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RESEARCH ARTICLE

Permutation groups, partition lattices and block structures

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Abstract

Let G be a finite transitive permutation group on Ω. The G-invariant partitions form a sublattice of the lattice of all partitions of Ω, having the further property that all its elements are uniform (that is, have all parts of the same size). If, in addition, all the equivalence relations defining the partitions commute, then the relations form an *orthogonal block structure*, a concept from statistics; in this case the lattice is modular. If it is distributive, then we have a *poset block structure*, whose automorphism group is a *generalised wreath product*. We examine permutation groups with these properties, which we call the *OB property* and *PB property* respectively, and in particular investigate when direct and wreath products of groups with these properties also have these properties.

A famous theorem on permutation groups asserts that a transitive imprimitive group G is embeddable in the wreath product of two factors obtained from the group (the group induced on a block by its setwise stabiliser, and the group induced on the set of blocks by G). We extend this theorem to groups with the PB property, embedding them into generalised wreath products. We show that the map from posets to generalised wreath products preserves intersections and inclusions.

We have included background and historical material on these concepts.

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1. Introduction

Let G be a finite transitive permutation group on a finite set Ω. Then the G-invariant partitions of Ω form a sublattice of the lattice of all partitions of Ω (ordered by refinement). The G-invariant partitions have the additional property that they are *uniform* (all parts have the same size).

In this paper all permutation groups will be finite and we are primarily interested in the class of permutation groups for which the equivalence relations corresponding to the G-invariant partitions commute pairwise. (We will see in Section 5.1 that, at least among transitive groups of small degree, the vast majority do satisfy this condition; for example, 1886 of the 1954 transitive groups of degree 16 do so.) Then the lattice of partitions which they form is called an *orthogonal block structure*, for short an OBS. This property can also be defined by saying that the subgroups containing a point stabiliser G_α commute pairwise. This implies that the lattice satisfies the *modular law*. It turns out that this property of a partition lattice was introduced, in the context of statistical design, by several different statisticians: see Section 3.

An orthogonal block structure gives rise, by an inclusion-exclusion argument, to an association scheme on Ω; we also explain this and its relevance to the study of permutation groups.

A more restrictive property requires that the lattice satisfies the *distributive law*. These structures are known, in the statistical context, as *poset block structures*. These are explained in Section 3. The simplest nontrivial cases are (i) a single nontrivial uniform partition and (ii) the rows and columns of a rectangle. These correspond to the imprimitive wreath product and the transitive direct product of two permutation groups.

This is related to an earlier permutation group construction, the so-called *generalised wreath product*. This takes as input data a partially ordered set M having a transitive permutation group associated with each of its elements, and produces a product which generalises both direct and wreath product (the cases where the poset is a 2-element antichain or 2-element chain respectively). The *Krasner–Kaloujnine theorem*, a well-known theorem in permutation group theory, describes the embedding of a transitive but imprimitive permutation group in a wreath product; we generalise this to embed a group whose invariant partitions form a poset block structure into a generalised wreath product over the poset.

We say that a transitive group G has the *OB property* (respectively *PB property*) if the G-invariant partitions form an orthogonal block structure (respectively a poset block structure). We investigate some properties of these groups, including their behaviour under direct and wreath products, and characterise the regular groups with the OB property (using a theorem of Iwasawa).

A summary of the paper follows. In Section 2, we give precise definitions of orthogonal and poset block structures and the generalised wreath product of a family of permutation groups indexed by a poset. Section 3 describes the history of these block structures in experimental design in statistics. Section 4

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contains our main results on permutation groups. We give somewhat informal descriptions here, since precise statements depend on the notions of generalised wreath product and the OB and PB properties.

(a) We show that a generalised wreath product of primitive permutation groups is pre-primitive and has the OB property, and we give a necessary and sufficient condition for it to have the PB property: the obstruction is the existence of incomparable elements in the poset whose associated groups are cyclic of the same prime order (Theorem 4.19).
(b) We show that a transitive group G which acts on a poset block structure (in particular, a transitive group with the PB property) can be embedded in a generalised wreath product, where the factors in the product can be defined in terms of the action of G (Theorem 4.21).
(c) The map from posets on the index set to generalised wreath products of families of groups preserves intersections and inclusions, where for a poset these refer to the set of ordered pairs comprising the relation. In particular, a generalised wreath product is the intersection of the iterated wreath products over all linear extensions of the poset (Theorem 4.24, Corollary 4.25).

We also examine the behaviour of OB and PB under direct and wreath product.

The final section, Section 5, describes some computational issues and gives some open problems.

Since the paper crosses boundaries between permutation groups, lattice theory and statistical design, we have given some introductory material on these topics (Section 2), as well as an account of the somewhat tangled history of their occurrence in statistics (Section 3).

2. Lattices of partitions

2.1. Partitions

Let Ω be a finite set. The set of all partitions of Ω is partially ordered by refinement: $\Pi_1 \preccurlyeq \Pi_2$ if each part of Π_1 is contained in a part of Π_2. With this order, the partitions form a lattice (a partially ordered set in which any two elements have a greatest lower bound or *meet*, and a least upper bound or *join*): the meet (also called *infimum*) $\Pi_1 \wedge \Pi_2$ is the partition whose parts are all nonempty intersections of parts of Π_1 and Π_2, and the join (also called *supremum*) $\Pi_1 \vee \Pi_2$ is the partition in which the part containing α consists of all points of Ω that can be reached from α by moving alternately within a part of Π_1 and within a part of Π_2.

Partitions can be considered also as equivalence relations. The composition $R_1 \circ R_2$ of two relations R_1 and R_2 is the relation in which α and β are related if and only if there exists γ with $(\alpha, \gamma) \in R_1$ and $(\gamma, \beta) \in R_2$.

In view of the natural correspondence between partitions and equivalence relations, we abuse notation by talking about the join $R_1 \vee R_2$ of two equivalence relations, or the composition $\Pi_1 \circ \Pi_2$ of two partitions.

