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Quantum stochastic integration and quantum stochastic differential equations

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Abstract

Quantum stochastic integrals are constructed using the non-commutative L^p-space theory of Haagerup. The existence and uniqueness of the solution to quantum stochastic differential equations driven by quasi-Wiener noises, or noises satisfying generalized standing hypotheses, is established as is the Markov behaviour of the solution. Various examples of the theory are discussed, and quantum Ornstein–Uhlenbeck processes are obtained as explicit solutions.

0. *Introduction*

A unified treatment of the various quantum stochastic theories has been given recently in [**11**] in terms of the non-commutative L^p-spaces introduced by Haagerup. In this paper, we pursue this analysis further. In particular, we show that it is possible to develop a theory of stochastic integration where the integrator is not necessarily a martingale, but instead satisfies so-called generalized standing hypotheses (of the kind considered by McShane). This analysis extends that of [**7**].

A further class of integrators is that given by those whose squares have a partiularly simple kind of Doob–Meyer decomposition (the quasi-Wiener processes). We show that for such processes, it is possible to integrate an L^r-process with respect to an L^p-integrator for any $r \in [2, \infty]$.

The existence and uniqueness of the solution of stochastic differential equations driven by the above processes is proven under the usual Lipschitz conditions on the coefficients [**3**]. The results here generalize those of [**7**].

We solve explicitly the simple special case corresponding to the classical Langevin equation driven by various quantum noises. Thus we obtain various fermion and boson analogues of the classical Ornstein–Uhlenbeck (velocity) process.

We show, further, that for either a boson theory or a fermion theory, the solution to a general quantum stochastic differential equation exhibits Markov behaviour. This generalizes the result of [**4**], where the Itô–Clifford theory was considered.

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The outline of the paper is as follows. In Section 1, we very briefly set up the notation and the definition of Haagerup's L^p-spaces, and in Section 2 we construct the various stochastic integrals. Contraction and isometry relations are obtained for them. We also discuss, in Section 2, various standard quantum stochastic theories and show how they fit into the scheme developed here.

Stochastic differential equations are discussed in Section 3, where the existence and uniqueness of the solution are proved. The examples of what might be called quantum Ornstein–Uhlenbeck (velocity) processes are obtained, in Section 4, as explicit solutions to a simple kind of quantum stochastic differential equation.

Finally, in Section 5, returning to the general case, we establish the Markov behaviour of the solution to quantum stochastic differential equations in the context of quasi-free boson and fermion theories. The method used is an extension of that of [**4**].

1. *L^p-spaces and L^p-processes*

The starting point of our considerations is a non-commutative stochastic base $(\mathscr{A}, \omega, (\mathscr{A}_s)_{s\in\mathbb{R}^+})$, where $\mathscr{A}$ is a von Neumann algebra, ω is a faithful normal state on $\mathscr{A}$ and $(\mathscr{A}_s)$ is an increasing net of von Neumann subalgebras of $\mathscr{A}$. Let $(\sigma_t^\omega)_{t\in\mathbb{R}}$ denote the modular automorphism group on $\mathscr{A}$ determined by ω. We assume that the algebras $\mathscr{A}_s$ are expected, that is, for any $s\in\mathbb{R}^+$ and $t\in\mathbb{R}$, $\sigma_t^\omega(\mathscr{A}_s)\subseteq\mathscr{A}_s$. By a theorem of Takesaki [**23**], this is equivalent to the existence of a unique (faithful, normal) ω-invariant conditional expectation E_s of $\mathscr{A}$ onto $\mathscr{A}_s$.

By passing to the GNS representation of $\mathscr{A}$ induced by ω, we may assume that $\mathscr{A}$ acts in a Hilbert space $\mathscr{H}$ and that ω is given by a cyclic and separating unit vector Ω in $\mathscr{H}$.

Let $R(\mathscr{A}, \sigma^\omega)$ denote the crossed product of $\mathscr{A}$ by the action (σ^ω) of $\mathbb{R}$, i.e. the von Neumann algebra acting in $L^2(\mathbb{R};\mathscr{H})\simeq L^2(\mathbb{R})\otimes\mathscr{H}$ and generated by the operators $\pi(x)$, $x\in\mathscr{A}$ and $\lambda(s)$, $s\in\mathbb{R}$, where

$$(\pi(x)\,\xi)(t) = \sigma_{-t}^\omega(x)\,\xi(t),$$
$$(\lambda(s)\,\xi)(t) = \xi(t-s)$$

for $\xi\in L^2(\mathbb{R};\mathscr{H})$.

Denote by $(\theta_s)_{s\in\mathbb{R}}$ the dual action (of $\mathbb{R}$ on $R(\mathscr{A},\sigma^\omega)$) defined by

$$\theta_s(\pi(x)) = \pi(x),$$
$$\theta_s(\lambda(t)) = e^{-ist}\lambda(t).$$

There is a canonical normal faithful semifinite trace τ on $R(\mathscr{A},\sigma^\omega)$ satisfying $\tau\circ\theta_s = e^{-s}\tau$ [**24**].

If ϕ is a normal weight on $\mathscr{A}$, we denote by $\tilde{\phi}$ the corresponding dual weight on $R(\mathscr{A},\sigma^\omega)$ [**14**]. Let h_ϕ be the generalized positive operator satisfying $\tilde{\phi} = \tau(h_\phi\,\cdot)$ [**14**]. Then $\lambda(t) = h_\omega^{it}$ for $t\in\mathbb{R}$. Put $\mathrm{tr}\,(h_\phi) = \phi(\mathbf{1})$.

Denote by $\tilde{\mathscr{A}}$ the algebra of τ-measurable operators associated with $R(\mathscr{A},\sigma^\omega)$, equipped with the measure topology. The L^p-spaces of Haagerup are defined by

$$L^p(\mathscr{A}) = \{h\in\tilde{\mathscr{A}}\colon \forall s\in\mathbb{R}\ \theta_s(h) = e^{-s/p}h\},\quad p\in[1,\infty].$$

Then $L^1_+(\mathscr{A}) = \{h_\phi\colon \phi\in\mathscr{A}_*^+\}$, and $L^\infty(\mathscr{A}) = \pi(\mathscr{A})\simeq\mathscr{A}$.

Put

$$\|h\|_p = \mathrm{tr}\,(|h|^p)^{1/p},$$

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for $h \in L^p(\mathscr{A})$, $1 \leqslant p < \infty$, and equip $L^\infty(\mathscr{A})$ with the norm of $\mathscr{A}$, denoted $\|\cdot\|_\infty$. Then for each $p \in [1, \infty]$, $L^p(\mathscr{A})$ is a Banach space with respect to the norm $\|\cdot\|_p$.

We note that if $p \neq r$ then $L^p(\mathscr{A}) \cap L^r(\mathscr{A}) = \{0\}$, and that for $1 \leqslant p < \infty$, $L^p(\mathscr{A})$ consists entirely of unbounded operators (apart from 0).

If ω is a trace on $\mathscr{A}$, then these L^p-spaces may be identified with the usual tracial L^p-spaces [**10**, **13**, **17**, **21**, **22**, **25**, **27**].

The restriction of the mapping $\phi \mapsto h_\phi$ to $\mathscr{A}_*^+$ can be extended to a linear isometry of $\mathscr{A}_*$ onto $L^1(\mathscr{A})$. The 'trace' can also be extended to $L^1(\mathscr{A})$ by setting $\mathrm{tr}\,(h_\phi) = \phi(\mathbf{1})$.

If $1/p + 1/q = 1$, $h \in L^p(\mathscr{A})$, $k \in L^q(\mathscr{A})$, then $hk \in L^1(\mathscr{A})$ and $\mathrm{tr}\,(hk) = \mathrm{tr}\,(kh)$. In particular, $L^2(\mathscr{A})$ is a Hilbert space with inner product $\langle h, k\rangle_{L^2(\mathscr{A})} = \mathrm{tr}\,(k^*h)$.

The algebra $\mathscr{A}$ acts on $L^2(\mathscr{A})$ via $k \mapsto xk$, $x \in \mathscr{A}$, $k \in L^2(\mathscr{A})$ and if J_p denotes the conjugate linear isometry on $L^p(\mathscr{A})$ given by $J_p k = k^*$, $k \in L^p(\mathscr{A})$, then $(\mathscr{A}, L^2(\mathscr{A}), J_2, L^2(\mathscr{A})_+)$ is a standard form of $\mathscr{A}$ [**12**].

For further information see [**13**, **25**].

The formula $x \mapsto \mathrm{tr}\,(kx) = \mathrm{tr}\,(xk)$ for $x \in \mathscr{A}$ and $k \in L^1(\mathscr{A})$ establishes an order isomorphism between $L^1(\mathscr{A})$ and $\mathscr{A}_*$. Thus, if $h \in L^2(\mathscr{A})$ then $|h|^2 \in L^1(\mathscr{A})$ and is identified with the positive normal functional

$$\begin{aligned} x \mapsto \mathrm{tr}\,(|h|^2 x) &= \mathrm{tr}\,(h^*hx) \\ &= \mathrm{tr}\,(hxh^*) \\ &= \mathrm{tr}\,((J_2 h)^* x J_2 h) \\ &= \langle xJ_2 h, J_2 h\rangle_{L^2(\mathscr{A})} \\ &= (xJ\zeta, J\zeta)_{\mathscr{H}}, \end{aligned}$$

where $\zeta \in \mathscr{H}$ corresponds to $h \in L^2(\mathscr{A})$. We see that $|h|^2$ corresponds precisely to $|\zeta|^2$ as defined in [**8**].

