and $m, n \in \mathbb{Z} \cup \{\pm\infty\}$.

Definition 2.4. The *conditional entropy of* f *with respect to a measurable partition* α *given* $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f, \alpha|\eta) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta).$$

The *conditional entropy of* f *given* $\eta \in \mathcal{P}^u$ is defined as

$$h_\mu(f|\eta) = \sup_{\alpha\in\mathcal{P}} h_\mu(f, \alpha|\eta),$$

and the *unstable metric entropy of* f is defined as

$$h^u_\mu(f) = \sup_{\eta\in\mathcal{P}^u} h_\mu(f|\eta).$$

We collect some facts on the partitions and covers, which will be used several times later in the paper.

LEMMA 2.5.
(i) *Let* $\alpha \in \mathcal{C}_M$ *with* $\operatorname{diam}(\alpha) < \varepsilon_0/\lambda$, *and* $\eta \in \mathcal{P}^u$. *Then* $\alpha_{-n}^{-1} \vee f^n\eta \succeq (f\alpha)^u$ *for any* $n \ge 1$.
(ii) *Let* $\mathcal{U} \in \mathcal{C}_M$ *with* $\operatorname{diam}(\mathcal{U}) < \varepsilon_0/\lambda$. *Fix* $N \ge 1$. *For any* $k \ge 1$ *and* $\alpha \succeq \mathcal{U}_0^{N-1}$ *we have* $f^{Nk}\alpha \vee \cdots \vee f^N\alpha \vee f^{Nk}\eta \succeq (f^N\alpha)^u$ *for* μ*-a.e.* x.

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Proof. Let $y \in (\alpha_{-n}^{-1} \vee f^n\eta)(x)$. Then $f^{-n}(y) \in \eta(f^{-n}(x))$ and hence

$$d^u(f^{-n}(x),\, f^{-n}(y)) < C_0\varepsilon_0.$$

Then $d^u(f^{-(n-1)}(x),\, f^{-(n-1)}(y)) < \lambda C_0\varepsilon_0 \ll L$. Then by $f^{-(n-1)}(y) \in \alpha(f^{-(n-1)}(x))$, we have $d(f^{-(n-1)}(x),\, f^{-(n-1)}(y)) < \varepsilon_0/\lambda$. As ε_0 is small enough, it follows that

$$d^u(f^{-(n-1)}(x),\, f^{-(n-1)}(y)) < C_0\varepsilon_0/\lambda.$$

By induction, we have $d^u(f^{-1}x,\, f^{-1}y) < C_0\varepsilon_0/\lambda$. By uniform expansion of W^u, $d^u(x,\, y) < C_0\varepsilon_0 < \bar{\varepsilon}_0$ and hence $y \in W^u(x,\, \bar{\varepsilon}_0)$. Recall that $y \in (f\alpha)(x)$, so we have $y \in (f\alpha)^u(x)$, which implies $\alpha_{-n}^{-1} \vee f^n\eta \succeq (f\alpha)^u$. This proves (i).

The proof of (ii) is in the same spirit to that of (i). If $y \in (f^{Nk}\alpha \vee \cdots \vee f^N\alpha \vee f^{Nk}\eta)(x)$, then $f^{-Nk}(y) \in \eta(f^{-Nk}(x))$ and $f^{-i}(y) \in U_i(f^{-i}(x))$ for any $1 \le i \le Nk$, where $U_i(f^{-i}(x))$ is an element in $\mathcal{U}$ containing $f^{-i}(x)$. By the definition of $\mathcal{P}^u$, we can suppose that η is bounded from above and with diameter $< C_0\varepsilon_0$. So $d^u(f^{-Nk}(x),\, f^{-Nk}(y)) < C_0\varepsilon_0$. Then $d^u(f^{-(Nk-1)}(x),\, f^{-(Nk-1)}(y)) < C_0\lambda\varepsilon_0$. Since $f^{-(Nk-1)}(y) \in U_{Nk-1}(f^{-(Nk-1)}(x))$, $d(f^{-(Nk-1)}(x),\, f^{-(Nk-1)}(y)) < \varepsilon_0$. It follows that $d^u(f^{-(Nk-1)}(x),\, f^{-(Nk-1)}(y)) < C_0\varepsilon_0$. By induction, we have $d^u(f^{-1}x,\, f^{-1}y) < C_0\varepsilon_0$. Thus $d^u(x,\, y) < C_0\lambda\varepsilon_0$. Recall that $y \in (f^N\alpha)(x)$, we have $y \in (f^N\alpha)^u(x)$, which implies $f^{Nk}\alpha \vee \cdots \vee f^N\alpha \vee f^{Nk}\eta \succeq (f^N\alpha)^u$. □

Remark 2.6. In Lemma 2.5(i), if $y \in (f\alpha)(x) \cap (f^n\eta)(x)$, then $f^{-1}(y) \in \alpha(f^{-1}(x))$ and $d(f^{-1}(x),\, f^{-1}(y)) < \varepsilon_0/\lambda$. However, it does not follow immediately that $d^u(f^{-1}(x),\, f^{-1}(y)) < C_0\varepsilon_0/\lambda$. This is because $y \in (f^n\eta)(x)$ only means y lies in the global unstable manifold of x, but maybe not in the local one. So we have to use the additional information $y \in \alpha_{-n}^{-1}(x)$ and do the induction as in the above proof.

2.3. *Increasing partitions.* For an ergodic measure μ with positive Lyapunov exponents $\lambda_1 > \lambda_2 > \cdots > \lambda_{\tilde{u}} > 0$, let $E^{(1)} \subset E^{(2)} \subset \cdots \subset E^{(\tilde{u})}$ denote the subbundles in the tangent bundle consisting of vectors whose Lyapunov exponents are greater than or equal to $\lambda_1, \lambda_2, \ldots, \lambda_{\tilde{u}}$, respectively. If f is $C^{1+\alpha}$, it is well known that for almost every x there exist unstable manifolds $W^{(1)}(x) \subset W^{(2)}(x) \subset \cdots \subset W^{(\tilde{u})}(x)$ such that if $y \in W^{(i)}(x)$, then $\limsup_{n\to\infty} -(1/n)\log d(f^{-n}y,\, f^{-n}x) \le -\lambda_i$ for any $1 \le i \le \tilde{u}$ (cf. [**1**, Ch. 7]). The entropies $h_\mu(f, \xi_i)$ are determined by a hierarchy of increasing partitions ξ_i, and $h(f, \xi_{\tilde{u}}) = h_\mu(f)$, the entropy of f with respect to μ (cf. [**18**, **19**]).

PROPOSITION 2.7. [**19**, Lemma 9.1.1] *Let f be $C^{1+\alpha}$ as above. Then there exist measurable partitions $\xi_1 \succeq \xi_2 \succeq \cdots \succeq \xi_{\tilde{u}}$ on M such that for each $1 \le i \le \tilde{u}$,*

(i) *ξ_i is subordinate to $W^{(i)}$;*

(ii) *ξ_i is increasing, i.e. $f^{-1}\xi_i \succeq \xi_i$;*

(iii) *ξ_i is a generator, i.e. $\bigvee_{n=1}^{\infty} f^{-n}\xi_i = \epsilon$, where ϵ is the partition of M into points.*

If there are u ($u \le \tilde{u}$) distinct Lyapunov exponents on the unstable subbundle, then the uth unstable foliation is the unstable foliation of the partially hyperbolic diffeomorphism f. Let us give more details on the measurable partition ξ_u, and the entropy $h(f, \xi_u)$. For convenience, write $\xi = \xi_u$. Then Proposition 2.7 is reformulated for partially hyperbolic

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diffeomorphisms as follows. One will see below from the construction of ξ that it is enough to assume f to be C^1.

PROPOSITION 2.8. *Let f be a C^1-partially hyperbolic diffeomorphism, and $\mu \in \mathcal{M}_f(M)$. Then there exists a measurable partition ξ of M such that:*

(i) *ξ is subordinate to W^u;*
(ii) *ξ is increasing, i.e. $f^{-1}\xi \succeq \xi$;*
(iii) *ξ is a generator, i.e. $\bigvee_{n=1}^{\infty} f^{-n}\xi = \epsilon$, where ϵ is the partition of M into points.*

There is an explicit way to construct an increasing measurable partition as in Proposition 2.8. Following the ideas in [**19**], we first construct an *open r-rectangle at x* with $r > 0$ as

$$\Pi(x, r) := \bigcup_{y \in W(x,r)} W^u(y, r),$$

where $W(x, r)$ is an open ball of radius r centered at x inside a local manifold W passing through x transversally to the unstable foliation W^u. Moreover, x and $\Pi(x, r)$ are taken in such a way that $\mu(\Pi(x, r)) > 0$ for any $r > 0$. Then define a partition $\hat{\xi}_x$ such that $\hat{\xi}_x(y) = W^u(\bar{y}, r)$ if $y \in \Pi(x, r)$, where $\bar{y} \in W(x, r)$ and $y \in W^u(\bar{y}, r)$, and $\hat{\xi}_x(y) = M \setminus \Pi(x, r)$ otherwise. Next take $\xi = \xi_x := \vee_{j \geq 0} f^j \hat{\xi}_x$. It has been proven (see e.g. [**19**]) that if μ is ergodic, then for almost every small real number $r > 0$ with respect to the Lebesgue measure, ξ is subordinate to unstable manifolds W^u. Therefore, it is a partition satisfying Proposition 2.8.

The above construction can be extended for a non-ergodic $\mu \in \mathcal{M}_f(M)$: take finitely many open rectangles $\Pi(x_i, r_i)$, $1 \leq i \leq N$ covering M, construct ξ_i, $1 \leq i \leq N$ as before and then set $\xi = \bigvee_{i=1}^{N} \xi_i$, which is the required partition. From now on, given $\mu \in \mathcal{M}_f(M)$, let $\mathcal{Q}^u$ denote the set of all measurable partitions ξ described in Proposition 2.8 satisfying the following property: there exists a subset $M' \subset M$ with $\mu(M') = 1$ such that for any $x \in M'$, $\xi(x)$ is an open neighborhood of x with diameter smaller than ε_0 in $W^u(x)$. Clearly, the construction above provide examples of partitions in $\mathcal{Q}^u$.

Given $\xi \in \mathcal{Q}^u$, there exists a family of conditional measures $\{\mu_x^{\xi}\}$ of μ with respect to ξ. By Remark 2.2, we can define

$$H_\mu(f^{-1}\xi|\xi) := \int_M -\log \mu_x^{\xi}((f^{-1}\xi)(x))\, d\mu(x).$$

It has been proven in [**19**] that $H_\mu(f^{-1}\xi|\xi)$ is finite, which is also denoted by $h_\mu(f, \xi)$. By [**18**, Lemma 3.1.2], $h_\mu(f, \xi)$ is independent of $\xi \in \mathcal{Q}^u$. Moreover, if f is $C^{1+\alpha}$ and there are no positive Lyapunov exponents in center direction, then $h_\mu(f, \xi) = h_\mu(f)$. One of the main results in [**11**] is that $h_\mu(f, \xi)$ coincides with the unstable metric entropy of f with respect to μ.

THEOREM 2.9. [**11**, Theorem A] *Let f be a C^1-partially hyperbolic diffeomorphism and $\mu \in \mathcal{M}_f^e(M)$. For any $\eta \in \mathcal{P}^u$ and $\xi \in \mathcal{Q}^u$,*

$$h_\mu^u(f) = h_\mu(f|\eta) = h_\mu(f, \xi).$$

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For any $\xi \in \mathcal{Q}^u$, $\alpha \in \mathcal{P}_M$, we can also define $h_\mu(f, \alpha|\xi)$ and $h_\mu(f|\xi)$ as in Definition 2.4 with η replaced by ξ, that is,

$$h_\mu(f, \alpha|\xi) = \lim_{n\to\infty} \sup \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi),$$
$$h_\mu(f|\xi) = \sup_{\alpha\in\mathcal{P}} h_\mu(f, \alpha|\xi).$$

2.4. *Unstable topological entropy and the unstable variational principle.* Recall that $W^u(x, \delta)$ denotes the open ball inside $W^u(x)$ with center x and radius δ with respect to d^u. We proceed to recall the definition of unstable topological entropy in [**11**]. Recall that $\mathcal{C}_M(\mathcal{C}_M^o)$ denotes the set of Borel (open) covers of M. Given $\mathcal{U} \in \mathcal{C}_M$, denote $\mathcal{U}_m^n := \bigvee_{i=m}^n f^{-i}\mathcal{U}$. For any $K \subset M$, set $N(K, \mathcal{U}) := \min\{\text{the cardinality of } \mathcal{V} : \mathcal{V} \subset \mathcal{U}, \bigcup_{V\in\mathcal{V}} \supset K\}$ and $H(\mathcal{U}|K) := \log N(K, \mathcal{U})$. We define

$$h_{\text{top}}(f|\overline{W^u(x, \delta)}) := \sup_{\mathcal{U}\in\mathcal{C}_M^o} \lim\sup_{n\to\infty} \frac{1}{n} H(\mathcal{U}_0^{n-1}|\overline{W^u(x, \delta)}).$$

The *unstable topological entropy* of f is then defined as

$$h_{\text{top}}^u(f) := \lim_{\delta\to 0} \sup_{x\in M} h_{\text{top}}(f|\overline{W^u(x, \delta)}).$$

In [**11**], the unstable topological entropy is defined equivalently by using separated sets and spanning sets on unstable manifolds.