Proposition 2.1. *$R_1 \circ R_2 = R_1 \vee R_2$ if and only if $R_1 \circ R_2 = R_2 \circ R_1$.*

Proof. Clearly $R_1 \circ R_2 \subseteq R_1 \vee R_2$.

Suppose that $R_1 \circ R_2 = R_2 \circ R_1$. If α and β lie in the same part of $R_1 \vee R_2$, then there is a path joining them, whose edges lie alternately in the same part of R_1 and of R_2. But any three consecutive steps $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in R_1$ and $(\alpha_2, \alpha_3) \in R_2$ can be shortened to two steps: for there exists β' with $(\alpha_1, \beta') \in R_2$ and $(\beta', \alpha_3) \in R_1$; then $(\beta', \alpha_4) \in R_1$ by transitivity. So $R_1 \vee R_2 = R_1 \circ R_2$.

Conversely, suppose that $R_1 \circ R_2 = R_1 \vee R_2$. Then $R_1 \circ R_2$ is symmetric, so it is equal to $R_2 \circ R_1$. □

This result was first proved in [18].

2.2. Lattices

A finite lattice is conveniently represented by its *Hasse diagram*: this is the plane diagram with a dot for each lattice element; if $a \prec b$ then b is higher than a in the plane; and if b covers a (that is, $a \prec b$ but there is no element c with $a \prec c \prec b$), then an edge joins a to b.

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Figure 1. *The modular law for commuting partitions.*

In a lattice, the *modular law* states that

$$a \leqslant c \text{ implies } a \vee (b \wedge c) = (a \vee b) \wedge c.$$

A lattice L is *modular* if this holds for all $a, b, c \in L$.

Proposition 2.2. *In a lattice of partitions, if every pair of partitions commute, then the lattice is modular.*

Proof. We are required to prove that $\Phi \leqslant \Psi$ implies $\Phi \vee (\Xi \wedge \Psi) = (\Phi \vee \Xi) \wedge \Psi$. In Figure 1, the dots represent points in Ω. Each edge is labelled by a partition of Ω. If an edge labelled Φ joins points α and β, this means that α and β are in the same part of Φ; and similarly for Ξ and Ψ.

Since $\Phi \leqslant \Psi$, any Φ-Ψ path can be replaced by a single Ψ edge. So, considering the paths from α to γ in the diagram on the left shows that $\Phi \vee (\Xi \wedge \Psi) \leqslant (\Phi \vee \Xi) \wedge \Psi$. Also, on the right, the Ψ-Φ path from θ to η implies that there is a Ψ edge between them. Thus there is a $\Xi \wedge \Psi$ path from θ to η, and hence a $(\Xi \wedge \Psi) \vee \Phi$ path from θ to ζ: this gives the reverse inequality. □

Proposition 2.2 is Theorem 9.11 in [5] and Proposition 8 in [19].

A lattice is *distributive* if it satisfies the conditions

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c),$$
$$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c),$$

for all a, b, c.

Proposition 2.3. *The following are true:*

(a) *Each of the two distributive laws implies the other.*
(b) *A distributive lattice is modular.*

Proof. (a) Suppose that the first law above holds. Then

$$\begin{aligned}(a \vee c) \wedge (b \vee c) &= ((a \vee c) \wedge b) \vee ((a \vee c) \wedge c)\\ &= (a \wedge b) \vee (c \wedge b) \vee c\\ &= (a \wedge b) \vee c.\end{aligned}$$

The proof of the other implication is similar.

(b) Suppose that L is distributive and let $a, b, c \in L$ with $a \leqslant c$. Then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c,$$

since $a \leqslant c$ implies $a \vee c = c$. □

Proposition 2.3 is a standard result in lattice theory and appears in [16] as Lemmas 4.2 and 4.3.

The fundamental theorem on distributive lattices states that every finite distributive lattice is isomorphic to a sublattice of the *Boolean lattice* of all subsets of a finite set. More precisely, a *down-set* in a partially ordered set $(M, \sqsubseteq)$ is a subset D of M with the property that, if $m \in D$ and $m' \sqsubseteq m$, then $m' \in D$. The down-sets form a lattice under the operations of intersection and union.

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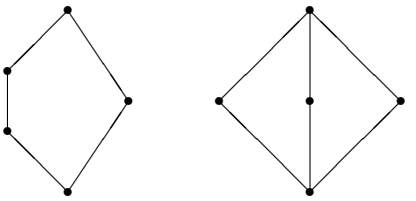


Figure 2. *The lattices P_5 (left) and N_3 (right).*

Theorem 2.4. *A finite distributive lattice L is isomorphic to the lattice of down-sets in a partially ordered set M. We can take M to be the set of join-indecomposable elements of L (elements m for which $m = m_1 \vee m_2$ implies $m = m_1$ or $m = m_2$).*

A proof of this theorem is in [12, p. 192]. We sometimes abbreviate 'join-indecomposable' to JI.

In particular, if M is an antichain (a poset in which any two elements are incomparable), then every subset is a down-set, and the corresponding lattice is the *Boolean lattice* on M.

There are well-known characterisations of these classes of lattices. The Hasse diagrams of P_5 and N_3 are shown in Figure 2.

Theorem 2.5. *The following are true:*

(a) *A lattice is modular if and only if it does not contain P_5 as a sublattice.*
(b) *A lattice is distributive if and only if it does not contain P_5 or N_3 as a sublattice.*

The proof of this theorem can be found in [16, p. 134].

2.3. Orthogonal block structures

The next definition comes from experimental design in statistics: see the discussion in Section 3. Our treatment follows [5].