The vector $h_\omega^{\frac{1}{2}} \in L^2(\mathscr{A})$ is cyclic and separating for $\mathscr{A}$ and so, by the uniqueness of the GNS representation, the cyclic representation $(\mathscr{A}, L^2(\mathscr{A}), h_\omega^{\frac{1}{2}})$ is unitarily equivalent to the original cyclic representation $(\mathscr{A}, \mathscr{H}, \Omega)$. Furthermore, by the uniqueness of the modular conjugation operator [**1**, **12**], J_2 corresponds to the modular conjugation operator J in $\mathscr{H}$ determined by Ω.

For any $s \in \mathbb{R}$, $R(\mathscr{A}_s, \sigma^\omega \restriction \mathscr{A}_s)$, $\tilde{\mathscr{A}_s}$ and $L^p(\mathscr{A}_s)$ can be canonically embedded into $R(\mathscr{A}, \sigma^\omega)$, $\tilde{\mathscr{A}}$ and $L^p(\mathscr{A})$, respectively. Furthermore, there is a canonical conditional expectation $\mathscr{E}_s$ (defined on a subset of $\tilde{\mathscr{A}}$) mapping $L^p(\mathscr{A})$ onto $L^p(\mathscr{A}_s)$ [**11**].

Let $S \subseteq \mathbb{R}^+$ be an index set. A stochastic process on S is a family $(X_s)_{s \in S}$ with $X_s \in \tilde{\mathscr{A}}$ for every $s \in S$. The stochastic process $(X_s)_{s \in S}$ is said to be adapted (to the filtration of the stochastic base) if $X_s \in \tilde{\mathscr{A}_s}$ for each $s \in S$. An L^p-process $(X_s)_{s \in S}$ is an adapted process such that $X_s \in L^p(\mathscr{A})$ for each $s \in S$, i.e. $X_s \in L^p(\mathscr{A}_s)$ for each $s \in S$. An L^p-process $(X_s)_{s \in S}$ is a martingale if $\mathscr{E}_s(X_t) = X_s$ for any $s, t \in S$, $s \leqslant t$.

2. *Stochastic integration*

We shall construct the stochastic integral for two important classes of stochastic processes, namely, those which satisfy certain L^p-inequalities (the generalized standing hypotheses), and those whose squares have a trivial Doob–Meyer decomposition (the quasi-Wiener processes).

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Definition 2·1. An L^p-process $(X_s)_{s\in\mathbb{R}^+}$ is said to satisfy the generalized standing hypotheses if for each $T>0$ there are Radon measures μ,ν on $\mathbb{R}^+$ such that for $0\leqslant s<t\leqslant T$

$$\|\mathscr{E}_s(X_t-X_s)\|_p\leqslant\mu([s,t)) \tag{2·1}$$

and

$$\|\mathscr{E}_s(|X_t-X_s|^2)\|_{p/2}\leqslant\nu([s,t)). \tag{2·2}$$

Any dominated martingale [11] and, in particular, any square-integrable martingale satisfies these hypotheses with $\mu=0$. If (X_s) is square-integrable, we can define ν by $\nu([s,t))=\operatorname{tr}(X_t^*X_t-X_s^*X_s)$. If a process satisfies the standing hypotheses as in [7] then it satisfies the generalized standing hypotheses; in this case we can take μ and ν to be proportional to Lebesgue measure on $\mathbb{R}^+$.

Suppose that $(X_s)_{s\in\mathbb{R}^+}$ is an L^p-process satisfying the generalized standing hypotheses. Fix $T>0$ and let $\mathscr{M}$ denote the collection of pairs of Radon measures (μ,ν) for which conditions (2·1) and (2·2) hold on $[0,T]$. Let $\mathscr{F}^T_{\mu,\nu}$ denote the set of adapted processes in $\mathscr{L}^1([0,T],\mu;L^q(\mathscr{A}))\cap\mathscr{L}^2([0,T],\nu;L^q(\mathscr{A}))$ where $1/p+1/q=\frac{1}{2}$. Endow $\mathscr{F}^T_{\mu,\nu}$ with the seminorm

$$\|f\|^T_{\mu,\nu}=2\int_{[0,T]}\|f(s)\|_q\,d\mu(s)+\left(\int_{[0,T]}\|f(s)\|_q^2\,d\nu(s)\right)^{\frac{1}{2}}$$

for $f\in\mathscr{F}^T_{\mu,\nu}$.

If $f=x\chi_{[a,b)}+f(T)\chi_{\{T\}}$, $0\leqslant a\leqslant b\leqslant T$, where $x\in L^q(\mathscr{A}_a)$, we set, as usual,

$$\int_{[0,T]}dXf=(X_b-X_a)x.$$

The stochastic integral is then extended to simple f by linearity.

THEOREM 2·2. *For any $f\in\mathscr{F}^T=\bigcup_{(\mu,\nu)\in\mathscr{M}}\mathscr{F}^T_{\mu,\nu}$ there is $(\mu,\nu)\in\mathscr{M}$ and a sequence (f_n) of simple processes in $\mathscr{F}^T_{\mu,\nu}$ such that $f_n\to f$ in $\mathscr{F}^T_{\mu,\nu}$ and $\int_{[0,T]}dX_sf_n(s)$ converges in $L^2(\mathscr{A})$. This limit is independent of (μ,ν) and the approximating sequence (f_n).*

We define the stochastic integral $\int_{[0,T]}dX_sf(s)$ as this limit. In the terminology of [11], $\mathscr{F}^T$ is an admissible class of integrands for (X_s).

LEMMA 2·3. *For any $f\in\mathscr{F}^T_{\mu,\nu}$, there is a sequence (f_n) of simple L^q-processes such that $\|f_n-f\|^T_{\mu,\nu}\to0$.*

Proof. Let $f\in\mathscr{F}^T_{\mu,\nu}$. Then since f is both μ and ν-measurable, it is $(\mu+\nu)$-measurable. We can now mimic the proof in [11, lemma 6·2]. ∎

LEMMA 2·4. *Let $\mathscr{A}_1,\dots,\mathscr{A}_m$ be expected von Neumann subalgebras of $\mathscr{A}$ and let Δ_j, $1\leqslant j\leqslant m$ be such that $\Delta_j\in L^p(\mathscr{A}_k)$ for all $j<k\leqslant m$. Then for any $u_1,\dots,u_m$ with $u_j\in L^q(\mathscr{A}_k)$, $1\leqslant j\leqslant k\leqslant m$, we have*

$$\left\|\sum_{j=1}^m\Delta_ju_j\right\|_2\leqslant2\sum_{j=1}^mC_j\|u_j\|_q+\left(\sum_{j=1}^mD_j\|u_j\|_q^2\right)^{\frac{1}{2}}, \tag{2·3}$$

where $C_j=\|\mathscr{E}_j\Delta_j\|_p$, $D_j=\|\mathscr{E}_j(\Delta_j^*\Delta_j)\|_{p/2}$, and $\mathscr{E}_j$ denotes the conditional expectation associated with $\mathscr{A}_j$.

Proof. This is essentially the same as that of [7, theorem 1·3], which in turn is a copy of that of McShane [19]. ∎

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Of course, it is the inequality (2·3) which allows progress beyond the integration of simple processes.

LEMMA 2·5. *For any simple L^q-process f on $[0, T]$ and any $(\mu, \nu) \in \mathscr{M}$,*

$$\left\| \int_{[0,T]} dX_s f(s) \right\|_2 \leqslant \|f\|^T_{\mu,\nu}. \tag{2·4}$$

Proof. We may write

$$f = \sum_{j=1}^{m} f(t_{j-1})\, \chi_{[t_{j-1}, t_j)} + f(T)\, \chi_{\{T\}}$$

for some $0 = t_0 < t_1 < \ldots < t_m = T$. Set $\Delta_j = X_{t_j} - X_{t_{j-1}}$ and $\mathscr{A}_j = \mathscr{A}_{t_{j-1}}$, $1 \leqslant j \leqslant m$. The result follows immediately from the definition of the stochastic integral of a simple process and lemma 2·4. ∎

Proof of Theorem 2·2. By lemmas 2·3 and 2·5, we deduce that for any $f \in \mathscr{F}^T$ there is $(\mu, \nu) \in \mathscr{M}$ and a sequence (f_n) of simple processes on $[0, T]$ such that $f_n \to f$ in $\mathscr{F}^T_{\mu,\nu}$ and the sequence $(\int_{[0,T]} dX_s f_n(s))$ of stochastic integrals is a Cauchy sequence in $L^2(\mathscr{A})$. Hence we may define

$$\int_{[0,T]} dX_s f(s) = \lim_n \int_{[0,T]} dX_s f_n(s).$$

To see that this is independent of the particular approximating sequence, we use the fact that $f \in \mathscr{F}^T_{\mu_1,\nu_1} \cap \mathscr{F}^T_{\mu_2,\nu_2}$ implies that $f \in \mathscr{F}^T_{\mu_1+\mu_2,\nu_1+\nu_2}$. Now we use an approximating sequence from $\mathscr{F}^T_{\mu_1+\mu_2,\nu_1+\nu_2}$ and deduces that this gives the same limit for the stochastic integral as any approximating sequence from $\mathscr{F}^T_{\mu_1,\nu_1}$ or from $\mathscr{F}^T_{\mu_2,\nu_2}$. ∎

Remark. It is easy to check that the stochastic integral satisfies the usual linearity properties, and is a martingale if (X_s) is. We also observe that the inequality (2·4) remains valid for any process $f \in \mathscr{F}^T_{\mu,\nu}$.