The following variational principle is proved in [**11**].

THEOREM 2.10. (Cf. [**11**, Theorem D]) *Let $f : M \to M$ be any C^1 partially hyperbolic diffeomorphism. Then*

$$h_{\text{top}}^u(f) = \sup_{\mu\in\mathcal{M}_f(M)} h_\mu^u(f) = \sup_{\mu\in\mathcal{M}_f^e(M)} h_\mu^u(f).$$

In [**11**], the ideas in Misiurewicz's proof of the classical variational principle [**23**] are adapted to prove Theorem 2.10, while the following crucial property of $\eta \in \mathcal{P}^u$ is used. Given any piece of unstable leaf $\overline{W^u(x, \delta)}$ with $h_{\text{top}}^u(f, \overline{W^u(x, \delta)})$ arbitrarily close to $h_{\text{top}}^u(f)$, there exists $\eta \in \mathcal{P}^u$ such that $\overline{W^u(x, \delta)} \subset \eta(x)$. This is just one example of the advantage of $\eta \in \mathcal{P}^u$ compared with $\xi \in \mathcal{Q}^u$ constructed in [**16**], since the elements of ξ there may have arbitrarily small size. Nevertheless, for partially hyperbolic diffeomorphisms, there is another approach to construct $\xi \in \mathcal{Q}^u$ given by Pesin–Sinai (see [**24**, p. 422]). By their approach, given any piece of unstable leaf $\overline{W^u(x, \delta)}$, we can construct $\xi \in \mathcal{Q}^u$ such that $\overline{W^u(x, \delta)} \subset \xi(x)$. Thus, we can give a unified proof of Theorem 2.10 using either $\eta \in \mathcal{Q}^u$ or $\xi \in \mathcal{Q}^u$ constructed in [**24**]. One can check the details of the proof using the latter partition, which is omitted here.

2.5. *Unstable topological pressure.* At last, we recall the definition of unstable topological pressure given in [**12**]. Let $\phi \in C(M, \mathbb{R})$. Put

$$P^u(f, \phi, \mathcal{U}, n, x, \delta) := \inf\left\{\sum_{B\in\mathcal{V}} \sup_{y\in B\cap\overline{W^u(x,\delta)}} \exp((S_n\phi)(y))|\mathcal{V} \in \mathcal{C}_M, \mathcal{V} \succeq \mathcal{U}_0^{n-1}\right\},$$

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where $(S_n\phi)(y) = \sum_{i=0}^{n-1} \phi^i(y)$. If $B \cap \overline{W^u(x,\delta)} = \emptyset$, we set $\exp((S_n\phi)(y)) = 0$. We define

$$P^u(f, \phi|\overline{W^u(x,\delta)}) := \sup_{\mathcal{U}\in\mathcal{C}_M^o} \limsup_{n\to\infty} \frac{1}{n} \log P^u(f, \phi, \mathcal{U}, n, x, \delta).$$

Then *unstable topological pressure* is defined as

$$P^u(f, \phi) := \lim_{\delta\to 0} \sup_{x\in M} P^u(f, \phi, \overline{W^u(x,\delta)}).$$

The following variational principle for unstable pressure is proved in [**12**].

THEOREM 2.11. (Cf. [**12**, Theorem A]) *For any $\phi \in C(M, \mathbb{R})$,*

$$\begin{aligned} P^u(f, \phi) &= \sup\left\{h_\mu^u(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}_f(M)\right\} \\ &= \sup\left\{h_\mu^u(f) + \int_M \phi\, d\mu : \mu \in \mathcal{M}_f^e(M)\right\}. \end{aligned}$$

3. *Local unstable entropies*

3.1. *Local unstable metric entropies.* Following the ideas of Romagnoli [**27**], we propose two notions of *local unstable metric entropies of f with respect to $\mu \in \mathcal{M}_f(M)$ and $\mathcal{U} \in \mathcal{C}_M$*, $h_\mu(f, \mathcal{U}|\zeta)$ and $h_\mu^+(f, \mathcal{U}|\zeta)$, where $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$ in this subsection.

3.1.1. *Local unstable metric entropy $h_\mu^+(f, \mathcal{U}|\zeta)$.* Recall that for any $\xi \in \mathcal{Q}^u$ and $\alpha \in \mathcal{P}_M$, we define $h_\mu(f, \alpha|\xi)$ as in Definition 2.4 with η replaced by ξ, i.e. $h_\mu(f, \alpha|\xi) = \limsup_{n\to\infty} (1/n) H_\mu(\alpha_0^{n-1}|\xi)$.

Definition 3.1. For any $\mathcal{U} \in \mathcal{C}_M$ and $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, define

$$h_\mu^+(f, \mathcal{U}|\zeta) = \inf_{\alpha\in\mathcal{P}_M, \alpha\succeq\mathcal{U}} h_\mu(f, \alpha|\zeta).$$

The following proposition is a crucial result toward studying $h_\mu^+(f, \mathcal{U}|\zeta)$. It says that $h_\mu(f, \alpha|\zeta)$ coincides with the unstable metric entropy $h_\mu^u(f)$ for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$ and $\alpha \in \mathcal{P}$, and the lim sup in the definition of $h_\mu(f, \alpha|\zeta)$ is in fact a lim. When μ is ergodic, the case for $\eta \in \mathcal{P}^u$ is [**11**, Theorem A], while the case for $\xi \in \mathcal{Q}^u$ is implicitly considered in [**11**, proof of Corollary B.1]. We provide an explicit and complete proof here, for *any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$ and any invariant measure μ*. Recall that $\lambda > \|T_x f|_{E_x^u}\| > 1$ for any $x \in M$, and ϵ_0, $\bar{\varepsilon}_0$ are small positive numbers defined at the beginning of §2.2.

PROPOSITION 3.2. *For any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$ and $\alpha \in \mathcal{P}$ with* $\operatorname{diam}(\alpha) < \epsilon_0/\lambda$,

$$h_\mu^u(f) = h_\mu(f|\zeta) = h_\mu(f, \alpha|\zeta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\zeta).$$

Proof. At first, we prove the proposition for $\eta \in \mathcal{P}^u$. We prepare a series of lemmata, some of which have independent importance.

LEMMA 3.3. *Let $\eta \in \mathcal{P}^u$ with $\eta = \beta^u$, $\beta \in \mathcal{P}$. Then $H_\mu(\eta|\zeta) = H_\mu(\beta|\zeta) \le H_\mu(\beta) < \infty$ for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$.*

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Proof. If $\eta = \beta^u$, $\beta \in \mathcal{P}$, since $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$ has diameter no more than $C_0\varepsilon_0$, then

$$\begin{aligned} H_\mu(\eta|\zeta) &= -\int_M \log \mu_x^\zeta(\eta(x))\, d\mu(x) \\ &= -\int_M \log \mu_x^\zeta(\beta(x) \cap W^u(x, \bar{\varepsilon}_0))\, d\mu(x) \\ &= -\int_M \log \mu_x^\zeta(\beta(x))\, d\mu(x) \\ &= H_\mu(\beta|\zeta) \le H_\mu(\beta) < \infty. \end{aligned}$$

□

Remark 3.4. We do not know if $H_\mu(\xi|\zeta)$ is finite for any $\xi \in \mathcal{Q}^u$, $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, as the elements of ξ could have arbitrarily small size. Lemma 3.3 will be technically important in the proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$.

LEMMA 3.5. *$h_\mu(f, \alpha|\eta) := \lim_{n\to\infty} (1/n) H_\mu(\alpha_0^{n-1}|\eta)$ exists for any $\eta \in \mathcal{P}^u$ and any $\alpha \in \mathcal{P}$ with* $\operatorname{diam}(\alpha) < \epsilon_0/\lambda$.

Proof. Let $\eta = \beta^u \in \mathcal{P}^u$. By Lemmas 2.3(ii), (iv), (v), and 2.5, one has

$$\begin{aligned} H_\mu(\alpha_0^{m+n-1}|\eta) &= H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(f^{-n}\alpha_0^{m-1}|\alpha_0^{n-1} \vee \eta) \\ &= H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|\alpha_{-n}^{-1} \vee f^n\eta) \\ &\le H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|(f\alpha)^u) \\ &\le H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|\eta) + H_\mu(\eta|(f\alpha)^u). \end{aligned} \tag{1}$$

Since $\eta = \beta^u$, $\eta(x) = \beta(x) \cap W^u(x, \bar{\varepsilon}_0)$. The diameter of $(f\alpha)^u(x)$ with respect to d^u is no more than $C_0\epsilon_0 < \bar{\varepsilon}_0$, so it follows that $(f\alpha)^u(x) \subset W^u(x, \bar{\varepsilon}_0)$. Then by Lemma 3.3, we have

$$H_\mu(\eta|(f\alpha)^u) \le H_\mu(\beta).$$

By (1), $H_\mu(\alpha_0^{m+n-1}|\eta) \le H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\alpha_0^{m-1}|\eta) + H_\mu(\beta)$. That is, $\{H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\beta)\}$ is a subadditive sequence. Hence we have

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = \inf_{n\in\mathbb{N}} \frac{H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\beta)}{n}. \tag{2}$$

□

LEMMA 3.6. *$h_\mu^u(f) = h_\mu(f|\eta) = h_\mu(f, \alpha|\eta)$ for any $\eta \in \mathcal{P}^u$ and $\alpha \in \mathcal{P}$ with* $\operatorname{diam}(\alpha) < \varepsilon_0/\lambda$.

Proof. First we claim that $\alpha_0^\infty \vee \eta = \epsilon$, where ϵ is the partition of M into points. Indeed, if $y \ne x$ and $y \in (\alpha_0^\infty \vee \eta)(x)$, then $y \in \eta(x)$ and $f^i(y) \in \alpha(f^i(x))$ for any $i \in \mathbb{N}$. Let $k \ge 1$ be the first natural number such that $d^u(f^k(x), f^k(y)) > C_0\varepsilon_0$. Then in fact $d^u(f^k(x), f^k(y)) \le \lambda d^u(f^{k-1}(x), f^{k-1}(y)) \le \lambda C_0\varepsilon_0$ by uniform expansion of W^u. A contradiction to $f^k(y) \in \alpha(f^k(x))$. This proves the claim.

Choose any $\beta \in \mathcal{P}$. For any $\rho > 0$, since $H_\mu(\beta|\alpha_0^\infty \vee \eta) = 0$ there exists $k \in \mathbb{N}$ such that $H_\mu(\beta|\alpha_0^{k-1} \vee \eta) < \rho$. Similarly, by Lemma 3.3, if $\eta = \gamma^u$ we have $H_\mu(\eta|\alpha_0^{k-1} \vee (f\alpha)^u) = H_\mu(\gamma|\alpha_0^{k-1} \vee (f\alpha)^u) < \rho$ for large enough k. By Lemma 2.3(iv), (ii),

$$\begin{aligned} H_\mu(\beta_0^{n-1}|\eta) &\le H_\mu(\beta_0^{n-1}|(\alpha_0^{k-1} \vee \eta)_0^{n-1}) + H_\mu((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta) \\ &\le nH_\mu(\beta|\alpha_0^{k-1} \vee \eta) + H_\mu((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta) \\ &< n\rho + H_\mu((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta). \end{aligned} \tag{3}$$

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Then by Lemma 2.3(ii), (iii),

$$\begin{aligned} H_\mu((\alpha_0^{k-1} \vee \eta)_0^{n-1}|\eta) &= H_\mu(\alpha_0^{n+k-2} \vee \eta_0^{n-1}|\eta) \\ &\le H_\mu(\alpha_0^{n+k-2}|\eta) + H_\mu(\eta_0^{n-1}|\alpha_0^{n+k-2} \vee \eta) \\ &\le H_\mu(\alpha_0^{n+k-2}|\eta) + \sum_{i=1}^{n-1} H_\mu(f^{-i}\eta|\alpha_0^{n+k-2} \vee \eta) \\ &\le H_\mu(\alpha_0^{n+k-2}|\eta) + \sum_{i=1}^{n-1} H_\mu(\eta|\alpha_{-i}^{n+k-i-2} \vee f^i\eta) \\ &\le H_\mu(\alpha_0^{n+k-2}|\eta) + \sum_{i=1}^{n-1} H_\mu(\eta|\alpha_0^{k-1} \vee (f\alpha)^u) \\ &\le H_\mu(\alpha_0^{n+k-2}|\eta) + (n-1)\rho. \end{aligned} \tag{4}$$

Here we used the fact that $\alpha_{-i}^{n+k-i-2} \vee f^i\eta \succeq \alpha_0^{k-1} \vee (f\alpha)^u$ by Lemma 2.5. Combining Lemma 3.5, (3), and (4), we have

$$\begin{aligned} h_\mu(f, \beta|\eta) &= \lim_{n\to\infty} \frac{1}{n} H_\mu(\beta_0^{n-1}|\eta) \le 2\rho + \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n+k-2}|\eta) \\ &= 2\rho + h_\mu(f, \alpha|\eta). \end{aligned}$$

Since $\rho > 0$ is arbitrary, $h_\mu(f, \beta|\eta) \le h_\mu(f, \alpha|\eta)$. Since $\beta \in \mathcal{P}$ is arbitrary, we have that $h_\mu(f, \alpha|\eta)$ is independent of $\alpha \in \mathcal{P}$ with $\operatorname{diam}(\alpha) < \varepsilon_0/\lambda$.