An *orthogonal block structure* $(\Omega, \mathcal{B})$ consists of a collection $\mathcal{B}$ of partitions of a single set Ω satisfying the conditions

(a) $\mathcal{B}$ is a sublattice of the partition lattice (that is, closed under meet and join);
(b) $\mathcal{B}$ contains the two extreme partitions (the equality partition E whose parts are singletons, and the universal partition U with just one part);
(c) every partition in $\mathcal{B}$ is uniform (that is, has all parts of the same size);
(d) any two partitions in $\mathcal{B}$ commute.

The set $\mathcal{B} = \{E, U\}$ is an orthogonal block structure, which we call *trivial*.

We remark that the definition in [5, Chapter 6] has a more complicated condition in place of our condition (d). With any partition Π is associated a subspace V_Π of the vector space $\mathbb{R}^\Omega$ consisting of functions which are constant on the parts of Π, and the operator P_Π of orthogonal projection of $\mathbb{R}^\Omega$ onto V_Π; two partitions Π_1 and Π_2 are said to be orthogonal if P_{Π_1} and P_{Π_2} commute. The remark at the top of page 153 of [5] notes that, in the presence of conditions (a)–(c), this is equivalent to our simpler condition (d).

An *association scheme* on Ω is a partition of Ω^2 into symmetric relations $S_0, S_1, \ldots, S_r$ having the properties that S_0 is the relation of equality and that the span over $\mathbb{R}$ of the zero-one relation matrices is an algebra. (Combinatorially this means that, given $i, j, k \in \{0, \ldots, r\}$ and $\alpha, \beta \in \Omega$ with $(\alpha, \beta) \in S_k$, the number p_{ij}^k of elements $\gamma \in \Omega$ such that $(\alpha, \gamma) \in S_i$ and $(\gamma, \beta) \in S_j$ is independent of the choice of $(\alpha, \beta) \in S_k$, depending only on i, j, k.)

An orthogonal block structure gives rise to an association scheme as follows. Let $R_0, R_1, \ldots, R_t$ be equivalence relations forming an OBS. For each i, let

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$$S_i = R_i \setminus \bigcup_{j:R_j \subset R_i} R_j.$$

Then the nonempty relations S_i are symmetric and partition Ω^2; after removing the empty ones and renumbering, we obtain an association scheme.

The nonequality relations in an association scheme are often thought of as graphs. We remark that, while in the association scheme associated with a primitive permutation group, all these graphs are connected, the association scheme associated with an orthogonal block structure is very different: all the graphs, except possibly the one associated with the universal relation U, are disconnected.

Note that, if two OBSs are isomorphic, then the association schemes obtained in this way are also isomorphic. The converse, however, is false, as the following example shows.

Example

Recall that two Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ over alphabets $\mathcal{A}$ and $\mathcal{B}$ are *orthogonal* if for every pair $(a, \beta) \in \mathcal{A} \times \mathcal{B}$ there exists a unique pair (i, j) such that $a_{ij} = a$ and $b_{ij} = \beta$. Note that it is a convention going back to Euler that the two alphabets are the Latin and Greek letters respectively when dealing with two orthogonal Latin squares.

Take a complete set of $q - 1$ mutually orthogonal Latin squares of order q. Take Ω to be the set of cells of the square and let $\mathcal{B}$ be the set containing the partitions E and U, and the partitions into rows, columns and letters of each of the squares. We claim that $\mathcal{B}$ is an orthogonal block structure.

Conditions (a) and (c) are straightforward to verify and (b) holds by construction, so we will only show that the partitions commute. Let $\Pi_1, \Pi_2 \in \mathcal{B}$. If Π_1 and Π_2 are the row and the column partitions, then they commute because any cell can be reached by first moving along a row and then a column or vice versa.

If Π_1 is the row partition and Π_2 is a letter partition, then we note that each letter appears in every row, and so if x_1 and x_2 are two cells in rows r_1 and r_2 respectively containing the letters a and b in the Latin square corresponding to Π_2, then we can reach x_2 from x_1 either by first moving to the cell in r_1 containing the letter b and then to x_2, or by first moving to the cell containing the letter a in r_2 and then moving along r_2 to x_2. Therefore Π_1 and Π_2 commute. An entirely similar argument shows that Π_1 and Π_2 commute if Π_1 is the column partition.

Finally, suppose that Π_1 and Π_2 are both letter partitions and let L_1 and L_2 be the Latin squares corresponding to Π_1 and Π_2 respectively. Moreover, let x_1 and x_2 be cells that contain the letters a and b in L_1 and the letters α and β in L_2. Orthogonality ensures that there is exactly one cell x_3 which contains the letter a in the L_1 and the letter β in L_2, so we can move from x_1 to x_2 via x_3. Similarly, there exists a unique cell x_4 that contains the letter α in L_2 and the letter b in L_1, so we can move from x_1 to x_2 via x_4, and so $\Pi_1 \circ \Pi_2 = \Pi_2 \circ \Pi_1$.

Since every pair of cells are either in the same row or column or carry the same letter in one of the squares, constructing the association scheme obtains the empty relation from the universal partition U. So the association scheme has $q + 1$ classes apart from the diagonal.

On the other hand, if we omit one of the Latin squares from the set, then the remaining ones give an OBS with q partitions apart from E and U; the last partition is recovered by deleting the pairs in all these from the relation U. So the association schemes are the same.

In particular, for $q = 2$, we obtain two orthogonal block structures, one of which is distributive and the other not, which give the same association scheme.

A similar inclusion-exclusion on subspaces of $\mathbb{R}^\Omega$ finds the orthogonal decomposition of $\mathbb{R}^\Omega$ into common eigenspaces for the matrices in the scheme.

Let us now record some remarks on association schemes.

- The product of two relation matrices is a linear combination of the relation matrices, hence symmetric; thus any two relation matrices commute, and the algebra associated with the association scheme (called its *Bose–Mesner algebra*) is commutative.

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- There is a more general notion, that of a *homogeneous coherent configuration*, defined as for association schemes but with the condition that every relation is symmetric replaced by the weaker condition that the converse of any relation in the configuration is another relation in the configuration. Some authors (including Hanaki and Miyamoto [21]) extend the usage of the term 'association scheme' to this more general situation; but we will not do so.