To obtain the right stochastic integral $\int_{[0,T]} f(s)\, dX_s$, one replaces (2·2) by the requirement that

$$\|\mathscr{E}_s(|X^*_t - X^*_s|^2)\|_{p/2} \leqslant \nu^*([s, t)) \tag{2·2*}$$

for some Radon measure ν^*.

If $f \in \mathscr{F}^T$ for every $T > 0$, then $\int_{[0,t]} dX_s f(s)$ is well-defined for all $t \geqslant 0$.

Definition 2·6. An L^p-process (X_s), with $p \geqslant 2$, is called a (left) quasi-Wiener process if there is a left-continuous increasing function $F: \mathbb{R}^+ \to \mathbb{R}$ such that the $L^{p/2}$-process $(X^*_s X_s - F(s)\, h^{2/p}_\omega)$ is a martingale.

Remarks. This terminology was introduced in [**11**] and was inspired by Levy's characterization of the classical Wiener process.

Squares of quasi-Wiener processes have a trivial Doob–Meyer decomposition:

$$X^*_s X_s = (X^*_s X_s - F(s)\, h^{2/p}_\omega) + F(s)\, h^{2/p}_\omega.$$

Many important processes are quasi-Wiener processes: the classical Wiener process, the fermion quantum field process and Wick martingales in the Itô–Clifford formalism, the creation and annihilation processes in quasi-free representations of the CAR and CCR algebras and exponential (Weyl) martingales (see [**2**, **3**, **5**, **26**] and the examples below).

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In [11], quasi-Wiener L^p-processes were treated as a special case of dominated L^p-martingales which permitted the integration of L^q-processes, where q is given by $1/p+1/q=\frac{1}{2}$. The stochastic integral used was, in fact, the contraction integral (see lemma 6·3 of [11]). The integral with respect to processes satisfying the generalized standing hypotheses above are also of this kind – the contraction property is given by the inequality (2·4). In the following, we shall define the stochastic integral with respect to quasi-Wiener processes using an isometry rather than a contraction. This will enable the integration of L^r-processes ($2 \leqslant r \leqslant \infty$) with respect to a quasi-Wiener L^p-process. Moreover, the two methods agree on the set of L^q-processes.

PROPOSITION 2·7. *Let (X_s) be a quasi-Wiener L^p-process and let f be a simple L^q-process on $[0, T]$. Then*

$$\left\| \int_{[0,T]} dX_s f(s) \right\|_2^2 = \int_{[0,T]} \|h_\omega^{1/p} f(s)\|_2^2 \, d\mu_F(s) \tag{2·5}$$

where μ_F is the Radon measure on $\mathbb{R}^+$ determined by the increasing function F.

Proof (cf. [2]). Writing f as

$$f = \sum_{i=1}^{n} f(t_{i-1}) \chi_{[t_{i-1}, t_i)} + f(T) \chi_{\{T\}}$$

with $0 = t_0 < t_1 < \ldots < t_n = T$, we have

$$\begin{aligned}
\left\| \int_{[0,T]} dX_s f(s) \right\|_2^2 &= \sum_{i,j=1}^{n} \operatorname{tr} (f(t_{j-1})^* (X_{t_j}^* - X_{t_{j-1}}^*)(X_{t_i} - X_{t_{i-1}}) f(t_{i-1})) \\
&= \sum_{i=1}^{n} \operatorname{tr} (f(t_{i-1})^* (X_{t_i}^* - X_{t_{i-1}}^*)(X_{t_i} - X_{t_{i-1}}) f(t_{i-1})) \\
&= \sum_{i=1}^{n} \operatorname{tr} (f(t_{i-1})^* \mathscr{E}_{t_{i-1}} (X_{t_i}^* X_{t_i} - X_{t_{i-1}}^* X_{t_{i-1}}) f(t_{i-1})) \\
&= \sum_{i=1}^{n} (F(t_i) - F(t_{i-1})) \operatorname{tr} (f(t_{i-1})^* h_\omega^{2/p} f(t_{i-1})) \\
&= \int_{[0,T]} \|h_\omega^{1/p} f(s)\|_2^2 \, d\mu_F(s). \quad \blacksquare
\end{aligned}$$

Let $\mathscr{K}_r$ denote the set of adapted processes in $\mathscr{L}^2_{\mathrm{loc}}(\mathbb{R}^+, \mu_F; L^r(\mathscr{A}))$.

LEMMA 2·8. *For any $r \geqslant 2$ and any $T > 0$, we have $h_\omega^{(r-2)/2r} \mathscr{K}_r \subseteq \mathscr{K}_2$ and*

$$\int_{[0,T]} \|h_\omega^{(r-2)/2r} f(s)\|_2^2 \, d\mu_F(s) \leqslant \int_{[0,T]} \|f(s)\|_r^2 \, d\mu_F(s)$$

for any $f \in \mathscr{K}_r$.

Proof. This is an application of Hölder's inequality: if $x \in L^r(\mathscr{A})$, then we have that $h_\omega^{(r-2)/2r} x \in L^2(\mathscr{A})$ and

$$\begin{aligned}
\|h_\omega^{(r-2)/2r} x\|_2 &\leqslant \|h_\omega^{(r-2)/2r}\|_{2r/(r-2)} \, \|x\|_r \\
&= \operatorname{tr} (h_\omega)^{(r-2)/2r} \, \|x\|_r \\
&= \|x\|_r.
\end{aligned}$$

The result follows. $\blacksquare$

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LEMMA 2·9. *For any $f\in\mathscr{K}_2$ and $T>0$, there is a sequence (f_n) of simple L^q-processes on $[0,T]$ such that*

$$\int_{[0,T]}\|h_\omega^{1/p}f_n(s)-f(s)\|_2^2\,d\mu_F(s)\to 0$$

as $n\to\infty$, where $1/p+1/q=\frac{1}{2}$.

Proof. The set of simple adapted L^2-processes is dense in the set of processes in $\mathscr{L}^2([0,T],d\mu_F;L^2(\mathscr{A}))$ with the seminorm

$$g\mapsto\left(\int_{[0,T]}\|g(s)\|_2^2\,d\mu_F(s)\right)^{\frac{1}{2}}$$

(see, for example, [**2**, **5**] and [**11**, lemma 6·2]). On the other hand, $h_\omega^{1/p}L^q(\mathscr{A}_s)$ is dense in $L^2(\mathscr{A}_s)$ for each $s\in\mathbb{R}^+$ and so the result follows. ∎

With the notation of Lemma 2·9, we see that the isometry relation (2·5) implies that $(\int_{[0,T]}dX_sf_n(s))$ is a Cauchy sequence in $L^2(\mathscr{A})$. This allows us to define $\int_{[0,T]}dX_sf(s)$ as its limit.

Definition 2·10. The stochastic integral with respect to the quasi-Wiener L^p-process (X_s) is defined for any $f\in\bigcup_{r\in[2,\infty]}\mathscr{K}_r$ as follows:

(1) if $f\in\mathscr{K}_2$, the integral $\int_{[0,T]}dX_sf(s)$ is the limit in $L^2(\mathscr{A})$ of the sequence

$$\left(\int_{[0,T]}dX_sf_n(s)\right)$$

where (f_n) is an approximating sequence as given in Lemma 2·9;

(2) if $f\in\mathscr{K}_r$, we put

$$\int_{[0,T]}dX_sf(s)=\int_{[0,T]}dX_s\,h_\omega^{(r-2)/2r}f(s),$$

where the right-hand side is defined according to (1).

Remarks. (*a*) For $f\in\mathscr{K}_2$ the integral is well-defined, as is easily checked. Furthermore, Lemma 2·8 shows that the integral is well-defined for any $r\in[2,\infty]$. (Note that if $r_1\neq r_2$ then $\mathscr{K}_{r_1}\cap\mathscr{K}_{r_2}$ consists only of the zero process.)

(*b*) There is no conflict between the definition 2·10 above and the earlier definition (resulting from Theorem 2·2) when $f\in\mathscr{K}_q$. In fact, $\mathscr{K}_q$ coincides with $\bigcap_{T>0}\mathscr{F}^T$; if $(\mu',\nu')\in\mathscr{M}$ then $\mu'\geqslant 0$ and $\nu'\geqslant\mu_F$, which implies that $\mathscr{F}^T=\mathscr{F}^T_{0,\mu_F}$, so that $\bigcap_{T>0}\mathscr{F}^T=\mathscr{K}_q$. Now, if $f\in\mathscr{K}_q$ and $T>0$, then by Lemma 2·3 there exists a sequence (f_n) of simple L^q-processes such that

$$\|f_n-f\|^T_{0,\mu_F}=\left(\int_{[0,T]}\|f_n(s)-f(s)\|_q^2\,d\mu_F(s)\right)^{\frac{1}{2}}\to 0,$$

as $n\to\infty$.

By Lemma 2·8,

$$\int_{[0,T]}\|h_\omega^{1/p}f_n(s)-h_\omega^{1/p}f(s)\|_2^2\,d\mu_F(s)\leqslant\int_{[0,T]}\|f_n(s)-f(s)\|_q^2\,d\mu_F(s)$$

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which shows that $(h_\omega^{1/p} f_n)$ is an approximating sequence for $h_\omega^{1/p} f$ in the sense of Definition 2·10 (Lemma 2·9). Hence both integrals are defined as the L^2-limit of the same sequence, namely $(\int_{[0,T]} dX_s f_n(s))$.