By Lemma 2.3(iv) we have

$$\begin{aligned} H_\mu(\alpha_0^{n-1}|\eta_1) &\le H_\mu(\alpha_0^{n-1}|\eta_2) + H_\mu(\eta_2|\eta_1), \\ H_\mu(\alpha_0^{n-1}|\eta_2) &\le H_\mu(\alpha_0^{n-1}|\eta_1) + H_\mu(\eta_1|\eta_2). \end{aligned}$$

Hence by using Lemma 3.3 we get

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta_1) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta_2).$$

Then $h_\mu(f|\eta)$ is independent of $\eta \in \mathcal{P}^u$. The proof of the lemma is complete. □

We state two lemmata here which will be used later. The proof is identical to that in [**11**]. We emphasize that equation (2) and Lemma 3.6 play an essential role in proving Lemma 3.8(b), i.e., the upper semi-continuity of $\mu \mapsto h_\mu^u(f)$.

LEMMA 3.7. [**11**, Proposition 2.14] *For any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$, the map $\mu \mapsto H_\mu(\alpha|\eta)$ from $\mathcal{M}(M)$ to $\mathbb{R}^+ \cup \{0\}$ is concave.*

Furthermore, the maps $\mu \mapsto h_\mu(f, \alpha|\eta)$ and $\mu \mapsto h_\mu^u(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ are affine.

Recall that for each partition $\alpha \in \mathcal{P}$, the partition η given by $\eta(x) = \alpha(x) \cap W^u(x, \bar{\varepsilon}_0)$ for any $x \in M$ is denoted by α^u. Conversely, for each partition $\eta \in \mathcal{P}^u$, there is a partition $\beta \in \mathcal{P}$ such that $\eta(x) = \beta(x) \cap W^u(x, \bar{\varepsilon}_0)$ for any $x \in M$. Denote such β by $\eta^{\mathcal{U}}$ and the set of Borel probability measures on M by $\mathcal{M}(M)$.

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LEMMA 3.8. [**11**, Proposition 2.15]

(a) *Let $\nu \in \mathcal{M}(M)$. For any $\alpha \in \mathcal{P}$ and $\eta \in \mathcal{P}^u$ with $\mu(\partial\alpha) = 0$ and $\mu(\partial\eta^u) = 0$, the map $\mu \mapsto H_\mu(\alpha|\eta)$ from $\mathcal{M}(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semi-continuous at μ, i.e.*

$$\limsup_{\nu\to\mu} H_\nu(\alpha|\eta) \le H_\mu(\alpha|\eta).$$

(b) *The unstable entropy map $\mu \mapsto h^u_\mu(f)$ from $\mathcal{M}_f(M)$ to $\mathbb{R}^+ \cup \{0\}$ is upper semi-continuous, i.e.*

$$\limsup_{\nu\to\mu} h^u_\nu(f) \le h^u_\mu(f).$$

We continue to the proof of Proposition 3.2 for $\xi \in \mathcal{Q}^u$. The above argument relies on the specific property of $\eta \in \mathcal{P}^u$, and unfortunately can not be applied for $\xi \in \mathcal{Q}^u$. Instead we refer to some results in [**11**]. The key is to relate $h_\mu(f, \alpha|\eta)$ with $h_\mu(f, \alpha|\xi)$.

LEMMA 3.9. *Let $\alpha \in \mathcal{P}$, $\eta \in \mathcal{P}^u$, and $\xi \in \mathcal{Q}^u$. Then for μ-almost every (a.e.) x,*

$$\liminf_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) = \liminf_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x),$$
$$\limsup_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(x) = \limsup_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x).$$

Proof. For μ-a.e. x,

$$\begin{aligned} &I_\mu(\alpha_0^{n-1}|\xi) + I_\mu(\eta|\alpha_0^{n-1} \vee \xi) = I_\mu(\alpha_0^{n-1} \vee \eta|\xi) = I_\mu(\alpha_0^{n-1}|\xi \vee \eta) + I_\mu(\eta|\xi), \\ &I_\mu(\alpha_0^{n-1}|\eta) + I_\mu(\xi|\alpha_0^{n-1} \vee \eta) = I_\mu(\alpha_0^{n-1} \vee \xi|\eta) = I_\mu(\alpha_0^{n-1}|\xi \vee \eta) + I_\mu(\xi|\eta). \end{aligned} \tag{5}$$

Since $\operatorname{diam}(\alpha_0^{n-1} \vee \xi)(x) \to 0$ and $\operatorname{diam}(\alpha_0^{n-1} \vee \eta)(x) \to 0$ as $n \to \infty$ for μ-a.e. x,

$$\lim_{n\to\infty} \frac{1}{n} I_\mu(\eta|\alpha_0^{n-1} \vee \xi)(x) = 0 = \lim_{n\to\infty} \frac{1}{n} I_\mu(\xi|\alpha_0^{n-1} \vee \eta)(x), \quad \mu\text{-a.e. } x,$$

Also, since $I_\mu(\xi|\eta)$ and $I_\mu(\eta|\xi)$ are finite μ-almost everywhere,

$$\lim_{n\to\infty} \frac{1}{n} I_\mu(\xi|\eta)(x) = 0 = \lim_{n\to\infty} \frac{1}{n} I_\mu(\eta|\xi)(x), \quad \mu\text{-a.e. } x.$$

By (5), for μ-almost every x,

$$I_\mu(\alpha_0^{n-1}|\xi) + I_\mu(\eta|\alpha_0^{n-1} \vee \xi) + I_\mu(\xi|\eta) = I_\mu(\alpha_0^{n-1}|\eta) + I_\mu(\xi|\alpha_0^{n-1} \vee \eta) + I_\mu(\eta|\xi).$$

Dividing by n, and taking lim inf and lim sup, we can get the equalities. □

Remark 3.10. The above argument only applies to the information function I_μ, but not to H_μ, since we do not know whether $H_\mu(\xi|\eta)$ is finite or not.

Theorem B in [**11**] established a version of the Shannon–McMillan–Breiman theorem for unstable metric entropy as follows.

THEOREM 3.11. [**11**, Theorem B] *Suppose that μ is an ergodic measure of f. Let $\eta \in \mathcal{P}^u$ be given. Then, for any partition α with $H_\mu(\alpha|\eta) < \infty$, we have*

$$\lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(x) = h_\mu(f, \alpha|\eta) \quad \mu\text{-a.e. } x \in M.$$

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By Lemmas 3.7 and 3.8, $\mu \mapsto h_\mu(f, \alpha|\eta)$ is affine and upper semi-continuous. Therefore, if $\mu = \int_{\mathcal{M}_f^e(M)} \nu \, d\tau(\nu)$ is the unique ergodic decomposition for $\mu \in \mathcal{M}_f(M)$ not necessarily ergodic, it follows from a classical result in convex analysis (cf. [**6**, Fact A.2.10 on p. 356]) and Theorem 3.11 that

$$h_\mu(f, \alpha|\eta) = \int_{\mathcal{M}_f^e(M)} h_\nu(f, \alpha|\eta) \, d\tau(\nu) = \int_{\mathcal{M}_f^e(M)} \lim_{n\to\infty} \frac{1}{n} I_\nu(\alpha_0^{n-1}|\eta) \, d\tau(\nu).$$

Let $\mathfrak{g}$ be the sub-σ-algebra consisting of all invariant subsets and let κ be a measurable partition such that $\mathfrak{g}$ is equivalent mod zero to the σ-algebra generated by κ. Recall the crucial fact that η refines κ by arguing similarly to [**16**, proof of Proposition 2.6]. Therefore, $I_\nu(\alpha_0^{n-1}|\eta)(x) = I_\mu(\alpha_0^{n-1}|\eta)(x)$ for μ-a.e. $x \in M$. Thus

$$h_\mu(f, \alpha|\eta) = \int \lim_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta) \, d\mu.$$

Hence by Fatou's lemma and Lemma 3.9,

$$\begin{aligned} h_\mu(f, \alpha|\xi) &\geq \liminf_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi) \geq \int \liminf_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi) \, d\mu \\ &= \int \liminf_{n\to\infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta) \, d\mu = h_\mu(f, \alpha|\eta). \end{aligned}$$

On the other hand, since

$$H_\mu(\alpha_0^{n-1}|\xi) \leq H_\mu(\alpha_0^{n-1}|\eta) + H_\mu(\eta|\xi),$$

and $H_\mu(\eta|\xi) < \infty$ by Lemma 3.3, we have

$$h_\mu(f, \alpha|\xi) = \limsup_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\xi) \leq \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) = h_\mu(f, \alpha|\eta).$$

Thus $h_\mu(f, \alpha|\xi) = h_\mu(f, \alpha|\eta)$. Combining this with Lemma 3.6 and Theorem 2.9, we proved Proposition 3.2. □

An immediate corollary of Proposition 3.2 is the following.

PROPOSITION 3.12. *If $\mathcal{U} \in \mathcal{C}_M$ has diameter small than ε_0/λ, then $h_\mu^+(f, \mathcal{U}|\zeta) = h_\mu(f|\zeta) = h_\mu^u(f)$ for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$.*

Obviously, $h_\mu^+(f, \mathcal{U}|\zeta)$ is independent of $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$. Therefore, we also write it as $h_\mu^{+,u}(f, \mathcal{U})$.

We prepare a lemma on $h_\mu^+(f^n, \mathcal{U}_0^{n-1}|\eta)$ here, which will be used in the proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$ in §3.3.

LEMMA 3.13. *$h_\mu^+(f^n, \mathcal{U}_0^{n-1}|\eta) = n h_\mu(f|\eta)$ for any $\eta \in \mathcal{P}^u$ and $n \in \mathbb{N}$.*

Note that $h_\mu^u(f^n) = n h_\mu^u(f)$, then $h_\mu(f^n|\eta) = n h_\mu(f|\eta)$ for any $\eta \in \mathcal{P}^u$ with sufficiently small diameter depending on n. But we can not obtain Lemma 3.13 directly from Proposition 3.12 since the constants λ and ε_0 for f^n vary as n changes.

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Proof. Let $\alpha \succeq \mathcal{U}_0^{n-1}$. First we claim that $\eta \vee \bigvee_{i=0}^{\infty} f^{-ni}\alpha = \epsilon$. Indeed, if $y \neq x$ and $y \in (\eta \vee \bigvee_{i=0}^{\infty} f^{-ni}\alpha)(x)$, then $y \in \eta(x)$ and $f^j(y) \in U_j(f^j(x))$ for any $j \in \mathbb{N}$ where $U_j(f^j(x))$ is an element in $\mathcal{U}$ containing $f^j(x)$. Let $k \geq 1$ be the first natural number such that $d^u(f^k(x), f^k(y)) > C_0\varepsilon_0$. Then $d^u(f^k(x), f^k(y)) \leq \lambda d^u(f^{k-1}(x), f^{k-1}(y)) \leq C_0\lambda\varepsilon_0$. A contradiction to $f^k(y) \in U_k(f^k(x))$. This proves the claim.