It was pointed out to us by the anonymous referee that partition lattices whose equivalence relations commute were independently introduced by G.-C. Rota and his students and collaborators, who called them *linear lattices*. Details can be found in [19], where it is explained how they arise in lattice theory and are connected with logic. The authors agree with the referee's view that Rota would have enjoyed the connection with OBSs.

We conclude this section by stating a result by Dubreil and Dubreil-Jacotin [18] that gives an alternative characterisation of partition lattices whose equivalence relations commute. We will say that two partitions Π_1 and Π_2 are *independent*, if for every $P_1 \in \Pi_1$ and $P_2 \in \Pi_2$ we have $P_1 \cap P_2 \neq \emptyset$.

Proposition 2.6. *Two partitions Π_1 and Π_2 commute if and only if for every part $P \in \Pi_1 \vee \Pi_2$, the restrictions $\Pi_1|_P$ and $\Pi_2|_P$ are independent.*

2.4. Crossing and nesting

Two methods of constructing new OBSs from old, both widely used in experimental design, are crossing and nesting, defined as follows.

Let $\mathcal{P}_1 = (\Omega_1, \mathcal{B}_1)$ and $\mathcal{P}_2 = (\Omega_2, \mathcal{B}_2)$ be orthogonal block structures. We think of the elements of $\mathcal{B}_1$ and $\mathcal{B}_2$ as equivalence relations. In each construction, we build a new OBS on $\Omega_1 \times \Omega_2$. For each pair $R_1 \in \mathcal{B}_1$ and $R_2 \in \mathcal{B}_2$, we define a relation $R_1 \times R_2$ to hold between two pairs (α_1, α_2) and (β_1, β_2) if and only if $(\alpha_1, \beta_1) \in R_1$ and $(\alpha_2, \beta_2) \in R_2$. It is clear that $R_1 \times R_2$ is an equivalence relation.

The first method uses the set of equivalence relations

$$\{R_1 \times R_2 : R_1 \in \mathcal{B}_1,\ R_2 \in \mathcal{B}_2\}.$$

This gives the set $\mathcal{B}_1 \times \mathcal{B}_2$ of equivalence relations on $\Omega_1 \times \Omega_2$. This is called *crossing* $\mathcal{P}_1$ and $\mathcal{P}_2$, and written $\mathcal{P}_1 \times \mathcal{P}_2$.

The second method uses the set of equivalence relations

$$\{R_1 \times U_2 : R_1 \in \mathcal{B}_1\} \cup \{E_1 \times R_2 : R_2 \in \mathcal{B}_2\},$$

where U_2 is the universal relation in Ω_2 and E_1 is the equality relation in Ω_1. This is called *nesting* $\mathcal{P}_2$ within $\mathcal{P}_1$, and written as $\mathcal{P}_1/\mathcal{P}_2$.

Of course, the roles of $\mathcal{P}_1$ and $\mathcal{P}_2$ can be reversed, to give $\mathcal{P}_2/\mathcal{P}_1$, with $\mathcal{P}_1$ nested within $\mathcal{P}_2$.

It is straightforward to show that, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are both closed under taking suprema and taking infima, then so are $\mathcal{P}_1 \times \mathcal{P}_2$, $\mathcal{P}_1/\mathcal{P}_2$ and $\mathcal{P}_2/\mathcal{P}_1$.

If R_1 and R_3 are in $\mathcal{B}_1$ and R_2 and R_4 are in $\mathcal{B}_2$ then $(R_1 \circ R_3) \times (R_2 \circ R_4) = (R_1 \times R_2) \circ (R_3 \times R_4)$. Therefore, since every two equivalence relations in $\mathcal{B}_1$ commute and every two equivalence relations in $\mathcal{B}_2$ commute, then the same is true for every two equivalence relations in each of $\mathcal{P}_1 \times \mathcal{P}_2$, $\mathcal{P}_1/\mathcal{P}_2$ and $\mathcal{P}_2/\mathcal{P}_1$.

Let $G \leq \mathrm{Sym}(\Gamma)$ and $H \leq \mathrm{Sym}(\Delta)$ be transitive permutation groups. Recall that the product action of the direct product $G \times H$ on $\Gamma \times \Delta$ is the action defined by $(\gamma, \delta)(g, h) = (\gamma g, \delta h)$ for all $(\gamma, \delta) \in \Gamma \times \Delta$ and $(g, h) \in G \times H$. For permutation group theorists, note the similarities between crossing and nesting on one hand, and direct product (with product action) and wreath product (with imprimitive action) on the other. Statisticians call the results of crossing and nesting trivial OBSs *row-column structures* and *block structures* respectively.

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Nelder [29] introduced the class of orthogonal block structures which can be obtained from trivial structures by repeatedly crossing and nesting, and called them *simple orthogonal block structures*. See Section 3.

2.5. Poset block structures

There is a class of OBSs, more general than the simple ones, effectively introduced in [26], and now called poset block structures, which we define.

A *poset block structure* is an orthogonal block structure in which the lattice of partitions is distributive. (We have seen in Proposition 2.3 that the distributive law is stronger than the modular law.)

Using the Fundamental Theorem on Distributive Lattices (Theorem 2.4), we can turn this abstract definition into something more useful. Recall that a distributive lattice L is the lattice of down-sets in a poset $(M, \sqsubseteq)$, where M can be recovered from L as the set of nonzero join-indecomposable elements (that is, JI elements different from E). Put $N = |M|$. Now we attach a finite set Ω_i of size $n_i > 1$ to each element $m_i \in M$, and take Ω to be the Cartesian product of the sets Ω_i for all $m_i \in M$. Now we need to define a partition Π_D for each down-set D in M. This is done as follows. Define a relation R_D on Ω by

$$R_D((\alpha_1, \ldots, \alpha_N), (\beta_1, \ldots, \beta_N)) \Leftrightarrow (\forall m_i \notin D)(\alpha_i = \beta_i),$$

where $\alpha_i, \beta_i \in \Omega_i$ for all $m_i \in M$. Then R_D is an equivalence relation on Ω, and we let Π_D be the corresponding partition. (The appearance of the poset M explains the name *poset block structures*.)