(*c*) One readily checks that the integral as defined by Definition 2·10 satisfies the usual linearity properties. Furthermore, the process $t \mapsto \int_{[0,t]} dX_s f(s)$ is an L^2-martingale.

(*d*) The isometry property for $f \in \mathscr{K}_2$ becomes

$$\left\| \int_{[0,T]} dX_s f(s) \right\|_2^2 = \int_{[0,T]} \|f(s)\|_2^2 \, d\mu_F(s).$$

This follows from the isometry relation (2·5) and Lemma 2·9 together with the construction of the stochastic integral.

(*e*) One can also consider a right quasi-Wiener L^p-process; i.e. an L^p-process (Y_s) such that $Y_s Y_s^* - G(s) h_\omega^{2/p}$ is a martingale for some left-continuous increasing function $G: \mathbb{R}^+ \to \mathbb{R}$. Instead of equation (2·5), we have

$$\left\| \int_{[0,T]} f(s) \, dY_s \right\|_2^2 = \int_{[0,T]} \|f(s) \, h_\omega^{1/p}\|_2^2 \, d\mu_G(s).$$

The stochastic integral $\int_{[0,T]} f(s) \, dY_s$ is then defined for any f belonging to the obvious analogue of $\mathscr{K}_r$, $r \in [2, \infty]$. Note that if (X_s) is a right quasi-Wiener process, then (X_s^*) is a left quasi-Wiener process (both having the same increasing function). Moreover, $(\int_{[0,T]} f(s) \, dY_s)^* = \int_{[0,T]} dY_s^* f(s)^*$.

(*f*) Suppose that (X_s) is a left quasi-Wiener L^p-process, where $2 < p \leqslant \infty$. For any α such that $1/p + 1/\alpha \leqslant \frac{1}{2}$ put $Y_s + X_s h_\omega^{1/\alpha}$. Then (Y_s) is a left quasi-Wiener $L^{\hat{p}}$-process, where $1/\hat{p} = 1/p + 1/\alpha$, with the same increasing function as for (X_s):

$$Y_s^* \, Y_s = h_\omega^{1/\alpha} X_s^* \, X_s h_\omega^{1/\alpha} = h_\omega^{2/\hat{p}} F(s).$$

For given $g \in \mathscr{K}_2$, there is a sequence (g_n) of simple $L^{\hat{q}}$-processes, where $1/\hat{p} = 1/\hat{q} = \frac{1}{2}$, such that

$$\int_{[0,T]} \|h_\omega^{1/\hat{p}} g_n - g\|_2^2 \, d\mu_F \to 0.$$

Now, $h_\omega^{1/\alpha} g_n$ is an L^q-process, and $h_\omega^{1/p} h_\omega^{1/\alpha} g_n = h_\omega^{1/\hat{p}} g_n$ so that

$$\int_{[0,T]} \|h_\omega^{1/p} (h_\omega^{1/\alpha} g_n) - g\|_2^2 \, d\mu_F \to 0.$$

Hence

$$\begin{aligned} \int_{[0,T]} dY \, g &= \lim_n \int_{[0,T]} dY \, g_n \\ &= \lim_n \int_{[0,T]} dX \, h_\omega^{1/\alpha} g_n \\ &= \int_{[0,T]} dX \, g. \end{aligned}$$

It follows that for any $f \in \mathscr{K}_r$

$$\int_{[0,T]} dY f = \int_{[0,T]} dX f.$$

An analogous remark applies also to the right quasi-Wiener processes.

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Examples

(*a*) *Classical Wiener process, free fermion field and fermion Wick martingales*

In any of these cases, we have a finite von Neumann algebra $\mathscr{A}$ – a commutative one for the Wiener process, and the hyperfinite II_1 factor for the free fermion field and Wick martingales [2] – and a tracial state ω on $\mathscr{A}$, given by a probability measure for the Wiener process and given by the Fock space vacuum vector for the fermion processes. The crossed product $R(\mathscr{A}, \sigma^\omega)$ is then isomorphic to $\mathscr{A} \otimes L^\infty(\mathbb{R})$ and $L^p(\mathscr{A})$ can be identified with $L^p(\mathscr{A}, \omega) \otimes \exp((\cdot)/p)$, where $L^p(\mathscr{A}, \omega)$ is the tracial L^p-space (see [25]). The operator h_ω corresponds to $\mathbf{1} \otimes \exp(\cdot)$ on $\mathscr{H} \otimes L^2(\mathbb{R})$.

Let $(B_s)_{s \in \mathbb{R}^+}$ be the classical Wiener process. Since $B_s \in L^p(\mathscr{A}, \omega)$ for any $p \in [2, \infty)$, we can define a continuum of non-commutative versions of the process: $X_s^{(p)} \equiv B_s \otimes \exp((\cdot)/p)$. Let $(\mathscr{A}_s)$ be the filtration generated by (B_s) and let $f: \mathbb{R}^+ \to L^2(\mathscr{A}, \omega)$ be adapted to the filtration. Then one checks that for any $p \in [2, \infty)$

$$\int_0^t f(s)\, dB_s = \int_{[0,t]} dX_s^{(p)} f(s) \otimes \exp((\cdot)/2)$$

$$= \int_{[0,t]} f(s) \otimes \exp((\cdot)/2)\, dX_s^{(p)}.$$

If $f: \mathbb{R}^+ \to L^r(\mathscr{A}, \omega)$ for some $r \geqslant 2$ and is adapted, we can integrate the process $s \mapsto f(s) \otimes \exp((\cdot)/r)$ and obtain the same result.

$(X_s^{(p)})$ is a quasi-Wiener L^p-process with increasing function $F(s) = s$.

The free fermion field is an $\mathscr{A}$-valued process (Ψ_s), which becomes a quasi-Wiener L^p-process when identified with $\Psi_s \otimes \exp((\cdot)/p)$. It has increasing function $F(s) = s$.

Let $u_1, \ldots, u_n \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$ be real-valued and for $s \geqslant 0$ let

$$W_s = :\Psi(u_1 \chi_{[0,s]}) \ldots \Psi(u_n \chi_{[0,s]}):$$

be the fermion Wick martingale in the Itô–Clifford theory [2, 3, 6]. Then $X_s^{(p)} = W_s \otimes \exp((\cdot)/p)$ is a quasi-Wiener L^p-process with increasing function

$$F(s) = \int_0^s \ldots \int_0^s |u(s_1, \ldots, s_n)|^2\, ds_1 \ldots ds_n,$$

where $u(s_1, \ldots, s_n) = (n!)^{\frac{1}{2}} A(u_n \otimes \ldots u_1)(s_1, \ldots, s_n)$ and A denotes the anti-symmetrization projection. In particular, $(X_s^{(p)})$ satisfies the generalized standing hypotheses with $\mu = 0$ and $\nu([s,t)) = F(t) - F(s)$. Evidently, in general, the measure ν (when restricted to any bounded interval) need not be bounded by a measure proportional to Lebesgue measure.

(*b*) *Quasi-free fermion creation and annihilation processes*

Let (b_s^*), (b_s) be the fermion creation and annihilation processes on $\mathbb{R}^+$ in the gauge-invariant quasi-free representation corresponding to the two-point functions

$$\omega(b_t^* b_s) = \int_0^{\min\{s,t\}} \rho(r)\, dr$$

where $0 < \rho < 1$ almost everywhere on $\mathbb{R}^+$. This representation is realized concretely

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on $\mathscr{H}$, the tensor product of two copies of the antisymmetric Fock space over $L^2(\mathbb{R}^+)$, and ω is given by the vector $\Omega = \Omega_0 \otimes \Omega_0$, where Ω_0 is the Fock vacuum vector (see [5, 9]). Here (b_s^*) and (b_s) are $\mathscr{A}$-valued processes. Thus (b_s^*) and (b_s) can be viewed as left quasi-Wiener L^∞-processes with increasing functions $s \mapsto \int_0^s (1-\rho(t))\,dt$ and $s \mapsto \int_0^s \rho(t)\,dt$, respectively. (See [5] for the calculation giving this.) They can also be viewed as right quasi-Wiener L^∞-processes with increasing functions as above, but interchanged. It follows directly from the definitions that, for $L^2(\mathscr{A})$-valued f, $\int_{[0,T]} db_s f$ corresponds to $\int_0^T db_s f$ as constructed in [5]; with a similar remark applying to $\int_{[0,T]} db_s^* f$.

Consider now the L^2-process $(b_s h_\omega^{\frac{1}{2}})$. By remark (f) above, $(b_s h_\omega^{\frac{1}{2}})$ is a left quasi-Wiener process with the same increasing function as for the L^∞-process (b_s). However, if g is an L^∞-valued process, the integrals $\int_{[0,T]} d(b_s h_\omega^{\frac{1}{2}})\, g$ and $\int_{[0,T]} db_s\, g h_\omega^{\frac{1}{2}}$ are not the same, in general; but it is true that $\int_{[0,T]} d(b_s h_\omega^{\frac{1}{2}})\, g = \int_{[0,T]} db_s (h_\omega^{\frac{1}{2}} g)$.