Choose any $\beta \in \mathcal{P}_M$. For any $\rho > 0$, since $H_\mu(\beta|\eta \vee \bigvee_{i=0}^{\infty} f^{-ni}\alpha) = 0$ there exists $k \in \mathbb{N}$ such that $H_\mu(\beta|\eta \vee \bigvee_{i=0}^{k-1} f^{-ni}\alpha) < \rho$. We have

$$\begin{aligned}
H_\mu\Big(\bigvee_{j=0}^{l-1} f^{-nj}\beta|\eta\Big) &\leq H_\mu\Big(\bigvee_{j=0}^{l-1} f^{-nj}\beta|\bigvee_{j=0}^{l-1} f^{-nj}\Big(\eta \vee \bigvee_{i=0}^{k-1} f^{-ni}\alpha\Big)\Big) \\
&\quad + H_\mu\Big(\bigvee_{j=0}^{l-1} f^{-nj}\Big(\eta \vee \bigvee_{i=0}^{k-1} f^{-ni}\alpha\Big)\Big|\eta\Big) \\
&\leq l H_\mu\Big(\beta|\eta \vee \bigvee_{i=0}^{k-1} f^{-ni}\alpha\Big) + H_\mu\Big(\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha \vee \bigvee_{j=0}^{l-1} f^{-nj}\eta|\eta\Big) \\
&< l\rho + H_\mu\Big(\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha|\eta\Big) + \sum_{j=1}^{l-1} H_\mu\Big(f^{-nj}\eta|\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha \vee \eta\Big) \\
&\leq l\rho + H_\mu\Big(\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha|\eta\Big) + \sum_{j=1}^{l-1} H_\mu\Big(\eta|\bigvee_{i=-j}^{k-j+l-2} f^{-ni}\alpha \vee f^{nj}\eta\Big) \\
&\leq l\rho + H_\mu\Big(\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha|\eta\Big) + \sum_{j=1}^{l-1} H_\mu\Big(\eta|\bigvee_{i=0}^{k-1} f^{-ni}\alpha \vee (f^n\alpha)^u\Big) \\
&\leq l\rho + H_\mu\Big(\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha|\eta\Big) + (l-1)\rho. \qquad (6)
\end{aligned}$$

In the last inequality we used the fact that $\bigvee_{i=-j}^{k-j+l-2} f^{-ni}\alpha \vee f^{nj}\eta \succeq \bigvee_{i=0}^{k-1} f^{-ni}\alpha \vee (f^n\alpha)^u$ by Lemma 2.5(ii). Dividing (6) by l, we have

$$\begin{aligned}
h_\mu(f^n, \beta|\eta) &= \lim_{l\to\infty} \frac{1}{l} H_\mu\Big(\bigvee_{j=0}^{l-1} f^{-nj}\beta|\eta\Big) \\
&\leq 2\rho + \lim_{l\to\infty} \frac{1}{l} H_\mu\Big(\bigvee_{i=0}^{k+l-2} f^{-ni}\alpha|\eta\Big) = 2\rho + h_\mu(f^n, \alpha|\eta).
\end{aligned}$$

Since $\beta \in \mathcal{P}$, $\rho > 0$, and $\alpha \succeq \mathcal{U}_0^{n-1}$ are arbitrary, we have $nh_\mu(f|\eta) = h_\mu(f^n|\eta) \leq h_\mu^+(f^n, \mathcal{U}_0^{n-1}|\eta)$. The other direction is obvious and the proof of the lemma is complete. □

Adapting an argument in [**20**], we prove the following.

PROPOSITION 3.14. *The local unstable entropy map $\mu \mapsto h_\mu^+(f, \mathcal{U}|\eta)$ is upper semi-continuous, where $\eta \in \mathcal{P}^u$ and $\mathcal{U} \in \mathcal{C}_M^o$.*

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Proof. By the proof of Lemma 3.5, we have

$$\begin{aligned} h_\mu^+(f,\mathcal{U}|\eta) &= \inf_{\alpha\in\mathcal{P}_M,\alpha\succeq\mathcal{U}} h_\mu(f,\alpha|\eta) \\ &= \inf_{\alpha\in\mathcal{P}_M,\alpha\succeq\mathcal{U}} \inf_{n\in\mathbb{N}} \frac{1}{n}(H_\mu(\alpha_0^{n-1}|\eta)+H_\mu(\beta)) \\ &= \inf_{n\in\mathbb{N}} \frac{1}{n} \inf_{\alpha\in\mathcal{P}_M,\alpha\succeq\mathcal{U}} (H_\mu(\alpha_0^{n-1}|\eta)+H_\mu(\beta)), \end{aligned}$$

where $\eta=\beta^u\in\mathcal{P}^u$ with $\mu(\partial\beta)=0$. By the property of upper semi-continuous functions, it suffices to prove that for each $n\in\mathbb{N}$, the map

$$\phi_n(\mu)=\inf_{\alpha\in\mathcal{P}_M,\alpha\succeq\mathcal{U}}(H_\mu(\alpha_0^{n-1}|\eta)+H_\mu(\beta))$$

is upper semi-continuous. By the definition of the upper semi-continuous function, it is suffice to prove that for each $\mu\in\mathcal{M}_f(M)$ and $\epsilon>0$,

$$\limsup_{\mu'\to\mu}\phi_n(\mu')\le\phi_n(\mu)+\epsilon.$$

Fix $\mu\in\mathcal{M}_f(M)$ and $\epsilon>0$. There exists an $\alpha\in\mathcal{P}$, $\alpha\succeq\mathcal{U}$ such that

$$H_\mu(\alpha_0^{n-1}|\eta)+H_\mu(\beta)\le\phi_n(\mu)+\epsilon/2.$$

Without loss of generality, we assume that $\alpha=\{A_1,\dots,A_d\}$, $A_m\subset U_m$, for all $m\in\{1,\dots,d\}$. Let $\mathcal{U}_{\mu,n}^*=\{\gamma\in\mathcal{P}_M:\gamma\succeq\mathcal{U}$ and $\mu(\bigcup_{C\in\gamma_0^{n-1}}\partial C)=0\}$. There exists a $\gamma=\{B_1,\dots,B_M\}\in\mathcal{U}_{\mu,n}^*$ with $H_\mu(\gamma_0^{n-1}|\alpha_0^{n-1})<\epsilon/2$ (see [**20**, proof of Theorem 1.1]). Since $\mu(\bigcup_{C\in\gamma_0^{n-1}}\partial C)=0$, then by Lemma 3.8(a), $\mu\mapsto H_\mu(\gamma_0^{n-1}|\eta)$ is upper semi-continuous. Since $\eta\in\mathcal{P}^u$ and W^u is a continuous foliation, η is a measurable partition with respect to any $\mu'\in\mathcal{M}_f(M)$. Then

$$\begin{aligned} \limsup_{\mu'\to\mu}\phi_n(\mu') &\le \limsup_{\mu'\to\mu} H_{\mu'}(\gamma_0^{n-1}|\eta)+H_{\mu'}(\beta) \\ &\le H_\mu(\gamma_0^{n-1}|\eta)+H_\mu(\beta) \\ &\le H_\mu(\alpha_0^{n-1}|\eta)+H_\mu(\gamma_0^{n-1}|\eta\vee\alpha_0^{n-1})+H_\mu(\beta) \\ &\le H_\mu(\alpha_0^{n-1}|\eta)+H_\mu(\gamma_0^{n-1}|\alpha_0^{n-1})+H_\mu(\beta) \\ &\le \phi_n(\mu)+\epsilon. \end{aligned}$$

□

In view of Proposition 3.12 for the case $\eta\in\mathcal{P}^u$ (for which Lemma 3.8(b) is not used in the proof) and Proposition 3.14, we verify again that the unstable entropy map $\mu\mapsto h_\mu^u(f)$ is upper semi-continuous (Lemma 3.8(b)). Recently, Yang [**29**] obtained a more general result, i.e., the upper semi-continuity of the unstable metric entropy with respect to both the invariant measures μ and the dynamical systems f. It was proved there by a different approach, that is, approximating $h_\mu(f,\xi)$, $\xi\in\mathcal{Q}^u$ by conditional entropies with respect to a sequence of finite partitions. Using this fact, Yang showed that the set of Gibbs u-states of $C^{1+\alpha}$ partially hyperbolic diffeomorphisms varies upper semi-continuously with respect to the diffeomorphisms in the C^1 topology.

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3.1.2. *Local unstable metric entropy $h_\mu(f, \mathcal{U}|\zeta)$.* We proceed to define another notion of local unstable metric entropy $h_\mu(f, \mathcal{U}|\zeta)$. Recall that we always suppose that $\mathcal{U} \in \mathcal{C}_M$ with $\operatorname{diam}(\mathcal{U}) \ll \varepsilon_0$, where ε_0 is defined at the beginning of §2.2. Define $H_\mu(\mathcal{U}|\zeta) = \inf_{\alpha \in \mathcal{P}_M, \alpha \succeq \mathcal{U}} H_\mu(\alpha|\zeta)$.

Definition 3.15. For $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, define

$$h_\mu(f, \mathcal{U}|\zeta) := \limsup_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\zeta).$$

We prove that for any $\eta \in \mathcal{P}^u$, the lim sup in the definition of $h_\mu(f, \mathcal{U}|\eta)$ is in fact a lim.

PROPOSITION 3.16. *For any $\eta \in \mathcal{P}^u$, $h_\mu(f, \mathcal{U}|\eta) = \lim_{n\to\infty} (1/n) H_\mu(\mathcal{U}_0^{n-1}|\eta)$.*

Proof. Let

$$\eta = \beta^u \in \mathcal{P}^u,$$

where $\beta \in \mathcal{P}$. Let $\alpha \succeq \mathcal{U}_0^{n-1}$, $\gamma \succeq \mathcal{U}_0^{m-1}$. Then by Lemma 2.3(ii), (v), (iv),

$$\begin{aligned} H_\mu(\mathcal{U}_0^{m+n-1}|\eta) &\le H_\mu(\alpha \vee f^{-n}\gamma|\eta) = H_\mu(\alpha|\eta) + H_\mu(f^{-n}\gamma|\alpha \vee \eta) \\ &= H_\mu(\alpha|\eta) + H_\mu(\gamma|f^n\alpha \vee f^n\eta) \\ &\le H_\mu(\alpha|\eta) + H_\mu(\gamma|(f^n\alpha)^u) \\ &\le H_\mu(\alpha|\eta) + H_\mu(\gamma|\eta) + H_\mu(\eta|(f^n\alpha)^u) \\ &\le H_\mu(\alpha|\eta) + H_\mu(\gamma|\eta) + H_\mu(\beta). \end{aligned}$$

We used the fact $H_\mu(\eta|(f^n\alpha)^u) \le H_\mu(\beta)$ by Lemmas 3.3 and 2.5(ii) with $k = 1$. Since $\alpha \succeq \mathcal{U}_0^{n-1}$, $\gamma \succeq \mathcal{U}_0^{m-1}$ are arbitrary, we have

$$H_\mu(\mathcal{U}_0^{m+n-1}|\eta) \le H_\mu(\mathcal{U}_0^{n-1}|\eta) + H_\mu(\mathcal{U}_0^{m-1}|\eta) + H_\mu(\beta).$$

That is, $\{H_\mu(\mathcal{U}_0^{n-1}|\eta) + H_\mu(\beta)\}$ is a subadditive sequence. Hence

$$\lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\eta)$$

exists. □

PROPOSITION 3.17. *$h_\mu(f, \mathcal{U}|\eta)$ is independent of $\eta \in \mathcal{P}^u$.*

Proof. For any $\alpha \succeq \mathcal{U}_0^{n-1}$, $H(\alpha|\eta_1) \le H(\alpha|\eta_2) + H(\eta_1|\eta_2)$. Taking infimum and limit, we have $h_\mu(f, \mathcal{U}|\eta_1) \le h_\mu(f, \mathcal{U}|\eta_2)$, as $H(\eta_1|\eta_2) < \infty$ for any $\eta_1, \eta_2 \in \mathcal{P}^u$ by Lemma 3.3. Similarly, $h_\mu(f, \mathcal{U}|\eta_2) \le h_\mu(f, \mathcal{U}|\eta_1)$. Thus $h_\mu(f, \mathcal{U}|\eta_1) = h_\mu(f, \mathcal{U}|\eta_2)$. □

3.1.3. *Relations between $h_\mu^+(f, \mathcal{U}|\zeta)$ and $h_\mu(f, \mathcal{U}|\zeta)$.* At first, the following relation between $h_\mu(f, \mathcal{U}|\zeta)$ and $h_\mu^+(f, \mathcal{U}|\zeta)$ is straightforward.

PROPOSITION 3.18. *$h_\mu(f, \mathcal{U}|\zeta) \le h_\mu^+(f, \mathcal{U}|\zeta)$ for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$.*

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Proof. For any $\alpha \succeq \mathcal{U}$, we have $\alpha_0^{n-1} \succeq \mathcal{U}_0^{n-1}$ and hence

$$\begin{aligned} h_\mu(f, \mathcal{U}|\zeta) &= \limsup_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\zeta) \le \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\zeta) \\ &= h_\mu(f, \alpha|\zeta). \end{aligned}$$

Since $\alpha \succeq \mathcal{U}$ is arbitrary, we have $h_\mu(f, \mathcal{U}|\zeta) \le h_\mu^+(f, \mathcal{U}|\zeta)$. □

The following lemmata will be useful in the proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$.

LEMMA 3.19. *For $\eta \in \mathcal{P}^u$, we have*
(1) $h_\mu(f, \mathcal{U}|\eta) = (1/N) h_\mu(f^N, \mathcal{U}_0^{N-1}|\eta)$ *for each* $N \in \mathbb{N}$;
(2) $h_\mu(f, \mathcal{U}|\eta) = \lim_{n\to\infty} (1/n) h_\mu^+(f^n, \mathcal{U}_0^{n-1}|\eta)$.