It is straightforward to check that

(a) the partitions E and U of Ω correspond to the empty set and the whole of M;
(b) if D_1 and D_2 are down-sets in M, then

$$\Pi_{D_1 \cap D_2} = \Pi_{D_1} \wedge \Pi_{D_2} \text{ and } \Pi_{D_1 \cup D_2} = \Pi_{D_1} \vee \Pi_{D_2}.$$

So the partitions Π_D form a lattice isomorphic to the given lattice L.

This is proved in [4, 34], where it is shown that every poset block structure (according to our definition) is given by this construction.

At this point, we mention a paper by Yan [38], whose title suggests that it concerns distributive lattices of commuting equivalence relations. In fact, both her hypotheses and her conclusion are much stronger than ours. In the case of uniform partitions, her theorem asserts the following: if Π_1 and Π_2 are commuting uniform equivalence relations such that every equivalence relation Ψ which commutes with both of them associates with them, in the sense that

$$\Psi \wedge (\Pi_1 \vee \Pi_2) = (\Psi \wedge \Pi_1) \vee (\Psi \wedge \Pi_2),$$

then Π_1 and Π_2 are comparable in the partial order. (This does not say that every distributive lattice of commuting partitions is a chain.)

Notation. For every $i \in \{1, \ldots, N\}$, let $A(i)$ denote the set $\{j \in \{1, \ldots, N\} : m_i \sqsubset m_j\}$ and $A[i]$ the set $\{j \in \{1, \ldots, N\} : m_i \sqsubseteq m_j\}$. Similarly, let $D(i)$ denote the set $\{j \in \{1, \ldots, N\} : m_j \sqsubset m_i\}$ and $D[i]$ the set $\{j \in \{1, \ldots, N\} : m_j \sqsubseteq m_i\}$. (Mnemonic: A = 'ancestor', D = 'descendant'.)

2.6. Generalised wreath products

Closely related to poset block structures is the notion of generalised wreath product. We now define this, following the notation used in [9].

We write Ω^i for the Cartesian product $\prod_{j \in A(i)} \Omega_j$ and π^i for the natural projection from Ω onto $\prod_{j \in A(i)} \Omega_j$. Finally, for every $m_i \in M$, let $G(m_i)$ be a permutation group on Ω_i, and let F_i denote the

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set of all functions from Ω^i into $G(m_i)$. Thus, if $f_i \in F_i$, then f_i allocates a permutation in $G(m_i)$ to each element of Ω^i.

The *generalised wreath product* G of the groups $G(m_1), \ldots, G(m_N)$ over the poset M is the group $\prod_{i=1}^{N} F_i$, and it acts on Ω in the following way: if $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$ and $f = (f_1, \ldots, f_N) \in G$, then

$$(\omega f)_i = \omega_i(\omega \pi^i f_i)$$

for $i = 1, \ldots, N$.

We note that, if M is the 2-element antichain $\{m_1, m_2\}$, then the generalised wreath product of $G(m_1)$ and $G(m_2)$ is their direct product; while if M is a 2-element chain, with $m_1 \sqsubset m_2$, then G is the wreath product $G(m_1) \wr G(m_2)$, in its imprimitive action.

The next result gives the automorphism group of a poset block structure.

Proposition 2.7. *The automorphism group of the poset block structure given above is the generalised wreath product of symmetric groups S_{n_i} over the poset $(M, \sqsubseteq)$.*

This is proved in [9].

The operations of crossing and nesting preserve the class of poset block structures: crossing corresponds to taking the disjoint union of the two posets (with no comparability between them); nesting corresponds to taking the ordered sum (with every element of the second poset below every element of the first).

Proposition 2.7 shows that poset block structures always have large automorphism groups. By contrast, orthogonal block structures may have no nontrivial automorphisms at all. Let L be a Latin square, with Ω the set of positions. Take the two trivial partitions and the three partitions into rows, columns and entries. Automorphisms of this structure are known as *autotopisms* in the Latin square literature; it is known that almost all Latin squares have trivial autotopism group: see [14, 28].

3. History in design of experiments

These ideas were developed gradually in the early days of design of statistical experiments. In order to describe them in a standard way, we will use some notation introduced by Nelder in [29]. If n is a positive integer, then we denote by $\underline{\underline{n}}$ any set of size n which has the *trivial block structure* $\{U, E\}$. (This notation is used in [5] but is replaced by $[n]$ in [4].)

3.1. Fisher and Yates at Rothamsted

Ronald Fisher was the first statistician at Rothamsted Experimental Station, working there from 1919 to 1933: see [6]. He advocated two, fairly simple, blocking structures. In the first, called a *block design*, the bk plots were partitioned into b blocks of size k, thus giving the orthogonal block structure $\underline{\underline{b}}/\underline{\underline{k}}$. In the second, called a *Latin square*, the n^2 plots formed a square array with n rows and n columns, to which n treatments were applied in such a way that each treatment occurred once in each row and once in each column. Ignoring the treatments, this gives the orthogonal block structure $\underline{\underline{n}} \times \underline{\underline{n}}$.

Frank Yates worked in the Statistics Department at Rothamsted Experimental Station from 1931 until 1968: see [6]. He gradually developed more and more complicated block structures for designed experiments. His paper on 'Complex Experiments' [39], read to the Royal Statistical Society in 1935, covers many of these. After describing block designs and Latin squares, he proposes 'splitting of plots' (page 197) into subplots in both cases. If the number of subplots per plot is s, this leads to the orthogonal block structures $\underline{\underline{b}}/\underline{\underline{k}}/\underline{\underline{s}}$ and $(\underline{\underline{n}} \times \underline{\underline{n}})/\underline{\underline{s}}$ (treatments are ignored in these block structures). These are all based on partially ordered sets (although he did not use this terminology), as shown in Figure 3.