We shall show now that $(b_s h_\omega^{\frac{1}{2}})$ is also a right quasi-Wiener L^2-process. Indeed, we note first that $(b_s h_\omega^{\frac{1}{2}})(b_s h_\omega^{\frac{1}{2}})^* = (J_2 b_s h_\omega^{\frac{1}{2}})^* (J_2 b_s h_\omega^{\frac{1}{2}})$. Now, the action of J_2 on $b_s h_\omega^{\frac{1}{2}}$ can be computed explicitly, via the equivalent action of $J = S\Delta^{-\frac{1}{2}}$ on $b_s \Omega$ in $\mathscr{H}$. In fact, from the known action of S and Δ, we obtain

$$\begin{aligned} J b_s \Omega &= J b(\chi_{[0,s]})\,\Omega \\ &= b^*(\chi_{[0,s]} \sqrt{[\rho/(1-\rho)]})\,\Omega. \end{aligned}$$

Hence, putting $\sqrt{[\rho/(1-\rho)]} = v$ and writing $b_s(v)$ for $b(\chi_{[0,s]} v)$,

$$(b_s h_\omega^{\frac{1}{2}})(b_s h_\omega^{\frac{1}{2}})^* = h_\omega^{\frac{1}{2}} b_s(v)\, b_s^*(v)\, h_\omega^{\frac{1}{2}}.$$

Now, using the CAR, we see that

$$b_s(v)\, b_s^*(v) = -b_s^*(v)\, b_s(v) + \int_0^s |v|^2\,dt.$$

Moreover, $(b_s^*(v)\, b_s(v) - \omega(b_s^*(v)\, b_s(v)))$ is a martingale [5]. It follows that $b_s h_\omega^{\frac{1}{2}}$ is a right quasi-Wiener process with increasing function

$$\begin{aligned} F(s) &= \int_0^s |v|^2\,dt - \omega(b_s^*(v)\, b_s(v)) \\ &= \int_0^s \rho/(1-\rho)\,dt - \int_0^s (\rho/(1-\rho))\,\rho\,dt \\ &= \int_0^s \rho\,dt. \end{aligned}$$

The (right integral) isometry relation (analogous to equation (2·5)) is then

$$\left\| \int_{[0,T]} f d(b_s h_\omega^{\frac{1}{2}}) \right\|_2^2 = \int_{[0,T]} \|f(s)\, h_\omega^{\frac{1}{2}}\|_2^2 \rho(s)\,ds$$

and $\int_{[0,T]} f d(b_s h_\omega^{\frac{1}{2}})$ corresponds to the stochastic integral $\int_0^T f db_s$ of [5], where f is L^∞-valued, and hence also when f is L^2-valued.

In an entirely analogous way, we can show that $(b_s^* h_\omega^{\frac{1}{2}})$ is a right quasi-Wiener process with increasing function $s \mapsto \int_0^s (1-\rho)\,dt$, and the right stochastic integral $\int_{[0,T]} f d(b_s^* h_\omega^{\frac{1}{2}})$ corresponds to $\int_0^T f db_s^*$ of [5].

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(c) *Quasi-free boson creation and annihilation processes and exponential martingales*

Let (a_s^*) and (a_s) be the boson creation and annihilation processes on $\mathbb{R}^+$ acting as usual on the tensor product of two copies of the symmetric Fock space over $L^2(\mathbb{R}^+)$ and with two-point functions

$$\omega(a_t^* a_s) = \int_0^{\min\{s,t\}} \tau(r)\, dr$$

where $\tau > 0$ almost everywhere on $\mathbb{R}^+$ and is locally bounded. We consider

$$\begin{aligned} |a_s h_\omega^{\frac{1}{2}}|^2(x) &= \operatorname{tr}((a_s h_\omega^{\frac{1}{2}})^* a_s h_\omega^{\frac{1}{2}} x) \\ &= \langle x J_2 a_s h_\omega^{\frac{1}{2}}, J_2 a_s h_\omega^{\frac{1}{2}} \rangle_{L^2(\mathscr{A})} \\ &= (xJa_s\Omega, Ja_s\Omega)_{\mathscr{H}} \end{aligned}$$

for $x \in \mathscr{A}$. Applying the technique of [**8**], we see that $(a_s h_\omega^{\frac{1}{2}})$ and $(a_s^* h_\omega^{\frac{1}{2}})$ are left quasi-Wiener L^2-processes with increasing functions $s \mapsto \int_0^s (1+\tau)\, dr$ and $s \mapsto \int_0^s \tau\, dr$, respectively.

Again, by using the explicit action of J_2, we see that

$$J_2 a_s h_\omega^{\frac{1}{2}} = a^*(\chi_{[0,s]} \sqrt{[\tau/(1+\tau)]})\, h_\omega^{\frac{1}{2}}$$

from which it follows [**5**] that $(a_s h_\omega^{\frac{1}{2}})$ is a right quasi-Wiener L^2-process with increasing function $s \mapsto \int_0^s \tau\, dr$. Similarly, $(a_s^* h_\omega^{\frac{1}{2}})$ is a right quasi-Wiener L^2-process with increasing function $s \mapsto \int_0^s (1+\tau)\, dr$. Moreover, for L^2-valued f, the stochastic integrals $\int_{[0,T]} f\, d(a_s h_\omega^{\frac{1}{2}})$ and $\int_{[0,T]} f\, d(a_s^* h_\omega^{\frac{1}{2}})$ correspond to $\int_0^T f\, da_s$ and $\int_0^T f\, da_s^*$ of [**5**]; the same isometry relation holding for each corresponding pair.

Finally, we note that the exponential martingales (U_s) and (V_s) of [**16**, **26**] are both left and right quasi-Wiener L^∞-processes with increasing function given by the map $s \mapsto \exp \int_0^s (1+2\tau(t))\, dt - 1$. Thus, the stochastic integrals $\int_{[0,T]} f\, d(U_s h_\omega^{\frac{1}{2}})$ and $\int_{[0,T]} f\, d(V_s h_\omega^{\frac{1}{2}})$ as constructed here, for an L^2-process f, correspond to those of [**26**].

3. *Stochastic differential equations*

In this section, we shall establish the existence and uniqueness of the solution to the stochastic differential equation

$$dY_t = dX_t F(Y_t, t) + G(Y_t, t)\, d\lambda(t) \tag{3·1}$$

for the classes of L^p-processes X_t discussed earlier. We suppose that λ is a non-atomic measure on $\mathbb{R}^+$.

THEOREM 3·1. *Let $0 \leqslant t_0 < T$ be given and let (X_s) be an L^p-process satisfying the generalized standing hypotheses, equations* (2·1), (2·2), *for non-atomic measures μ and ν. Suppose that F and G are continuous maps from $L^2(\mathscr{A}) \times \mathbb{R}^+$ into $L^q(\mathscr{A})$ and $L^2(\mathscr{A})$, respectively, with $1/p + 1/q = \frac{1}{2}$, and such that $F(L^2(\mathscr{A}_s), s) \subseteq L^q(\mathscr{A}_s)$ and $G(L^2(\mathscr{A}_s), s) \subseteq L^2(\mathscr{A}_s)$ for any $t_0 \leqslant s \leqslant T$. Further suppose that F and G are Lipschitz maps, i.e. there is a constant $K > 0$ such that*

$$\|F(X,s) - F(Y,s)\|_q + \|G(X,s) - G(Y,s)\|_2 \leqslant K \|X - Y\|_2 \tag{3·2}$$

for any $X, Y \in L^2(\mathscr{A}_s)$, and all $t_0 \leqslant s \leqslant T$.

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Then, for given $Y_0 \in L^2(\mathscr{A}_{t_0})$, there is a unique L^2-process (Y_s) on $[t_0, T]$ satisfying the stochastic differential equation (3·1) *with* $Y_{t_0} = Y_0$.

Proof. The proof is a straightforward extension of that in [3, 7]. For $t_0 \leqslant s \leqslant T$, set $Y_t^{(0)} = Y_0$ and define $Y_t^{(n)}$ inductively by

$$Y_t^{(n+1)} = Y_0 + \int_{t_0}^t dX_s F(Y_s^{(n)}, s) + \int_{t_0}^t G(Y_s^{(n)}, s)\, d\lambda(s). \tag{3·3}$$

Then, by induction, we see that the hypotheses of the theorem ensure that $Y^{(n)}$ is well-defined and is a continuous L^2-process. Using equations (2·4) and (3·2), and writing ΔF for $F(Y_s^{(n)}, s) - F(Y_s^{(n-1)}, s)$ etc., we have

$$\begin{aligned}
\|Y_t^{(n+1)} - Y_t^{(n)}\|_2 &\leqslant \left\| \int_{t_0}^t dX_s\, \Delta F \right\|_2 + \left\| \int_{t_0}^t \Delta G\, d\lambda(s) \right\|_2 \\
&\leqslant 2 \int_{t_0}^t \|\Delta F\|_q\, d\mu(s) + \left(\int_{t_0}^t \|\Delta F\|_q^2\, d\nu(s) \right)^{\frac{1}{2}} + \int_{t_0}^t \|\Delta G\|_2\, d\lambda(s) \\
&\leqslant 2K \int_{t_0}^t \|\Delta Y\|_2\, d\mu(s) + K \left(\int_{t_0}^t \|\Delta Y\|_2^2\, d\nu(s) \right)^{\frac{1}{2}} + K \int_{t_0}^t \|\Delta Y\|_2\, d\lambda(s).
\end{aligned}$$