Proof. For part (1), by Proposition 3.16, $h_\mu(f, \mathcal{U}|\eta) = \lim_{n\to\infty} (1/n) H_\mu(\mathcal{U}_0^{n-1}|\eta)$. Thus

$$\frac{1}{N} h_\mu(f^N, \mathcal{U}_0^{N-1}|\eta) = \lim_{k\to\infty} \frac{1}{kN} H_\mu(\mathcal{U}_0^{kN-1}|\eta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\eta) = h_\mu(f, \mathcal{U}|\eta).$$

For part (2), recall that $h_\mu^+(f, \mathcal{U}|\eta) = \inf_{\alpha\in\mathcal{P}_M, \alpha\succeq\mathcal{U}} h_\mu(f, \alpha|\eta)$. Combining with Lemma 2.5(ii), we have

$$\begin{aligned} \frac{1}{N} h_\mu^+(f^N, \mathcal{U}_0^{N-1}|\eta) &= \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} h_\mu(f^N, \alpha|\eta) \\ &= \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} \lim_{k\to\infty} \frac{1}{k} H_\mu(\alpha \vee f^{-N}\alpha \vee \cdots \vee f^{-N(k-1)}\alpha|\eta) \\ &= \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} \lim_{k\to\infty} \frac{1}{k} (H_\mu(\alpha|\eta) + H_\mu(f^{-N}\alpha|\alpha \vee \eta) \\ &\quad + \cdots + H_\mu(f^{-N(k-1)}\alpha|\alpha \vee \cdots \vee f^{-N(k-2)}\alpha \vee \eta)) \\ &= \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} \lim_{k\to\infty} \frac{1}{k} (H_\mu(\alpha|\eta) + H_\mu(\alpha|f^N\alpha \vee f^N\eta) \\ &\quad + \cdots + H_\mu(\alpha|f^{N(k-1)}\alpha \vee \cdots \vee f^N\alpha \vee f^{N(k-1)}\eta)) \\ &\le \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} \lim_{k\to\infty} \frac{1}{k} (H_\mu(\alpha|\eta) + (k-1) H_\mu(\alpha|(f^N\alpha)^u) \quad \text{by Lemma 2.5(ii)} \\ &= \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} H_\mu(\alpha|(f^N\alpha)^u) \\ &\le \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} H_\mu(\alpha|\eta) + \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} H_\mu(\eta|(f^N\alpha)^u)) \quad \text{by Lemma 2.3(iv)} \\ &= \frac{1}{N} H_\mu(\mathcal{U}_0^{N-1}|\eta) + \inf_{\alpha\succeq\mathcal{U}_0^{N-1}} \frac{1}{N} H_\mu(\eta|(f^N\alpha)^u). \end{aligned}$$

By Lemma 3.3, $H_\mu(\eta|(f^N\alpha)^u) \le H(\beta) < \infty$, where $\beta \in \mathcal{P}$ is such that $\eta = \beta^u$. Thus $\limsup_{N\to\infty} (1/N) h_\mu^+(f^N, \mathcal{U}_0^{N-1}|\eta) \le \limsup_{N\to\infty} (1/N) H_\mu(\mathcal{U}_0^{N-1}|\eta) = h_\mu(f, \mathcal{U}|\eta)$.

On the other hand, by part (1) and Proposition 3.18,

$$h_\mu(f, \mathcal{U}|\eta) = \liminf_{N\to\infty} \frac{1}{N} h_\mu(f^N, \mathcal{U}_0^{N-1}|\eta) \le \liminf_{N\to\infty} \frac{1}{N} h_\mu^+(f^N, \mathcal{U}_0^{N-1}|\eta).$$

Part (2) follows. □

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3.2. *Local unstable topological entropy.* Following the definition of unstable topological entropy $h^u_{\text{top}}(f)$, it is natural to make the following definition.

Definition 3.20. For any $\mathcal{U} \in \mathcal{C}_M$, we define

$$h^u_{\text{top}}(f, \mathcal{U}) := \lim_{\delta \to 0} \sup_{x \in M} h_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)}),$$

where

$$h_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)}) := \limsup_{n \to \infty} \frac{1}{n} H(\mathcal{U}_0^{n-1} | \overline{W^u(x, \delta)}).$$

$h^u_{\text{top}}(f, \mathcal{U})$ is called the *local unstable topological entropy of f with respect to a Borel cover $\mathcal{U} \in \mathcal{C}_M$*. Since we are only interested in $\mathcal{U} \in \mathcal{C}_M$ with sufficiently small diameter, we suppose that $\text{diam}(\mathcal{U}) \ll \epsilon_0$ in the remainder of the paper. We show that the size of local unstable manifold that we use in the definition is irrelevant.

PROPOSITION 3.21. *$h^u_{\text{top}}(f, \mathcal{U}) = \sup_{x \in M} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)})$ for any $\delta > 0$.*

Proof. Since $\delta \mapsto \sup_{x \in M} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)})$ is increasing, it is clear that $h^u_{\text{top}}(f, \mathcal{U}) \leq \sup_{x \in M} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)})$ for any $\delta > 0$.

Fix $\delta > 0$. For any $\rho > 0$, there exists $y \in M$ such that

$$\sup_{x \in M} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)}) \leq h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(y, \delta)}) + \frac{\rho}{2}. \tag{7}$$

We choose $\delta_1 > 0$ small enough such that $\delta_1 < \delta$ and

$$\sup_{x \in M} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta_1)}) \leq h^u_{\text{top}}(f, \mathcal{U}) + \frac{\rho}{2}. \tag{8}$$

There exists a positive integer K, which depends on δ, δ_1, and the Riemannian structure on $\overline{W^u(y, \delta)}$ such that

$$\overline{W^u(y, \delta)} \subset \bigcup_{j=1}^{K} \overline{W^u(y_j, \delta_1)},$$

where $y_j \in \overline{W^u(y, \delta)}$. It follows that there is some y_i, $1 \leq i \leq K$ such that

$$\begin{aligned} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(y, \delta)}) &= \limsup_{n \to \infty} \frac{1}{n} \log N(\overline{W^u(y, \delta)}, \mathcal{U}_0^{n-1}) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log \left(\sum_{j=1}^{K} N(\overline{W^u(y_j, \delta_1)}, \mathcal{U}_0^{n-1}) \right) \\ &\leq \limsup_{n \to \infty} \frac{1}{n} \log K N(\overline{W^u(y_i, \delta_1)}, \mathcal{U}_0^{n-1}) \\ &= h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(y_i, \delta_1)}). \end{aligned} \tag{9}$$

By (7), (9), and (8), we have $\sup_{x \in M} h^u_{\text{top}}(f, \mathcal{U} | \overline{W^u(x, \delta)}) \leq h^u_{\text{top}}(f, \mathcal{U}) + \rho$. Since $\rho > 0$ is arbitrary, the proposition follows. □

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3.3. *Proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$.* Now we turn to study the relationship between the local unstable topological entropy and the local unstable metric entropies. A local unstable variational inequality is as follows.

PROPOSITION 3.22. *Let μ be any f-invariant measure and $\mathcal{U} \in \mathcal{C}_M$ with sufficiently small diameter. Then, for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$,*

$$h^u_\mu(f, \mathcal{U}|\zeta) \le h^u_{\text{top}}(f, \mathcal{U}).$$

Proof. For any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, the diameter of $\zeta(x)$ is bounded from above for every $x \in M'$ where $\mu(M') = 1$ (see the definition of $\mathcal{P}^u$ and $\mathcal{Q}^u$). Fix $\delta > 0$ such that $\zeta(x) \subset \overline{W^u(x, \delta)}$ for every $x \in M'$. At first, we show that for each $\mathcal{V} = \{V_1, V_2, \dots, V_k\} \in \mathcal{C}_M$ there exists $\alpha \in \mathcal{P}_M$ finer than $\mathcal{V}$ such that $H_\mu(\alpha|\zeta) \le \int_M \log N(\overline{W^u(y, \delta)}, \mathcal{V})\, d\mu(y)$.

Let $\mu = \int_M \mu^\zeta_y \, d\mu(y)$ be the disintegration of μ with respect to ζ. For each $y \in M'$, there exists $I_y \subset \{1, \dots, k\}$ with minimal cardinality no more than $N(\overline{W^u(y, \delta)}, \mathcal{V})$ such that $\bigcup_{i \in I_y} V_i \supset \zeta(y)$. Hence we can find $y_1, \dots, y_s \in M'$ such that for each $y \in M'$, $I_y = I_{y_i}$ for some $i \in \{1, \dots, s\}$. For $i = 1, \dots s$, define $D_i = \{y \in M' : \bigcup_{j \in I_{y_i}} V_j \supset \zeta(y)\}$. Clearly, D_i is open for $i = 1, \dots, s$ and $\mu(\bigcup_{i=1}^s D_i) = 1$.

Fix $i \in \{1, \dots, s\}$. Assume $I_{y_i} = \{k_1 < \cdots < k_{t_i}\}$, where $t_i \le N(\overline{W^u(y, \delta)}, \mathcal{V})$ for any $y \in D_i$. Take $\alpha_{y_i} = \{W_1(y_i), \dots, W_{t_i}(y_i)\}$, where

$$W_1(y_i) = V_{k_1},\ W_2(y_i) = V_{k_2} \setminus V_{k_1}, \dots, W_{t_i}(y_i) = V_{k_{t_i}} \Big\backslash \bigcup_{j=1}^{t_i - 1} V_{k_j}.$$

For any $y \in D_i$, since $\mu^\zeta_y(\bigcup_{j=1}^{t_i} W_j(y_i)) = \mu^\zeta_y(\bigcup_{j \in I_{y_i}} V_j) = 1$, α_{y_i} can be considered as a finite partition of $\zeta(y)$ and

$$H_{\mu^\zeta_y}(\alpha_{y_i}) \le \log t_i \le \log N(\overline{W^u(y, \delta)}, \mathcal{V}). \tag{10}$$

Let $C_1 = D_1$, $C_i = D_i \setminus \bigcup_{j=1}^{i-1} D_j$, $i = 2, \dots, s$ and

$$A = M \Big\backslash \bigcup_{i=1}^{s} \Big(\Big(\bigcup_{y \in C_i} \zeta(y) \Big) \cap \bigcup_{j=1}^{t_i} W_j(y_i) \Big).$$

Set $A_l = A \cap (V_l \setminus \bigcup_{j=1}^{l-1} V_j)$, $l = 1, \dots, k$. Then put

$$\alpha = \Big\{ \Big(\bigcup_{y \in C_1} \zeta(y) \Big) \cap W_1(y_1), \dots, \Big(\bigcup_{y \in C_1} \zeta(y) \Big) \cap W_{t_1}(y_1), \dots, \Big(\bigcup_{y \in C_s} \zeta(y) \Big) \cap W_1(y_s), \dots, \Big(\bigcup_{y \in C_s} \zeta(y) \Big) \cap W_s(y_s), A_1, \dots, A_k \Big\}.$$

Clearly, α is a finite measurable partition of M finer than $\mathcal{V}$. Now for $y \in C_i$, $i = 1, 2, \dots, s$, we have $H_{\mu^\zeta_y}(\alpha) = H_{\mu^\zeta_y}(\alpha_y) \le \log N(\overline{W^u(y_i, \delta)}, \mathcal{V})$ by inequality (10).

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Hence

$$\begin{aligned}
H_\mu(\alpha|\zeta) &= \int_M -\log \mu_y^\zeta(\alpha(y))\, d\mu(y) \\
&= \int_M \left(\int_{\zeta(y)} -\log \mu_y^\zeta(\alpha(x))\, d\mu_y^\zeta(x) \right) d\mu(y) \\
&= \int_M H_{\mu_y^\zeta}(\alpha)\, d\mu(y) \\
&= \int_{\bigcup_{i=1}^s C_i} H_{\mu_y^\zeta}(\alpha)\, d\mu(y) \\
&\le \int_M \log N(\overline{W^u(y,\delta)}, \mathcal{V})\, d\mu(y).
\end{aligned}$$

Thus $H_\mu(\mathcal{V}|\zeta) \le \int_M \log N(\overline{W^u(y,\delta)}, \mathcal{V})\, d\mu(y)$. By Fatou's lemma,

$$\begin{aligned}
h_\mu^u(f,\mathcal{U}|\zeta) &= \limsup_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\zeta) \\
&\le \limsup_{n\to\infty} \int_M \frac{1}{n} \log N(\overline{W^u(y,\delta)}, \mathcal{U}_0^{n-1})\, d\mu(y) \\
&\le \int_M \limsup_{n\to\infty} \frac{1}{n} \log N(\overline{W^u(y,\delta)}, \mathcal{U}_0^{n-1})\, d\mu(y) \\
&= \int_M h_{\mathrm{top}}^u(f,\mathcal{U}|\overline{W^u(y,\delta)})\, d\mu(y) \\
&\le \sup_{x\in M} h_{\mathrm{top}}^u(f,\mathcal{U}|\overline{W^u(x,\delta)}) = h_{\mathrm{top}}^u(f,\mathcal{U}).
\end{aligned}$$

This ends the proof of Proposition 3.22. □

We are ready to prove Theorem 1.1 for $\eta \in \mathcal{P}^u$, i.e., $h_\mu(f,\mathcal{U}|\eta) = h_\mu^+(f,\mathcal{U}|\eta) = h_\mu(f|\eta) = h_\mu^u(f)$ and $h_{\mathrm{top}}^u(f,\mathcal{U}) = h_{\mathrm{top}}^u(f)$ for $\mathcal{U} \in \mathcal{C}_M^o$ with sufficiently small diameter.

Proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$. Take $\mathcal{U} \in \mathcal{C}_M$ with $\mathrm{diam}(\mathcal{U}) \ll \epsilon_0$. Then it follows from Proposition 3.12 that $h_\mu^+(f,\mathcal{U}|\eta) = h_\mu(f|\eta) = h_\mu^u(f)$.

It is clear that by Proposition 3.18 $h_\mu(f,\mathcal{U}|\eta) \le h_\mu^+(f,\mathcal{U}|\eta) = h_\mu(f|\eta)$. For the other direction, by Lemmas 3.19 and 3.13 we have

$$h_\mu(f,\mathcal{U}|\eta) = \lim_{n\to\infty} \frac{1}{n} h_\mu^+(f^n, \mathcal{U}_0^{n-1}|\eta) = h_\mu(f|\eta).$$

For unstable topological entropy, it is clear that $h_{\mathrm{top}}^u(f,\mathcal{U}) \le h_{\mathrm{top}}^u(f)$ for any $\mathcal{U} \in \mathcal{C}_M^o$ by definition. For the other direction, by Proposition 3.22 we have

$$h_\mu(f|\eta) = h_\mu(f,\mathcal{U}|\eta) \le h_{\mathrm{top}}^u(f,\mathcal{U})$$

for any $\mu \in \mathcal{M}_f(M)$. Now we use the unstable variational principle Theorem 2.10 to see that

$$h_{\mathrm{top}}^u(f) = \sup\{h_\mu(f|\eta) : \mu \in \mathcal{M}_f(M)\} \le h_{\mathrm{top}}^u(f,\mathcal{U}).$$

This completes the proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$. □

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Remark 3.23. We do not know if there is a direct proof of Theorem 1.1 as above for $\xi \in \mathcal{Q}^u$. For any $\xi \in \mathcal{Q}^u$, $\eta \in \mathcal{P}^u$ and $\alpha \succeq \mathcal{U}_0^{n-1}$,

$$H(\alpha|\xi) \le H(\alpha|\eta) + H(\eta|\xi).$$

Since $H(\eta|\xi) < \infty$ by Lemma 3.3, it follows that

$$h_\mu(f, \mathcal{U}|\xi) \le h_\mu(f, \mathcal{U}|\eta).$$

However, the reverse inequality cannot be proved in this way, since we do not know if $H(\xi|\eta) < \infty$. Instead, we introduce and study unstable topological conditional entropy and unstable tail entropy in §4. The complete proof of Theorem 1.1 for $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$ is provided at the end of §4.

4. *Unstable topological conditional entropy and unstable tail entropy*

In this section, we define the notions of unstable topological conditional entropy and unstable tail entropy. We complete the proof of Theorem 1.1 by showing that both of them vanish (Theorem 1.3).

4.1. *Unstable topological conditional entropy.* Let $\mathcal{U} \in \mathcal{C}_M$ and $Y \subset M$. Set $N(Y, \mathcal{U}) := \min\{\text{the cardinality of } \mathcal{V} : \mathcal{V} \subset \mathcal{U}, \bigcup_{V\in\mathcal{V}} \supset Y\}$. Considering only the unstable direction, we can define

$$N^u(Y, \mathcal{U}) := \sup_{y\in Y} N(Y \cap \overline{W^u(y, \delta)}, \mathcal{U}),$$

where $0 < \delta < \varepsilon_0$ is an arbitrary small number. δ is fixed and omitted from the notation since it is irrelevant in the argument. We put $N^u(\emptyset, \mathcal{U}) = 1$ and write $N^u(\mathcal{U}) := N^u(M, \mathcal{U}) = \sup_{y\in M} N(\overline{W^u(y, \delta)}, \mathcal{U})$. For any two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_M$, we then define

$$N^u(\mathcal{U}|\mathcal{V}) := \max_{V\in\mathcal{V}} N^u(V, \mathcal{U}).$$

The above definition is an adaption of the classical topological conditional entropy in the sense of Misiurewicz (cf. [**22**]). Note that $N^u(\mathcal{U}|\Theta) = N^u(\mathcal{U})$ where $\Theta = \{M\} \in \mathcal{C}_M$.

We have the following proposition collecting some basic properties of the function N^u. The proof of them is straightforward and hence omitted here.

PROPOSITION 4.1.

(1) $N^u(Y, \mathcal{A}) \le N^u(Z, \mathcal{B})$ *for any* $Y \subset Z$, $\mathcal{B} \succeq \mathcal{A}$;
(2) $N^u(\mathcal{A}|\mathcal{B}) \le N^u(\mathcal{C}|\mathcal{D})$ *for any* $\mathcal{C} \succeq \mathcal{A}$, $\mathcal{B} \succeq \mathcal{D}$;
(3) $\sup_{y\in Y} N(f^{-1}(Y) \cap f^{-1}(\overline{W^u(y, \delta)}), f^{-1}\mathcal{A}) = N^u(Y, \mathcal{A})$;
(4) $N^u(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) \le N^u(\mathcal{A}|\mathcal{C})N^u(\mathcal{B}|\mathcal{A} \vee \mathcal{C})$;
(5) $N^u(\mathcal{A} \vee \mathcal{B}|\mathcal{C} \vee \mathcal{D}) \le N^u(\mathcal{A}|\mathcal{C})N^u(\mathcal{B}|\mathcal{D})$;
(6) $N^u(Y, \mathcal{A} \vee \mathcal{B}) \le N^u(Y, \mathcal{A})N^u(Y, \mathcal{B})$;
(7) $N^u(\mathcal{A}) \le N^u(\mathcal{B})N^u(\mathcal{A}|\mathcal{B})$;
(8) $N^u(\mathcal{A}|\mathcal{B}) \le N^u(\mathcal{A}|\mathcal{C})N^u(\mathcal{C}|\mathcal{B})$.

LEMMA 4.2. *If* $\operatorname{diam}(\mathcal{B}) < \varepsilon_0 \ll \delta$, *then* $\lim_{n\to\infty} (1/n) \log N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1})$ *exists.*

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Proof. We claim that $b \cap f^n\overline{W^u(y,\delta)} = b \cap \overline{W^u(f^n y,\delta)}$ for any $b \in \mathcal{B}_{-n}^{m-1}$ and $y \in f^{-n}b$. Indeed, if $z \in b \cap f^n\overline{W^u(y,\delta)}$, then $d(f^{-i}(z), f^{n-i}(y)) < \varepsilon_0 (0 \le i \le n)$ since $b \in \mathcal{B}_{-n}^{m-1}$. This forces $d^u(z, f^n(y)) < C_0\varepsilon_0$ and hence $z \in b \cap \overline{W^u(f^n y,\delta)}$. In view of Proposition 4.1(3) and (4), we have

$$\begin{aligned}
N^u(\mathcal{A}_0^{m+n-1}|\mathcal{B}_0^{m+n-1}) &\le N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{m+n-1})N^u(f^{-n}\mathcal{A}_0^{m-1}|\mathcal{B}_0^{m+n-1}) \\
&\le N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1}) \sup_{b\in\mathcal{B}_{-n}^{m-1}} \sup_{y\in f^{-n}b} N(f^{-n}b \cap \overline{W^u(y,\delta)}, f^{-n}\mathcal{A}_0^{m-1}) \\
&= N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1}) \sup_{b\in\mathcal{B}_{-n}^{m-1}} \sup_{y\in f^{-n}b} N(b \cap f^n\overline{W^u(y,\delta)}, \mathcal{A}_0^{m-1}) \\
&= N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1}) \sup_{b\in\mathcal{B}_{-n}^{m-1}} \sup_{f^n y\in b} N(b \cap \overline{W^u(f^n y,\delta)}, \mathcal{A}_0^{m-1}) \\
&= N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1})N^u(\mathcal{A}_0^{m-1}|\mathcal{B}_{-n}^{m-1}) \le N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1})N^u(\mathcal{A}_0^{m-1}|\mathcal{B}_0^{m-1}).
\end{aligned}$$

Thus $\log N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1})$ is subadditive. So

$$\lim_{n\to\infty}\frac{1}{n}\log N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1}) = \inf_{n\in\mathbb{N}}\frac{1}{n}\log N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1}). \qquad \square$$

Definition 4.3. The *unstable conditional entropy of f on the cover $\mathcal{A}$ with respect to the cover $\mathcal{B}$* is defined as

$$h^u(f,\mathcal{A}|\mathcal{B}) = \lim_{n\to\infty}\frac{1}{n}\log N^u(\mathcal{A}_0^{n-1}|\mathcal{B}_0^{n-1}).$$

Notice that $h^u(f,\mathcal{A}|\mathcal{B}) \le \log N^u(\mathcal{A}|\mathcal{B})$. For completeness, we also give a separate definition when $\mathcal{B} = \Theta$, which is related to the local unstable topological entropy $h^u_{\mathrm{top}}(f,\mathcal{A})$.

Definition 4.4. $h^u(f,Y,\mathcal{A}) := \limsup_{n\to\infty}(1/n)\log N^u(Y,\mathcal{A}_0^{n-1})$.

Notice that $h^u(f,\mathcal{A}|\Theta) = h^u(f,M,\mathcal{A})$. Compare with Definition 3.20, we know that $h^u(f,M,\mathcal{A}) \ge h^u_{\mathrm{top}}(f,\mathcal{A})$. However, they are not equal in general.

Using Proposition 4.1, we obtain some properties of unstable conditional entropy as follows.

PROPOSITION 4.5.

(1) $h^u(f,Y,\mathcal{A}) \le h^u(f,Z,\mathcal{B})$ *for any* $Y \subset Z$, $\mathcal{B} \succeq \mathcal{A}$;
(2) $h^u(f,\mathcal{A}|\mathcal{B}) \le h^u(f,\mathcal{C}|\mathcal{D})$ *for any* $\mathcal{C} \succeq \mathcal{A}$, $\mathcal{B} \succeq \mathcal{D}$;
(3) $h^u(f,M,\mathcal{A}) \le h^u(f,M,\mathcal{B}) + h^u(f,\mathcal{A}|\mathcal{B})$ *when* $\mathrm{diam}(\mathcal{B}) < \varepsilon_0$;
(4) $h^u_{\mathrm{top}}(f,\mathcal{A}) \le h^u_{\mathrm{top}}(f,\mathcal{B}) + h^u(f,\mathcal{A}|\mathcal{B})$ *when* $\mathrm{diam}(\mathcal{B}) < \varepsilon_0$;
(5) $h^u(f,\mathcal{A}|\mathcal{B}) \le h^u(f,\mathcal{A}|\mathcal{C}) + h^u(f,\mathcal{C}|\mathcal{B})$ *when* $\mathrm{diam}(\mathcal{B}), \mathrm{diam}(\mathcal{C}) < \varepsilon_0$.

We define the *unstable conditional entropy of f with respect to a Borel cover $\mathcal{B}$* as

$$h^u(f|\mathcal{B}) := \sup_{\mathcal{A}\in\mathcal{C}_M^o} h^u(f,\mathcal{A}|\mathcal{B}).$$

For $Y \subset M$, we define

$$h^u(f,Y) := \sup_{\mathcal{A}\in\mathcal{C}_M^o} h^u(f,Y,\mathcal{A}).$$

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It is clear that $h^u(f|\mathcal{A}) \le h^u(f|\mathcal{B})$ if $\mathcal{A} \succeq \mathcal{B}$ by Proposition 4.5(2). One can show that $h^u(f, M) \ge h^u_{\text{top}}(f)$. Further, the following property is important.