Yates also suggests 'two 4×4 Latin squares with subplots' (page 201), which gives the orthogonal block structure $\underline{\underline{2}}/(\underline{\underline{4}} \times \underline{\underline{4}})/\underline{\underline{2}}$; splitting each row of an $\underline{\underline{r}} \times \underline{\underline{c}}$ rectangle into two subrows, which gives the

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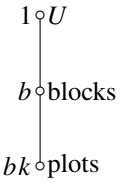
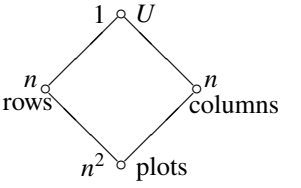
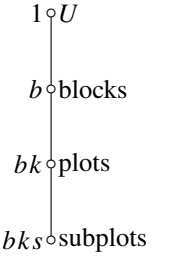
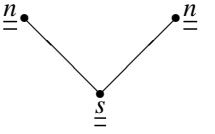
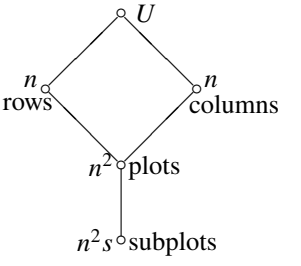
Verbal description	Hasse diagram of poset	Hasse diagram of OBS
Block design		
Latin square		
Split-plot design		
Latin square with split plots		

Figure 3. *Orthogonal block structures mentioned by Yates in [39].*

orthogonal block structure $(\underline{r}/\underline{2}) \times \underline{c}$ (page 202); and a collection of four 5×5 Latin squares (page 218), which gives the orthogonal block structure $\underline{4}/(\underline{5} \times \underline{5})$. These are shown in Figure 4.

3.2. Nelder's simple orthogonal block structures

John Nelder worked in the Statistics Section of the UK's National Vegetable Research Station from 1951 to 1968. In two papers [29, 30] in 1965 he introduced the class of orthogonal block structures which can be obtained from trivial structures by repeated crossing and nesting, and called them *simple orthogonal block structures*. In that year, he also visited CSIRO (the Commonwealth Scientific and Industrial Research Organisation) at the Waite Campus of the University of Adelaide in South Australia, where he worked with Graham Wilkinson to start developing the statistical software GenStat. He and colleagues developed GenStat further while he was Head of the Statistics Department at Rothamsted

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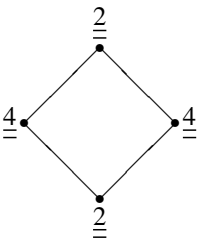
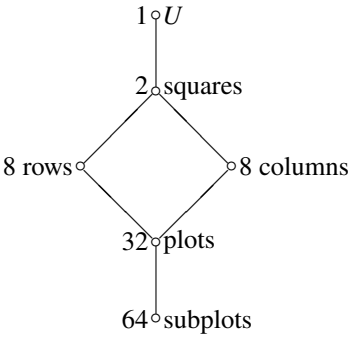
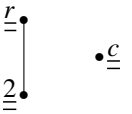
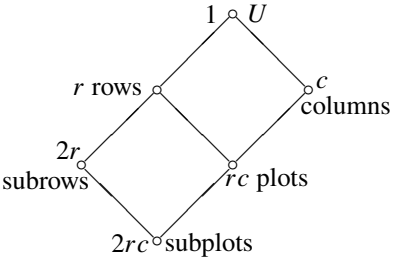
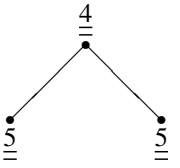
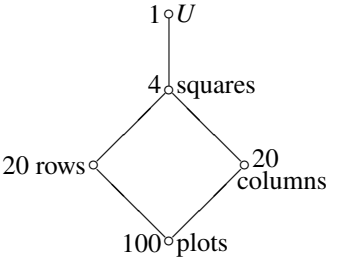
Verbal description	Hasse diagram of poset	Hasse diagram of OBS
Two Latin squares with subplots		
Splitting rows of a rectangle		
Four Latin squares of order five		

Figure 4. *More orthogonal block structures mentioned by Yates.*

Experimental Station from 1968 to 1984. The benefit of iterated crossing and nesting is that each block structure can be described by a simple formula, which can be input as a line in the program used to analyse the data obtained from an experiment.

3.3. Statisticians at Iowa State University

In parallel with Nelder's work was the work of Oscar Kempthorne and his colleagues. Kempthorne worked at the Statistics Department at Rothamsted Experimental Station from 1941 to 1946. He spent most of the rest of his career at Iowa State University. While there, he obtained a grant from the Aeronautical Research Laboratory to work with his colleagues on various problems in the design of experiments.

Their technical report [26] was completed in November 1961, and consisted of 218 typed pages. It uses the phrases 'experimental structure' and 'response structure' for what we call 'block structure'. Sometimes the treatments were also included in this structure. Chapter 3 is based on the PhD theses of Zyskind [40] and Throckmorton [36]; part of this was later published as [41].