Hence, for some constant c (depending on T), we have

$$\begin{aligned}
\|Y_t^{(n+1)} - Y_t^{(n)}\|_2^2 &\leqslant c \int_{t_0}^t \|Y_s^{(n)} - Y_s^{(n-1)}\|_2^2\, d\kappa \\
&\leqslant \sup_{t_0 \leqslant s \leqslant T} \|Y_s^{(1)} - Y^{(0)}\|^2\, c^n \kappa^{\otimes n}(\Delta_n^T),
\end{aligned}$$

where $\kappa = \mu + \nu + \lambda$ and $\Delta_n^T = \{(t_1, \ldots, t_n) : t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = T\}$. From this, it follows that, for some constant $C > 1$,

$$\begin{aligned}
\|Y_t^{(n+1)} - Y_t^{(k+1)}\|_2 &\leqslant \sum_{m=k+1}^n \|Y_t^{(m+1)} - Y_t^{(m)}\|_2 \\
&\leqslant \sum_{m=k+1}^n C^m (\kappa^{\otimes m}(\Delta_m^T))^{\frac{1}{2}} \\
&= \sum_{m=k+1}^n C^m (\kappa([0, T])^m / m!)^{\frac{1}{2}},
\end{aligned}$$

and we deduce that $(Y_t^{(n)})$ is L^2-Cauchy, uniformly on $[t_0, T]$. It follows that there is a continuous L^2-process (Y_t) on $[t_0, T]$ such that $Y_t^{(n)} \to Y_t$ in $L^2(\mathscr{A})$, uniformly on $[t_0, T]$. It is now straightforward to check that Y_t satisfies the stochastic differential equation (3·1) and that the solution is unique (see [3]). ∎

Remark. Precisely the same method establishes the existence and uniqueness of the solution to the stochastic differential equation

$$dY_t = \sum_{i=1}^m dX_t^i F_i(Y_t, t) + G(Y_t, t)\, d\lambda(t)$$

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under analogous hypotheses on the integrators $X_t^1, \ldots, X_t^m$ and the coefficients $F_1, \ldots, F_m$ and G. In particular, the result holds for integrators which are L^2-martingales. One could also incorporate right stochastic integrals as well. It is also straightforward to establish the stability with respect to perturbations of the initial condition and of the coefficients (see [**3**]).

For quasi-Wiener processes, we can obtain a somewhat stronger result.

THEOREM 3·2. *Let $X^1, \ldots, X^m$ be quasi-Wiener L^p-processes such that their associated increasing functions are continuous. Suppose that, for $1 \leqslant i \leqslant m$, F_i is a continuous adapted map from $L^2(\mathscr{A}) \times \mathbb{R}^+$ into $L^{r_i}(\mathscr{A})$ with $2 \leqslant r_i \leqslant \infty$ and such that, for given $0 \leqslant t_0 \leqslant T$, there is a constant $K > 0$ such that*

$$\|F_i(X,t) - F_i(Y,t)\|_{r_i} \leqslant K\|X - Y\|_2$$

for any $X, Y \in L^2(\mathscr{A})$ and all $t_0 \leqslant t \leqslant T$. Let G be as in Theorem 3·1. Then, for any given $Y_0 \in L^2(\mathscr{A}_{t_0})$, there is a unique solution on $[t_0, T]$ to the stochastic differential equation

$$dY_t = \sum_{i=1}^{m} dX_t^i F_i(Y_t, t) + G(Y_t, t)\, d\lambda(t) \tag{3·4}$$

with $Y_{t_0} = Y_0$.

Proof. The proof is just as before, but using equation (2·5). ∎

Remark. Theorem 3·1 is a generalization of the corresponding theorem from [**7**]. Theorem 3·2 is applicable when the integrators are the classical Wiener process, the free fermion quantum field or a fermion Wick martingale, quasi-free boson or fermion creation or annihilation processes, or boson exponential (Weyl) martingales, as described earlier.

4. *Quantum Ornstein–Uhlenbeck processes*

We will consider a particular example of the preceding analysis. Specifically, we shall consider the stochastic differential equation

$$dY_t = \gamma_1(t)\, \mathbf{1}\, dX_t^1 + \gamma_2(t)\, \mathbf{1}\, dX_t^2 + g(t)\, Y_t\, dt + f(t)\, h_\omega^{\frac{1}{2}}\, dt, \tag{4·1}$$

where (X_t^1) and (X_t^2) are certain (quantum) martingales and γ_1, γ_2, f and g are continuous functions on $\mathbb{R}^+$. Thus we can think of this as a quantum Ornstein–Uhlenbeck stochastic differential equation.

Fix $u, v \in L^2_{\mathrm{loc}}(\mathbb{R}^+)$. We consider four cases (and their corresponding filtrations).

Case 1. *Classical Wiener theory*

Let $X_t^1 = B_t(u)\, h_\omega^{\frac{1}{2}} \equiv B(\chi_{[0,t]} u)\, h_\omega^{\frac{1}{2}}$, where $B_t(u)$ is the classical Wiener process with $\mathbb{E}((B_t(u))^2) = \int_0^t |u(s)|^2\, ds$. Put $\gamma_2 = 0$, and write $\gamma_1 = \gamma$. This case yields the usual classical Ornstein–Uhlenbeck (velocity) process [**20**].

Case 2. *Itô–Clifford theory*

Let $X_t^1 = \Psi_t(u)\, h_\omega^{\frac{1}{2}}$ where $\Psi_t(u)$ is the fermion quantum field as in [**2**]. Put $\gamma_2 = 0$ and write $\gamma_1 = \gamma$.

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Case 3. *Quasi-free fermion theory*

Let $X_t^1 = b_t^*(u)\, h_\omega^{\frac{1}{2}}$, $X_t^2 = b_t(v)\, h_\omega^{\frac{1}{2}}$ where $b_t^*(u) = b^*(u\chi_{[0,t]})$ and $b_t(v) = b(v\chi_{[0,t]})$ are the fermion creation and annihilation processes in the quasi-free representation as discussed earlier [5].

Case 4. *Quasi-free boson theory*

Let $X_t^1 = a_t^*(u)\, h_\omega^{\frac{1}{2}}$, $X_t^2 = a_t(v)\, h_\omega^{\frac{1}{2}}$ where $a_t^*(u) = a^*(u\chi_{[0,t]})$ and $a_t(v) = a(v\chi_{[0,t]})$ are the boson and annihilation processes in the quasi-free representation as before [5].

As we have already noted, in Cases 1 and 2 the filtration lives in a finite von Neumann algebra, whereas in Cases 3 and 4 it lives (typically) in a type III von Neumann algebra. (Case 3 leads to a type II_1 algebra when $\rho = \frac{1}{2}$.)

THEOREM 4·1. *For given* $t_0 \geqslant 0$ *and* $Y_0 \in L^2(\mathscr{A}_{t_0})$ *the solution to equation* (4·1) *on* $[t_0, \infty)$ *and with* $Y_{t_0} = Y_0$ *is given as follows*:

Case 1

$$Y_t = \exp\left\{\int_{t_0}^t g(s)\,ds\right\} Y_0 + \int_{t_0}^t f(\tau)\exp\left\{\int_\tau^t g(s)\,ds\right\} d\tau\, h_\omega^{\frac{1}{2}} + B(\gamma u\chi_{[t_0,t]}\exp\left\{\int_\cdot^t g(s)\,ds\right\}) h_\omega^{\frac{1}{2}};$$

Case 2

$$Y_t = \exp\left\{\int_{t_0}^t g(s)\,ds\right\} Y_0 + \int_{t_0}^t f(\tau)\exp\left\{\int_\tau^t g(s)\,ds\right\} d\tau\, h_\omega^{\frac{1}{2}} + \Psi(\gamma u\chi_{[t_0,t]}\exp\left\{\int_\cdot^t g(s)\,ds\right\}) h_\omega^{\frac{1}{2}};$$

Case 3

$$Y_t = \exp\left\{\int_{t_0}^t g(s)\,ds\right\} Y_0 + \int_{t_0}^t f(\tau)\exp\left\{\int_\tau^t g(s)\,ds\right\} d\tau\, h_\omega^{\frac{1}{2}}$$
$$+ b^*\left(\gamma_1 u\chi_{[t_0,t]}\exp\left\{\int_\cdot^t g(s)\,ds\right\}\right) h_\omega^{\frac{1}{2}} + b\left(\overline{\gamma_2} v\chi_{[t_0,t]}\exp\left\{\int_\cdot^t \overline{g(s)}\,ds\right\}\right) h_\omega^{\frac{1}{2}};$$

Case 4

$$Y_t = \exp\left\{\int_{t_0}^t g(s)\,ds\right\} Y_0 + \int_{t_0}^t f(\tau)\exp\left\{\int_\tau^t g(s)\,ds\right\} d\tau\, h_\omega^{\frac{1}{2}}$$
$$+ a^*(\gamma_1 u\chi_{[t_0,t]}\exp\left\{\int_\cdot^t g(s)\,ds\right\}) h_\omega^{\frac{1}{2}} + a\left(\overline{\gamma_2} v\chi_{[t_0,t]}\exp\left\{\int_\cdot^t \overline{g(s)}\,ds\right\}\right) h_\omega^{\frac{1}{2}}$$

for $t_0 \leqslant t$.