PROPOSITION 4.6. *$h^u(f, M) \le h^u(f, M, \mathcal{B}) + h^u(f|\mathcal{B})$ and $h^u_{\text{top}}(f) \le h^u_{\text{top}}(f, \mathcal{B}) + h^u(f|\mathcal{B})$ when $\mathcal{B} \in \mathcal{C}_M$ with $\operatorname{diam}(\mathcal{B}) < \varepsilon_0$.*

Proof. In view of Proposition 4.5(3),

$$\begin{aligned} h^u(f, M) &= \sup_{\mathcal{A}\in\mathcal{C}^o_M} h^u(f, M, \mathcal{A}) \le h^u(f, M, \mathcal{B}) + \sup_{\mathcal{A}\in\mathcal{C}^o_M} h^u(f, \mathcal{A}|\mathcal{B}) \\ &= h^u(f, M, \mathcal{B}) + h^u(f|\mathcal{B}). \end{aligned}$$

The second inequality is proved similarly using Proposition 4.5(4). □

Definition 4.7. Define the *unstable topological conditional entropy of f* in the sense of Misiurewicz as

$$h^{*u}(f) := \inf_{\mathcal{B}\in\mathcal{C}^o_M} h^u(f|\mathcal{B}).$$

4.2. *Unstable tail entropy.* Now we turn to define the unstable tail entropy of f in the sense of Bowen. For $y \in W^u(x)$, $d^u_n(x, y) := \max_{0\le i\le n-1} d^u(f^i(x), f^i(y)) = d^u(f^{n-1}(x), f^{n-1}(y))$. A set $E \subset W^u(x)$ is called an (n, ϵ') *u-spanning set* of $Y \cap \overline{W^u(x, \delta)}$ if $Y \cap \overline{W^u(x, \delta)} \subset \bigcup_{y\in E} B^u_n(y, \epsilon')$, where $B^u_n(y, \epsilon') = \{z \in W^u(x) : d^u_n(y, z) \le \epsilon'\}$ is the (n, ϵ') *u-Bowen ball around y*. Denote the smallest cardinality of an (n, ϵ') u-spanning set of $Y \cap \overline{W^u(x, \delta)}$ by $R^u_n(Y \cap \overline{W^u(x, \delta)}, \epsilon')$. Define

$$r^u_n(Y, \epsilon') := \sup_{y\in Y} R^u_n(Y \cap \overline{W^u(y, \delta)}, \epsilon').$$

Further, put

$$\bar{r}^u(Y, \epsilon') := \limsup_{n\to\infty} \frac{1}{n} \log r^u_n(Y, \epsilon'), \quad \tilde{h}^u(f, Y) := \lim_{\epsilon'\to 0} \bar{r}^u(Y, \epsilon').$$

Now consider $Y = \Phi(x, \epsilon) := \bigcap_{n=1}^{\infty} B_n(x, \epsilon)$, where $B_n(x, \epsilon)$ is the (n, ϵ) Bowen ball around x.

Definition 4.8. We define

$$h^{*u}(f, \epsilon) = \sup_{x\in M} \tilde{h}^u(f, \Phi(x, \epsilon)).$$

PROPOSITION 4.9. *The unstable tail topological entropy $h^{*u}(f, \epsilon)$ vanishes for sufficiently small ϵ.*

Proof. We claim that for $x, y \in M$, $\overline{W^u(y, \delta)} \cap \Phi(x, \epsilon_0)$ contains at most one point. Indeed, if $z, w \in \overline{W^u(y, \delta)} \cap \Phi(x, \epsilon_0)$, then $d(f^k(z), f^k(w)) \le \epsilon_0$ for any $k \ge 0$. Thus $d^u(f^k(z), f^k(w)) \le C_0\epsilon_0$ for any $k \ge 0$. This is impossible due to the expansion along unstable manifolds unless $z = w$. So

$$r^u_n(\Phi(x, \epsilon_0), \epsilon) = \sup_{y\in\Phi(x,\epsilon_0)} R^u_n(\Phi(x, \epsilon_0) \cap \overline{W^u(y, \delta)}, \epsilon) = 1.$$

Hence $\bar{r}^u(\Phi(x, \epsilon_0), \epsilon) = 0$ and $\tilde{h}^u(f, \Phi(x, \epsilon_0)) = 0$. We have $h^{*u}(f, \epsilon_0) = 0$ for sufficiently small $\epsilon_0 > 0$. □

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4.3. *Proof of Theorem 1.3.* In this subsection, we prove Theorem 1.3, i.e., both the unstable topological conditional entropy and the unstable tail entropy vanish. By Proposition 4.9, it remains to show that the unstable topological conditional entropy equals zero. The following proposition demonstrates the relationship between the unstable topological conditional entropy and the unstable tail entropy.

PROPOSITION 4.10. *If $Y \subset M$, $\mathcal{A}, \mathcal{E} \in \mathcal{C}_M^o$ with $C_0 diam(\mathcal{A}) < \epsilon < L/2$ where L is the Lebesgue number for $\mathcal{E}$, then*

$$N^u(Y, \mathcal{E}_0^{n-1}) \le r_n^u(Y, \epsilon) \le N^u(Y, \mathcal{A}_0^{n-1}).$$

Proof. Let $y \in Y$ and let $\mathcal{C} \subset \mathcal{A}_0^{n-1}$ be such that $Y \cap \overline{W^u(y, \delta)} \subset \bigcup_{C\in\mathcal{C}} C$ and $\operatorname{card} \mathcal{C} = N(Y \cap \overline{W^u(y, \delta)}, \mathcal{A}_0^{n-1})$. Pick one point from each $C \in \mathcal{C}$ and let E_n denote the set thus obtained. If $x \in C = \bigcap_0^{n-1} f^{-k} A_k$ for $A_k \in \mathcal{A}$, $k = 0, \dots, n-1$, we know $f^k(C) \subset A_k \subset B(f^k(x), \epsilon/C_0)$. Thus $C \cap \overline{W^u(y, \delta)} \subset B_n^u(x, \epsilon)$. Hence E_n is an (n, ϵ) u-spanning set of $Y \cap \overline{W^u(x, \delta)}$. As $y \in Y$ is arbitrary, $r_n^u(Y, \epsilon) \le N^u(Y, \mathcal{A}_0^{n-1})$.

Let $y \in Y$. If E_n is a minimal (n, ϵ) u-spanning set of $Y \cap \overline{W^u(y, \delta)}$, then for any $x \in E_n$, $B_n^u(x, \epsilon) \subset E(x)$ for an element $E(x)$ in $\mathcal{E}_0^{n-1}$ containing x. Then $Y \cap \overline{W^u(y, \delta)} \subset \bigcup_{x\in E_n} E(x)$. As $y \in Y$ is arbitrary, we have $N^u(Y, \mathcal{E}_0^{n-1}) \le r_n^u(Y, \epsilon)$. □

Now we prove a stronger version of the first part of Theorem 1.3, namely, if $\mathcal{A} \in \mathcal{C}_M$ with $\operatorname{diam}(\mathcal{A}) \ll \epsilon_0$, then $h^u(f|\mathcal{A}) = 0$. For any $\epsilon > 0$ let $\mathcal{D} \in \mathcal{C}_M^o$ with Lebesgue number 3ϵ. Then using Proposition 4.10

$$\begin{aligned} N^u(\mathcal{D}_0^{n-1}|\mathcal{A}_0^{n-1}) &\le \sup_{x\in M} N^u(B_n(x, \epsilon_0), \mathcal{D}_0^{n-1}) \\ &\le \sup_{x\in M} r_n^u(B_n(x, \epsilon_0), \epsilon). \end{aligned} \tag{11}$$

Fix $y \in B_n(x, \epsilon_0)$. Let $z \in B_n(x, \epsilon_0) \cap \overline{W^u(y, \delta)}$. Then $d(f^k(z), f^k(y)) \le 2\epsilon_0$ for any $0 \le k \le n-1$. Thus $d^u(f^k(z), f^k(y)) \le 2C_0\epsilon_0$ for any $0 \le k \le n-1$. That is

$$B_n(x, \epsilon_0) \cap \overline{W^u(y, \delta)} \subset B_n^u(y, 2C_0\epsilon_0).$$

Observe that $B_n^u(y, 2C_0\epsilon_0) \subset f^{-n} B^u(f^n(y), 2C_0\epsilon_0)$. One can find z_i, $1 \le i \le N$ with $B^u(f^n(y), 2C_0\epsilon_0) \subset \bigcup_{i=1}^N B^u(f^n(z_i), \epsilon)$, where $N = D(2C_0\epsilon_0/\epsilon)^k$, $k = \dim E^u$ for some $D > 1$. Thus $B_n(x, \epsilon_0) \cap \overline{W^u(y, \delta)} \subset \bigcup_{i=1}^N B_n^u(z_i, \epsilon)$. Hence $r_n^u(B_n(x, \epsilon_0), \epsilon) \le D(2C_0\epsilon_0/\epsilon)^k$. By (11), $N^u(\mathcal{D}_0^{n-1}|\mathcal{A}_0^{n-1}) \le D(2C_0\epsilon_0/\epsilon)^k$. It follows that $h^u(f, \mathcal{D}|\mathcal{A}) = 0$. Since $\mathcal{D} \in C_M^o$ has arbitrarily small diameter, we have $h^u(f|\mathcal{A}) = 0$. The proof of Theorem 1.3 is complete.

4.4. *Completing the Proof of Theorem 1.1.* In this subsection, we complete the proof of Theorem 1.1 for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, that is, if $\mathcal{U}$ is an open cover with sufficiently small diameter, then $h_\mu(f, \mathcal{U}|\zeta) = h_\mu^+(f, \mathcal{U}|\zeta) = h_\mu(f|\zeta) = h_\mu^u(f)$ and $h_{\text{top}}^u(f, \mathcal{U}) = h_{\text{top}}^u(f)$ for any $\mu \in \mathcal{M}_f(M)$ and $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$.

PROPOSITION 4.11. *For any $\mu \in \mathcal{M}_f(M)$, $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, and $\mathcal{U}, \mathcal{V} \in \mathcal{C}_M$ with sufficiently small diameter,*

$$H_\mu(\mathcal{V}|\zeta) \le H_\mu(\mathcal{U}|\zeta) + \log N^u(\mathcal{V}|\mathcal{U}).$$

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Proof. By the definition of $\mathcal{P}^u$ and $\mathcal{Q}^u$, for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, the diameter of $\zeta(x)$ is bounded from above for every $x \in M'$, where $\mu(M') = 1$. Fix $\delta > 0$ such that $\zeta(x) \subset \overline{W^u(x, \delta)}$ for every $x \in M'$.

Let $\mathcal{V} = \{V_1, \ldots, V_m\}$ and $\beta \in \mathcal{P}_M$ with $\beta \succeq \mathcal{U}$. Let $B \in \beta$. For each $y \in B \cap M'$, there exists $I_y \subset \{1, \ldots, m\}$ with minimal cardinality no more than $N^u(\mathcal{V}|\beta)$ such that $\bigcup_{i \in I_y} V_i \supset B \cap \zeta(y)$. Hence we can find $y_1, \ldots, y_s \in B$ such that for each $y \in B$, $I_y = I_{y_i}$ for some $i \in \{1, \ldots, s\}$. For $i = 1, \ldots s$, define $D_i = \{y \in B : \bigcup_{j \in I_{y_i}} V_j \supset B \cap \zeta(y)\}$. Let $C_1 = D_1$, $C_i = D_i \setminus \bigcup_{j=1}^{i-1} D_j$, $i = 2, \ldots, s$. Note that if $y \in C_i \cap M'$, then $\zeta(y) \subset C_i$.