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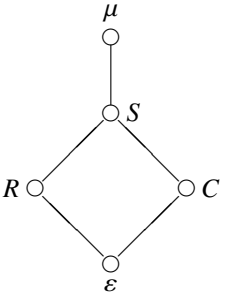
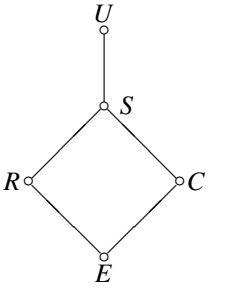
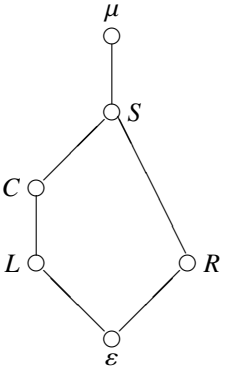
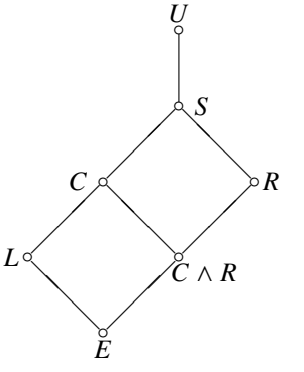
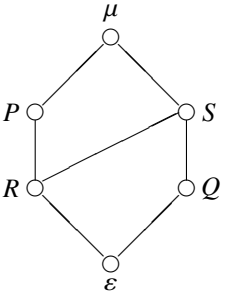
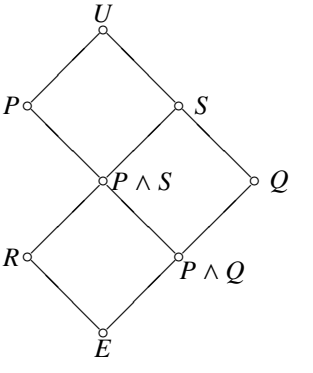
Symbolic representation	Structure diagram	Hasse diagram of OBS
$S\colon RC$		
$S\colon (R)(C\colon L)$		
$(S\colon Q)(P)$ and $(SP\colon R)$		

Figure 5. *Some orthogonal block structures in [26].*

With hindsight, it seems that they were trying to define poset block structures, but they managed to confuse the poset M of coordinates with the lattice of partitions. They denoted the universal partition U by μ, and the equality partition E by ε. They used complicated formulae, called *symbolic representations*, to explain the poset M, but then included μ and ε in the corresponding Hasse diagram, which they called the *structure diagram*. They dealt with all posets of size at most four, and showed 16 of the 63 posets of size five.

Figure 5 shows three of their block structures. The first of these is also in Figure 4; the last one cannot be obtained by crossing and nesting, so it needs two formulae.

3.4. Unifying the theory

In [34], Speed and Bailey aimed to combine the two approaches by explaining Nelder's 'simple orthogonal block structures' and Throckmorton's 'complete balanced block structures' as 'association schemes

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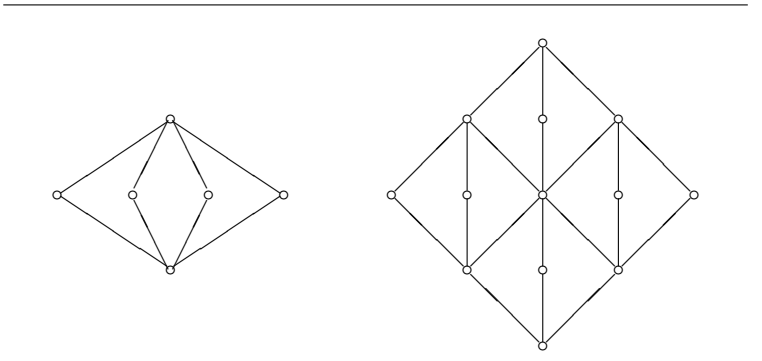


Figure 6. *Hasse diagrams of two nondistributive orthogonal block structures.*

derived from finite distributive lattices of commuting uniform equivalence relations'. They noted that the words 'permutable' and 'permuting' were sometimes used in place of 'commuting'. Each partition is defined by a 'hereditary' subset of the poset M. This is the dual notion to down-set. A subset H of M is hereditary if, whenever $m \in H$ and $m \sqsubseteq m'$, then $m' \in H$. Then $\Omega = \Omega_1 \times \cdots \times \Omega_N$ (where $N = |M|$). Two elements $(\alpha_1, \ldots, \alpha_N)$ and $(\beta_1, \ldots, \beta_N)$ are in the same part of the partition Π_H if and only if $\alpha_i = \beta_i$ for all i in H.

To match the partial order on partitions to the partial order $\subseteq$ on subsets of M, they defined $\preccurlyeq$ in the opposite way to what we do here. They proved that every distributive block structure is isomorphic to a poset block structure, but did not use the latter term, even though they showed that the construction depends on a partially ordered set.

They also explained that most of the theory extends to what we now call an *orthogonal block structure*, where the lattice is modular but not necessarily distributive. Figure 6 shows the corresponding Hasse diagrams in their two examples. In the one on the left, the nontrivial partitions form the rows, columns and the Greek and Latin letter partitions of a pair of mutually orthogonal Latin squares. Note that the underlying set has size n^2 with $n \notin \{1, 2, 6\}$, since it is well known that there exists a pair of two mutually orthogonal Latin squares of order n if and only if $n \notin \{1, 2, 6\}$. One way of achieving the one on the right is to use some carefully chosen subgroups of the elementary abelian group of order 16.

In [2], Bailey restricted attention to distributive block structures, using the term 'ancestral subset' in place of 'hereditary subset' and drawing the Hasse diagrams in the way consistent with our current use of the refinement partial order $\preccurlyeq$. This cited [40] as well as [36], and commented that Holland [22] 'defines the automorphism group of a poset block structure to be a *generalised wreath product*'. The explicit form for such a group was given in [9], following the arguments in [22].

Paper [9] gives a formal definition of *poset block structure* and an automorphism of such a structure. It shows that, in the finite case, the automorphism group is the generalised wreath product of the relevant symmetric groups. The argument draws on work of Wells [37] for semi-groups. The paper also states that, in the finite case, the generalised wreath product of permutation groups is the same as that constructed by [22, 33].

In [35], Speed and Bailey discuss *factorial dispersion models*, which are statistical models whose underlying structure is a poset block structure. Now hereditary subsets are called *filters* and the refinement partial order is shown in the same way as we do here.