Proof. From Section 3, we know that there is a unique solution and that it may be obtained as the limit of an iterative scheme. Let $Y_t^{(0)} = Y_0$ and set

$$Y_t^{(n+1)} = Y_0 + \int_{t_0}^t f(\tau)\,d\tau\, h_\omega^{\frac{1}{2}} + \int_{t_0}^t g(\tau)\, Y_\tau^{(n)}\,d\tau + \int_{t_0}^t \gamma_1(\tau)\,\mathbf{1}\,dX_\tau^1 + \int_{t_0}^t \gamma_2(\tau)\,\mathbf{1}\,dX_\tau^2$$

for $n \geqslant 1$. Then we know that $Y_t^{(n)}$ converges to the solution Y_t in $L^2(\mathscr{A})$, uniformly on compact sets. Now, we can write $Y_t^{(n+1)}$ as

$$Y_t^{(n+1)} = S_1^{(n+1)}\, Y_0 + S_2^{(n+1)}\, h_\omega^{\frac{1}{2}} + S_3^{(n+1)} + S_4^{(n+1)},$$

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where

$$S_1^{(n+1)} = 1 + \int_{t_0}^{t} g(s)\,ds + \int_{t_0}^{t} g(s_1) \int_{t_0}^{s_1} g(s_0)\,ds_0\,ds_1 + \dots + \int_{t_0}^{t} g(s_n) \int_{t_0}^{s_n} g(s_{n-1}) \dots \int_{t_0}^{s_1} g(s_0)\,ds_0\,ds_1 \dots ds_n,$$

$$S_2^{(n+1)} = \int_{t_0}^{t} f(s)\,ds + \int_{t_0}^{t} g(s_1) \int_{t_0}^{s_1} f(s_0)\,ds_0\,ds_1 + \dots + \int_{t_0}^{t} g(s_n) \int_{t_0}^{s_n} g(s_{n-1}) \dots \int_{t_0}^{s_2} g(s_1) \int_{t_0}^{s_1} f(s_0)\,ds_0 \dots ds_n,$$

$$S_3^{(n+1)} = \int_{t_0}^{t} \gamma_1(s)\,\mathbf{1}\,dX_s^1 + \int_{t_0}^{t} g(s_1) \int_{t_0}^{s_1} \gamma_1(s_0)\,\mathbf{1}\,dX_{s_0}^1\,ds_1 + \dots + \int_{t_0}^{t} g(s_n) \int_{t_0}^{s_n} g(s_{n-1}) \dots \int_{t_0}^{s_2} g(s_1) \int_{t_0}^{s_1} \gamma_1(s_0)\,\mathbf{1}\,dX_{s_0}^1\,ds_1 \dots ds_n,$$

$$S_4^{(n+1)} = \int_{t_0}^{t} \gamma_2(s)\,\mathbf{1}\,dX_s^2 + \int_{t_0}^{t} g(s_1) \int_{t_0}^{s_1} \gamma_2(s_0)\,\mathbf{1}\,dX_{s_0}^2\,ds_1 + \dots + \int_{t_0}^{t} g(s_n) \int_{t_0}^{s_n} g(s_{n-1}) \dots \int_{t_0}^{s_2} g(s_1) \int_{t_0}^{s_1} \gamma_2(s_0)\,\mathbf{1}\,dX_{s_0}^2\,ds_1 \dots ds_n.$$

As $n \to \infty$, we notice that $S_1^{(n+1)}$ converges to the solution of the differential equation $dx(t) = g(t)\,x(t)\,dt$ with $x(t_0) = 1$; i.e. $S_1^{(n+1)}$ converges to $\exp \int_{t_0}^{t} g(s)\,ds$ as $n \to \infty$.

Furthermore, as $n \to \infty$, $S_2^{(n+1)}$ converges to the solution to the differential equation

$$dy(t) = f(t)\,dt + g(t)\,y(t)\,dt$$

with $y(t_0) = 0$. That is, $S_2^{(n+1)}$ converges to

$$\int_{t_0}^{t} f(\tau) \exp\left\{ \int_{\tau}^{t} g(s)\,ds \right\} d\tau$$

as $n \to \infty$.

We must now discuss the explicit behaviour of $S_3^{(n+1)}$ and $S_4^{(n+1)}$, as $n \to \infty$, in the four cases under consideration. For illustration, we will only write out the analysis for $S_4^{(n+1)}$ and for Case 3, the quasi-free fermion theory.

First we note that for any $t \geqslant t_0$

$$\int_{t_0}^{t} \gamma_2(s)\,\mathbf{1}\,d(b_s(v)\,h_\omega^{\frac{1}{2}}) = b(\overline{\gamma_2}\,v\chi_{[t_0,t]})\,h_\omega^{\frac{1}{2}}.$$

This is a direct consequence of the conjugate-linearity and continuity of the map $b(\cdot)\,h_\omega^{\frac{1}{2}}$ on $L^2(\mathbb{R}^+)$ (see [**6**]).

Furthermore, for any f in $L^2(\mathbb{R}^r)$, we can write

$$b(f)\,h_\omega^{\frac{1}{2}} = \int_0^\infty \overline{f(s)}\,b(s)\,h_\omega^{\frac{1}{2}}\,ds,$$

where $b(s)$ is considered, as usual, as an operator-valued distribution (see, for example, [**15**]).

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The general term of $S_4^{(n+1)}$ is then

$$\int_{t_0}^{t} g(s_k) \int_{t_0}^{s_k} g(s_{k-1}) \dots \int_{t_0}^{s_2} g(s_1) \int_0^\infty \gamma_2(s_0)\, \overline{v(s_0)}\, \chi_{[t_0, s_1]}(s_0)\, b(s_0)\, h_\omega^{\frac{1}{2}}\, ds_0\, ds_1, \dots, ds_k$$

$$= \int_0^\infty \Big\{ \int_{s_{k-1}}^{t} g(s_k) \int_{s_{k-2}}^{t} g(s_{k-1}) \dots$$

$$\int_{s_1}^{t} g(s_2) \int_{s_0}^{t} g(s_1)\, ds_1 \dots ds_k \Big\} \gamma_2(s_0)\, \overline{v(s_0)}\, \chi_{[t_0, t]}(s_0)\, b(s_0)\, h_\omega^{\frac{1}{2}}\, ds_0$$

$$= \int_0^\infty \Big\{ \int_{s_0}^{t} g(s_k) \int_{s_0}^{s_k} g(s_{k-1}) \dots$$

$$\int_{s_0}^{s_2} g(s_1)\, ds_1 \dots ds_k \Big\} \gamma_2(s_0)\, \overline{v(s_0)}\, \chi_{[t_0, t]}(s_0)\, b(s_0)\, h_\omega^{\frac{1}{2}}\, ds_0.$$

The above manipulations are justified, for example, by considering the integrals as weak integrals on $L^2(\mathscr{A})$. Now, arguing as for $S_1^{(n+1)}$, we see that

$$\Big\{ 1 + \int_{s_0}^{t} g(s_1)\, ds_1 + \dots + \int_{s_0}^{t} g(s_n) \int_{s_0}^{s_{n-1}} g(s_{n-2}) \dots \int_{s_0}^{s_2} g(s_1)\, ds_1 \dots ds_k \Big\} \gamma_2(s_0)\, \overline{v(s_0)}\, \chi_{[t_0, t]}(s_0)$$

converges in $L^2(\mathbb{R}^+)$ (in fact, uniformly) as a function of s_0 to

$$\exp\Big(\int_{s_0}^{t} g(s)\, ds \Big) \gamma_2(s_0)\, \overline{v(s_0)}\, \chi_{[t_0, t]}(s_0)$$

as $n \to \infty$.

Hence

$$S_4^{(n+1)}\, h_\omega^{\frac{1}{2}} \to \int_0^\infty \gamma_2(s_0)\, \overline{v(s_0)} \exp\Big(\int_{s_0}^{t} g(s)\, ds \Big) \chi_{[t_0, t]}(s_0)\, b(s_0)\, h_\omega^{\frac{1}{2}}\, ds_0$$

$$= b\Big(\overline{\gamma_2(\cdot)}\, v(\cdot)\, \chi_{[t_0, t]}(\cdot) \exp\Big(\int_{\cdot}^{t} \overline{g(s)}\, ds \Big) \Big) h_\omega^{\frac{1}{2}},$$

as $n \to \infty$.

The limit of $S_3^{(n+1)}\, h_\omega^{\frac{1}{2}}$ is similarly treated, as are Cases 1, 2 and 4. ∎

Let us consider a special case of Case 4. Suppose that $f = 0$, $g = -\alpha$, $\gamma_1 = \gamma_2 = \sigma$, $t_0 = 0$, $Y_{t_0} = c h_\omega^{\frac{1}{2}}$, $u = v = 1$, where α, σ and c are constants. The stochastic differential equation becomes

$$dY_t = \sigma\, da_t^*\, h_\omega^{\frac{1}{2}} + \sigma\, da_t\, h_\omega^{\frac{1}{2}} - \alpha Y_t\, dt$$

with initial condition $Y_0 = c h_\omega^{\frac{1}{2}}$. The solution is

$$Y_t = e^{-\alpha t} c h_\omega^{\frac{1}{2}} + \sigma a^*(\chi_{[0,t]}\, e^{-\alpha(t-\cdot)})\, h_\omega^{\frac{1}{2}} + \sigma a(\chi_{[0,t]}\, e^{-\alpha(t-\cdot)})\, h_\omega^{\frac{1}{2}}.$$

The corresponding operator solution is

$$Z_t = e^{-\alpha t} c + \sigma a^*(\chi_{[0,t]}\, e^{-\alpha(t-\cdot)}) + \sigma a(\chi_{[0,t]}\, e^{-\alpha(t-\cdot)}).$$

(See [**5**, **18**] for the construction of an operator from a vector.) Thus, if we set

$$\Phi_t = \sigma a^*(\chi_{[0,t]} e^{-\alpha(t-\cdot)}) + \sigma a(\chi_{[0,t]}\, e^{-\alpha(t-\cdot)}),$$

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then (Φ_t) is a Gaussian process (with independent increments) with mean zero and variance

$$\operatorname{var}\Phi_t = \sigma^2 e^{-2\alpha t}\int_0^t (1+2\tau(s))\, e^{2\alpha s}\, ds.$$

If τ is constant, then we see that (Z_t) is Gaussian with mean $c\,e^{-\alpha t}$ and variance $\sigma^2(1+2\tau)(1-e^{-2\alpha t})/2\alpha$.