Fix $i \in \{1, \ldots, s\}$. Assume $I_{y_i} = \{k_1 < \cdots < k_{t_i}\}$, where $t_i \le N^u(\mathcal{V}|\beta)$. Take $\alpha_i = \{W_1(y_i), \ldots, W_{t_i}(y_i)\}$, where

$$W_1(y_i) = C_i \cap V_{k_1},\ W_2(y_i) = C_i \cap (V_{k_2} \setminus V_{k_1}),\ \ldots,\ W_{t_i}(y_i) = C_i \cap \Bigg(V_{k_{t_i}} \setminus \bigcup_{j=1}^{t_i - 1} V_{k_j}\Bigg).$$

Then α_i is a finite partition of C_i and hence $\gamma_B = \alpha_1 \cup \cdots \cup \alpha_s$ is a partition of B. Suppose that $\beta = \{B_1, \ldots, B_n\}$. Then $\gamma = \gamma_{B_1} \cup \cdots \cup \gamma_{B_n}$ is a partition of M with $\gamma \succeq \mathcal{V}$. From the construction above we know $N(\gamma|\beta \vee \zeta) \le N^u(\mathcal{V}|\beta)$, which is no more than $N^u(\mathcal{V}|\mathcal{U})$. Here

$$N(\gamma|\beta \vee \zeta) := \max_{B \in \beta} \sup_{y \in B \cap M'} N(B \cap \zeta(y), \gamma).$$

It follows that

$$\begin{aligned} H_\mu(\gamma|\zeta) &\le H_\mu(\beta|\zeta) + H_\mu(\gamma|\beta \vee \zeta) \\ &= H_\mu(\beta|\zeta) + \int_{M'} H_{\mu_x^{\beta \vee \zeta}}(\gamma)\, d\mu(x) \\ &\le H_\mu(\beta|\zeta) + \log N(\gamma|\beta \vee \zeta) \\ &\le H_\mu(\beta|\zeta) + \log N^u(\mathcal{V}|\mathcal{U}). \end{aligned}$$

Hence $H_\mu(\mathcal{V}|\zeta) \le H_\mu(\gamma|\zeta) \le H_\mu(\beta|\zeta) + \log N^u(\mathcal{V}|\mathcal{U})$. Since $\beta \succeq \mathcal{U}$ is arbitrary, we have $H_\mu(\mathcal{V}|\zeta) \le H_\mu(\mathcal{U}|\zeta) + \log N^u(\mathcal{V}|\mathcal{U})$. □

PROPOSITION 4.12. *For any $\mu \in \mathcal{M}_f(M)$, $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, $\mathcal{U} \in \mathcal{C}_M$ with sufficiently small diameter,*

$$h_\mu(f|\zeta) \le h_\mu(f, \mathcal{U}|\zeta) + h^u(f|\mathcal{U}).$$

Proof. In view of Proposition 4.11 we have

$$H_\mu(\mathcal{V}_0^{n-1}|\zeta) \le H_\mu(\mathcal{U}_0^{n-1}|\zeta) + \log N^u(\mathcal{V}_0^{n-1}|\mathcal{U}_0^{n-1}).$$

Dividing by n and letting $n \to \infty$

$$h_\mu(f, \mathcal{V}|\zeta) \le h_\mu(f, \mathcal{U}|\zeta) + h^u(f, \mathcal{V}|\mathcal{U}).$$

Taking supremum over $\mathcal{V} \in \mathcal{C}_M^o$

$$h_\mu(f|\zeta) \le h_\mu(f, \mathcal{U}|\zeta) + h^u(f|\mathcal{U}).$$ □

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Proof of Theorem 1.1. Let $\mathcal{U} \in \mathcal{C}_M$ with sufficiently small diameter. As a corollary of Proposition 4.12 and Theorem 1.3, we have $h_\mu(f|\zeta) = h_\mu(f, \mathcal{U}|\zeta)$. As $h_\mu(f, \mathcal{U}|\zeta) \le h_\mu^+(f, \mathcal{U}|\zeta) \le h_\mu(f|\zeta)$, we also have $h_\mu(f|\zeta) = h_\mu^+(f, \mathcal{U}|\zeta)$, which is already proved in Proposition 3.12. $h^u_{\mathrm{top}}(f) \le h^u_{\mathrm{top}}(f, \mathcal{U})$ is already proved in §3.3 (in the proof of Theorem 1.1 for $\eta \in \mathcal{P}^u$) using the unstable variational principle. Nevertheless, it also follows from Proposition 4.6 and Theorem 1.3. If $\mathcal{U} \in \mathcal{C}_M^o$, then $h^u_{\mathrm{top}}(f, \mathcal{U}) \le h^u_{\mathrm{top}}(f)$ from the definition. This completes the proof of Theorem 1.1 for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$. □

5. *Generalization to the case of unstable pressure*

We make some generalizations of the previous results to the case of unstable pressure in this section. Recall that

$$P^u(f, \phi, \mathcal{U}, n, x, \delta) := \inf \left\{ \sum_{B \in \mathcal{V}} \sup_{y \in B \cap \overline{W^u(x,\delta)}} \exp((S_n\phi)(y)) | \mathcal{V} \in \mathcal{C}_M,\ \mathcal{V} \succeq \mathcal{U}_0^{n-1} \right\}$$

and $P^u(f, \phi, \mathcal{U}|\overline{W^u(x, \delta)}) := \limsup_{n\to\infty} (1/n) \log P^u(f, \phi, \mathcal{U}, n, x, \delta)$. The *unstable topological pressure of f with respect to $\phi \in C(M, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_M$* is defined as

$$P^u(f, \phi, \mathcal{U}) := \lim_{\delta\to 0} \sup_{x\in M} P^u(f, \phi, \mathcal{U}|\overline{W^u(x, \delta)}).$$

Similarly to Proposition 3.21, we can prove that

$$P^u(f, \phi, \mathcal{U}) = \sup_{x\in M} P^u(f, \phi, \mathcal{U}|\overline{W^u(x, \delta)}).$$

For $\mathcal{V} \in \mathcal{C}_M$, we let α be the Borel partition generated by $\mathcal{V}$ and let

$$\mathcal{P}^*(\mathcal{V}) = \{\beta \in \mathcal{P}_M : \beta \succeq \mathcal{V} \text{ and each atom of } \beta \text{ is the union of some atoms of } \alpha\}.$$

LEMMA 5.1. *Let $\mathcal{U} \in \mathcal{C}_M$ and $\phi \in C(M, \mathbb{R})$, then*

$$\inf_{\mathcal{V}\in\mathcal{C}_M, \mathcal{V}\succeq\mathcal{U}} \left\{ \sum_{B\in\mathcal{V}} \sup_{y\in B\cap\overline{W^u(x,\delta)}} \phi(y) \right\} = \min_{\beta\in\mathcal{P}^*(\mathcal{U})} \left\{ \sum_{B\in\beta} \sup_{y\in B\cap\overline{W^u(x,\delta)}} \phi(y) \right\}.$$

Proof. This is Lemma 2.1 in [**21**] with $M = \overline{W^u(x, \delta)}$. □

PROPOSITION 5.2. *For any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$, $h_\mu(f, \mathcal{U}|\zeta) + \int_M \phi\, d\mu \le P^u(f, \phi, \mathcal{U})$.*

Proof. Let $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$. We have

$$\begin{aligned} h_\mu(f, \mathcal{U}|\zeta) + \int_M \phi\, d\mu &= \limsup_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{U}_0^{n-1}|\zeta) + \int_M \phi\, d\mu \\ &= \limsup_{n\to\infty} \frac{1}{n} \int_M \left(H_{\mu_x^\zeta}(\mathcal{U}_0^{n-1}) + \int_M S_n\phi\, d\mu_x^\zeta \right) d\mu(x). \end{aligned} \tag{12}$$

Let $\delta > 0$ be such that $\zeta(x) \subset \overline{W^u(x, \delta)}$ for every $x \in M'$, where $\mu(M') = 1$. For any $\beta \in \mathcal{P}_M$ let $\beta_x = \{C : C = B \cap \zeta(x) \text{ for some } B \in \beta\}$ for any $x \in M'$. Then by Lemma 5.1,

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there exists a $\beta \in \mathcal{P}^*(\mathcal{U}_0^{n-1})$ such that

$$\begin{aligned}
\log P^u(f, \phi, \mathcal{U}, n, x, \delta) &= \log\Big(\sum_{B\in\beta} \sup_{y\in B\cap\overline{W^u(x,\delta)}} \exp((S_n\phi)(y))\Big) \\
&\geq \log\Big(\sum_{C\in\beta_x} \sup_{y\in C} \exp((S_n\phi)(y))\Big) \\
&\geq \sum_{C\in\beta_x} \mu_x^\zeta(C)\Big(\sup_{y\in C}(S_n\phi)(y) - \log \mu_x^\zeta(C)\Big) \\
&= H_{\mu_x^\zeta}(\beta_x) + \sum_{C\in\beta_x} \mu_x^\zeta(C) \sup_{y\in C}(S_n\phi)(y) \\
&\geq H_{\mu_x^\zeta}(\beta_x) + \int_M S_n\phi \, d\mu_x^\zeta \\
&\geq H_{\mu_x^\zeta}(\mathcal{U}_0^{n-1}) + \int_M S_n\phi \, d\mu_x^\zeta. \qquad (13)
\end{aligned}$$

Combining (12), (13), and by Fatou's lemma,

$$\begin{aligned}
h_\mu(f, \mathcal{U}|\zeta) + \int_M \phi \, d\mu &\leq \limsup_{n\to\infty} \frac{1}{n} \int_M \log P^u(f, \phi, \mathcal{U}, n, x, \delta) \, d\mu(x) \\
&\leq \int_M \limsup_{n\to\infty} \frac{1}{n} \log P^u(f, \phi, \mathcal{U}, n, x, \delta) \, d\mu(x) \\
&= \int_M P^u(f, \phi, \mathcal{U}|\overline{W^u(x, \delta)}) \, d\mu(x) \\
&\leq P^u(f, \phi, \mathcal{U}). \qquad (14)
\end{aligned}$$

This finishes the proof of Proposition 5.2. □

Now we prove Theorem 1.4, that is, if $\mathcal{U}$ is an open cover of M with sufficiently small diameter, then $P^u(f, \phi, \mathcal{U}) = P^u(f, \phi)$ for any $\phi \in C(M, \mathbb{R})$.

Proof of Theorem 1.4. Combining Proposition 5.2, Theorem 1.1, and the variational principle for unstable pressure in [**12**], we have

$$\begin{aligned}
P^u(f, \phi) &= \sup\left\{h_\mu(f|\zeta) + \int_M \phi \, d\mu : \mu \in \mathcal{M}_f(M)\right\} \\
&= \sup\left\{h_\mu(f, \mathcal{U}|\zeta) + \int_M \phi \, d\mu : \mu \in \mathcal{M}_f(M)\right\} \\
&\leq P^u(f, \phi, \mathcal{U}).
\end{aligned}$$

It is clear that $P^u(f, \phi, \mathcal{U}) \leq P^u(f, \phi)$ by definition if $\mathcal{U} \in \mathcal{C}_M^o$. The proof of Theorem 1.4 is complete. □

Though the notions of local unstable entropies and pressure are defined analogously as the relative local entropies in [**14**] and [**21**], there are significant differences between them since one uses the increasing partition $\xi \in \mathcal{Q}^u$ while the other uses the invariant partition given by fibers of the factor map as the conditionals. The following example illustrates such a difference.

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Remark 5.3. Consider a factor map π between two topological dynamical systems (X, T) and (Y, S). The following classical relative inner variational principle for pressure was obtained by Ledrappier and Walters [**7**, **17**]: for any $\nu \in \mathcal{M}_S(Y)$ and $\phi \in C(X, T)$,

$$\sup_{\mu \in \mathcal{M}_T(X)} \left\{ h_\mu(T, X|Y) + \int_X \phi(x)\, d\mu(x) : \pi\mu = \nu \right\} = \int_Y P(T, \phi, y)\, d\nu(y).$$

It is curious whether there is an inner variational principle for unstable pressure in our setting. In fact, we have proved one variational inequality in (14):

$$h_\mu(f, \mathcal{U}|\zeta) + \int_M \phi\, d\mu \le \int_M P^u(f, \phi, \mathcal{U}|\overline{W^u(x, \delta)})\, d\mu(x)$$

for any $\zeta \in \mathcal{P}^u \cup \mathcal{Q}^u$. However, the reverse direction is not true in general. We provide a counterexample here. Let $f = f_A : \mathbb{T}^2 \to \mathbb{T}^2$ be a linear toral automorphism, where $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Let $\phi(x) = 0$ for any $x \in \mathbb{T}^2$. Let $\mu = \delta_0$ be the Dirac measure supported on the origin $0 \in \mathbb{T}^2$, which is fixed by f. It is clear that $h_\mu(f, \mathcal{U}|\zeta) = 0$ for any $\mathcal{U} \in \mathcal{C}^o_{\mathbb{T}^2}$. However by the variational principle for unstable entropy in [**11**]

$$\begin{aligned} h^u_{\text{top}}(f) &= \sup\{h_\nu(f, \zeta) : \nu \in \mathcal{M}_f(\mathbb{T}^2)\} = \sup\{h_\nu(f) : \nu \in \mathcal{M}_f(\mathbb{T}^2)\} \\ &= h_{\text{top}}(f) = \log \lambda, \end{aligned}$$

where $\lambda = (3 + \sqrt{5})/2$. Thus by Theorem 1.1, $h^u_{\text{top}}(f, \mathcal{U}) = \log \lambda$ for any $\mathcal{U} \in \mathcal{C}^o_{\mathbb{T}^2}$ with small diameter. By Proposition 3.21, $h^u_{\text{top}}(f, \mathcal{U}) = \sup_{x \in \mathbb{T}^2} h^u_{\text{top}}(f, \mathcal{U}|\overline{W^u(x, \delta)})$. By linearity, we know that $h^u_{\text{top}}(f, \mathcal{U}|\overline{W^u(x, \delta)})$ is independent of $x \in \mathbb{T}^2$. Thus

$$h^u_{\text{top}}(f, \mathcal{U}|\overline{W^u(0, \delta)}) = h^u_{\text{top}}(f, \mathcal{U}) = \log \lambda.$$

Hence $\int_M h^u_{\text{top}}(f, \mathcal{U}|\overline{W^u(x, \delta)})\, d\mu(x) = \log \lambda$. We get

$$h_\mu(f, \mathcal{U}|\zeta) < \int_M h^u_{\text{top}}(f, \mathcal{U}|\overline{W^u(x, \delta)})\, d\mu(x).$$

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