This is to be compared with the Ornstein–Uhlenbeck (velocity) process [**20**].

5. *Markov processes*

In [**4**], it was shown that the Itô–Clifford fermion field process and, more generally, the solutions to stochastic differential equations were Markov processes. We shall show that a similar result also holds within the formalism of the quasi-free stochastic theory.

For any subset $S \subseteq L^2(\mathscr{A})$, we denote by $\mathfrak{A}_S$ the smallest expected von Neumann subalgebra of $\mathscr{A}$ such that $S \subseteq L^2(\mathfrak{A}_S)$. Evidently, if $S \subseteq L^2(\mathscr{A}_t)$, for some t, then $\mathfrak{A}_S \subseteq \mathscr{A}_t$. If ω is a tracial state on $\mathscr{A}$, then $L^2(\mathscr{A}) \simeq L^2(\mathscr{A},\omega)$ and $\mathfrak{A}_S$ is the von Neumann subalgebra of $\mathscr{A}$ generated by the operators $\{\zeta\colon \zeta\in S\}$. Indeed, if $\zeta\in L^2(\mathscr{A},\omega)$ and $\mathscr{B}$ is a von Neumann subalgebra of $\mathscr{A}$, then $\zeta\in L^2(\mathscr{B},\omega)$ if and only if ζ is affiliated to $\mathscr{B}$. Thus, $\mathfrak{A}_S$ is the analogue of the σ-algebra generated by a family S of random variables.

Let F_1, F_2 and G be continuous Lipschitz maps from $L^2(\mathscr{A})\times\mathbb{R}^+$ into $L^2(\mathscr{A})$ as in Theorem 3·2, and suppose further that $F_1(\zeta,s)\in L^2(\mathfrak{A}_{\{\zeta\}})$ for any $\zeta\in L^2(\mathscr{A}_s)$, $t_0 \leqslant s \leqslant T$, with a similar condition holding for F_2 and G.

It follows that F_1 (and F_2 and G) is adapted in that $\zeta\in L^2(\mathscr{A}_s) \Rightarrow F_1(\zeta,s)\in L^2(\mathscr{A}_s)$.

We will treat the fermion and boson theories (almost) simultaneously. Let (A_t^*, A_t) denote either of the quasi-free fermion or boson creation and annihilation pair of processes (b_s^*, b_s) or (a_s^*, a_s). Then $(A_t^* h_\omega^{\frac12}, A_t h_\omega^{\frac12})$ is a pair of (left and right) quasi-Wiener L^2-processes.

For given $X_0\in L^2(\mathscr{A}_{t_0})$, let (X_t) denote the L^2-solution, on $[t_0, T]$, of the stochastic differential equation

$$dX_t = F_1(X_t,t)\, d(A_t^* h_\omega^{\frac12}) + F_2(X_t,t)\, d(A_t h_\omega^{\frac12}) + G(X_t,t)\, d\lambda \tag{5·1}$$

with $X_{t_0} = X_0$, as guaranteed by Theorem 3·2. (We could also consider left integrals, or a combination of both left and right integrals. The method of proof is the same.)

Let $\mathscr{B}_s = \mathfrak{A}_{\{X_s\}}$ and $\mathscr{B}_{[a,b]} = \mathfrak{A}_{\{X_t\colon a\leqslant t\leqslant b\}}$, and let $\mathscr{D}_s$ denote the (expected) von Neumann subalgebra of $\mathscr{A}$ generated by the differences $A_r^* - A_s^*$, $t_0 \leqslant s \leqslant r \leqslant T$.

Proposition 5·1. *For any $t_0 \leqslant s \leqslant t \leqslant T$, X_t belongs to $L^2(\mathscr{B}_s \vee \mathscr{D}_s)$.*

Proof. First we note that the von Neumann algebra generated by two expected subalgebras is itself expected; so $\mathscr{B}_s \vee \mathscr{D}_s$ is expected. Define $X_t^{(n)}$ inductively by $X_t^{(0)} = X_s$,

$$X_t^{(n+1)} = X_s + \int_{[s,t]} F_1(X_r^{(n)},r)\, d(A_r^* h_\omega^{\frac12}) + \int_{[s,t]} F_2(X_r^{(n)},r)\, d(A_r h_\omega^{\frac12}) + \int_{[s,t]} G(X_r^{(n)},r)\, d\lambda(r).$$

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By the definition of $\mathscr{B}_s$, $X_t^{(0)} = X_s \in L^2(\mathscr{B}_s) \subseteq L^2(\mathscr{B}_s \vee \mathscr{D}_s)$. Now, if $X_r^{(n)} \in L^2(\mathscr{B}_s \vee \mathscr{D}_s)$ for $r \geqslant s$ we have $\mathfrak{A}_{\{X_r^{(n)}\}} \subseteq \mathscr{B}_s \vee \mathscr{D}_s$, by definition, and hence $F_i(X_r^{(n)}, r)$, $i = 1, 2$, and $G(X_r^{(n)}, r) \in L^2(\mathscr{B}_s \vee \mathscr{D}_s)$. It then follows easily from the definition of the integrals that $X_r^{(n+1)} \in L^2(\mathscr{B}_s \vee \mathscr{D}_s)$. By the uniqueness of the solution, $X_t^{(n)} \to X_t$ as $n \to \infty$, for any $s \leqslant t \leqslant T$, so that $X_t \in L^2(\mathscr{B}_s \vee \mathscr{D}_s)$. ∎

Let β denote the (spatial) fermion parity automorphism of $\mathscr{A}$. Then $\beta(\mathscr{A}_s) = \mathscr{A}_s$, for all s, and β commutes with the modular automorphism group (σ_t^ω). Hence if $\mathscr{C}$ is an expected subalgebra of $\mathscr{A}$, then so is $\beta(\mathscr{C})$, and hence so is $\mathscr{C} \vee \beta(\mathscr{C})$. Write $\mathscr{C}^\beta$ for $\mathscr{C} \vee \beta(\mathscr{C})$. Then $\beta(\mathscr{C}^\beta) = \mathscr{C}^\beta$.

The following theorem is the quasi-free boson and fermion analogue of [**4**, theorem 2·2].

THEOREM 5·2. *The solution* (X_t) *of equation* (5·1) *is a Markov process on* $[t_0, T]$ *in the sense that*

(1) *boson case*: $\mathscr{E}(Y \mid \mathscr{B}_{[t_0, s]}) = \mathscr{E}(Y \mid \mathscr{B}_s)$ *for any* $Y \in L^1(\mathscr{B}_{[s, T]})$,

(2) *fermion case*: $\mathscr{E}(Y \mid \mathscr{B}^\beta_{[t_0, s]}) = \mathscr{E}(Y \mid \mathscr{B}^\beta_s)$ *for any* $Y \in L^1(\mathscr{B}^\beta_{[s, T]})$,

where $\mathscr{E}(\cdot \mid \mathscr{C})$ *denotes the canonical extension of the conditional expectation* $E(\cdot \mid \mathscr{C})$ *onto* $\mathscr{C}$.

Proof. By the L^1-continuity of $\mathscr{E}$ it is enough to prove the result for Y of the form $h_\omega Z$ with $Z \in \mathscr{B}_{[s, T]}$ (or $\mathscr{B}^\beta_{[s, T]}$). Now, by Proposition 5·1, $X_t \in L^2(\mathscr{B}_s \vee \mathscr{D}_s)$ for all $s \leqslant t \leqslant T$, and so $\mathscr{B}_{[s, T]} \subseteq \mathscr{B}_s \vee \mathscr{D}_s$ (and hence, in the fermion case, $\mathscr{B}^\beta_{[s, T]} \subseteq \mathscr{B}^\beta_s \vee \mathscr{D}_s$). Therefore, by linearity and the σ-weak continuity of $\mathscr{E}$, we may suppose that Z has the form

$$Z = B_1 D_1 \dots B_n D_n$$

for some $B_1, \dots, B_n \in \mathscr{B}_s$ (or $\mathscr{B}^\beta_s$) and $D_1, \dots, D_n \in \mathscr{D}_s$.

Now we use the CCR or the CAR to deduce that such Z can be written as $Z = BD$ with $B \in \mathscr{B}_s$ (or $\mathscr{B}^\beta_s$) and $D \in \mathscr{D}_s$. But ω is a product state so that $\mathscr{B}_{[t_0, s]}$ (or $\mathscr{B}^\beta_{[t_0, s]}$) and $\mathscr{D}_s$ are independent:

$$\omega(BD) = \omega(B)\,\omega(D)$$

for any $B \in \mathscr{B}_{[t_0, s]}$ (or $\mathscr{B}^\beta_{[t_0, s]}$) and $D \in \mathscr{D}_s$.

Hence, for bosons,

$$\begin{aligned}
\mathscr{E}(h_\omega BD \mid \mathscr{B}_{[t_0, s]}) &= h_\omega E(BD \mid \mathscr{B}_{[t_0, s]}) \\
&= h_\omega BE(D \mid \mathscr{B}_{[t_0, s]}) \\
&= \omega(D)\, h_\omega B \\
&= h_\omega BE(D \mid \mathscr{B}_s) \\
&= \mathscr{E}(h_\omega BD \mid \mathscr{B}_s).
\end{aligned}$$

Adding the superscript β yields the result for fermions. ∎

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