Papers [2, 9, 35] have the disadvantage that the partial order on the subsets of M is the wrong way up for inclusion. In the current paper, our use of down-sets rather than hereditary subsets gets round this problem.

In [23], Houtman and Speed extend the meaning of 'orthogonal block structure' to mean a particular desirable property of covariance matrices. This is even more general than their being based on an association scheme, so we do not use that meaning here.

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The survey paper [4] explains the combinatorial aspects of all these ideas in more detail. It notes that a 'complete balanced response structure' is not necessarily a poset block structure, but can always be extended to one by the inclusion of infima.

It also discusses automorphisms. In the present paper, an automorphism of a poset block structure is a permutation of the base-set Ω which preserves each of the relevant partitions. In [4, 8], this is called a 'strong automorphism', while a 'weak automorphism' preserves the set of these partitions. These are called 'strict automorphism' and 'automorphism', respectively, in [11].

If there are nonidentity weak automorphisms, then under suitable conditions we can extend our group by adjoining these. We do not discuss this here, but note that three of the types of primitive group in the celebrated O'Nan–Scott theorem [32] can be realised in this way: affine groups, wreath products with product action, and diagonal groups.

3.5. Statistics and group theory

Why do statisticians care about these groups? First, because of the need to randomise. An experimental design is an allocation of treatments to the elements of the base-set Ω. To avoid possible bias, this allocation is then randomised by applying a permutation chosen at random from the automorphism group of the block structure. Denote by Y_α the random variable for the response on plot α. The method of randomisation allows us to assume that the covariance of Y_α and Y_β is equal to the covariance of Y_γ and Y_δ (but unknown in advance) if and only if (γ, δ) is in the same orbit of the action of the automorphism group on $\Omega \times \Omega$ as at least one of (α, β) and (β, α).

For the full generalised wreath product of symmetric groups, these orbits on pairs are precisely the association classes of the association scheme described in [34]. Thus the eigenspaces of the covariance matrix are known in advance of data collection. These eigenspaces are called *strata* in [29, 30]. Data can be projected onto each stratum for a straightforward analysis.

Now suppose that each symmetric group G_i in the generalised wreath product is replaced by a subgroup H_i. Lemma 11 in [9] shows that the eigenspaces are known in advance if and only if the permutation character of the generalised wreath product is multiplicity-free (or a slight weakening of this, because the covariance-matrix must be symmetric). In particular, so long as each subgroup H_i is doubly transitive then the strata are the same as they are for the generalised wreath product of symmetric groups.

Paper [34] concludes with acknowledgements to several people, including P. J. Cameron and D. E. Taylor. These two had explained to the authors of [34] the importance of having a permutation character which is multiplicity-free.

4. Permutation groups

In this section, we consider transitive permutation groups, and say that such a group G has the *OB property* (respectively, the *PB property*) if the G-invariant partitions form an orthogonal block structure (respectively, a poset block structure). We examine the behaviour of these properties under various products of permutation groups. Our major result is a proof that any transitive group G with the PB property is embeddable in a generalised wreath product of transitive groups extracted from G.

4.1. Introduction to OB groups

Let G be a transitive permutation group on Ω. The set of all G-invariant partitions satisfies the first three of the four conditions listed in Section 2.3 for an orthogonal block structure. When does it satisfy the fourth? We will say that G has the OB property if the fourth condition holds.

We observe that, for a given point $\alpha \in \Omega$, there is a natural order-preserving bijection between G-invariant partitions of Ω and subgroups of G containing G_α: if $G_\alpha \leq H \leq G$, then αH is a part of a G-invariant partition; in the other direction, if Π is a G-invariant partition, the corresponding subgroup

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is the setwise stabiliser of the part of Π containing α. If Π_1 and Π_2 correspond to H and K, then $\Pi_1 \wedge \Pi_2$ corresponds to $H \cap K$, and $\Pi_1 \vee \Pi_2$ corresponds to $\langle H, K\rangle$. (The result for join is in [1], and for meet [17, Theorem 1.5A].)

Theorem 4.1. *Suppose that G-invariant partitions Π_1 and Π_2 correspond to subgroups H and K containing G_α. Then Π_1 and Π_2 commute if and only if $HK = KH$.*

Proof. Suppose that $HK = KH$. Then HK is a subgroup, and is equal to $\langle H, K\rangle$. The points β such that $(\alpha, \beta) \in \Pi_1 \circ \Pi_2$ (respectively, $\Pi_2 \circ \Pi_1$, $\Pi_1 \vee \Pi_2$) are those that can be reached from α by applying an element of HK (respectively, KH, $\langle H, K\rangle$). So the three relations are all equal.

Conversely, suppose that Π_1 and Π_2 are the G-invariant partitions corresponding to H and K, and that $\Pi_1 \circ \Pi_2 = \Pi_1 \vee \Pi_2$. In particular, this holds for the part containing α. So any point in this part can be reached from α by first moving to a point β in the same part of Π_1, then to a point γ in the same part of Π_2 as β. Since the stabiliser of the part of Π_1 containing α is H, we have $\beta = \alpha h$ for some $h \in H$. Then the part of Π_2 containing β is obtained by mapping the part containing α by h, so its stabiliser is K^h; so $\gamma = \beta h^{-1} k h$ for some $k \in K$. Thus $\gamma = \alpha k h$. We conclude that the part of $\Pi_1 \vee \Pi_2$ containing α is αKH. Because the partitions commute, this part is also equal to αHK. We now claim that this implies that $HK = KH$.

Let $g_1 \in HK$. Then $\alpha g_1 \in \alpha HK = \alpha KH$, so there exists some $g_2 \in KH$ such that $\alpha g_1 = \alpha g_2$. It follows that $\alpha g_1 g_2^{-1} = \alpha$, and so $g_1 g_2^{-1} \in G_\alpha$, which in turn implies that $g_1 \in G_\alpha g_2 \subseteq G_\alpha KH$. But since $G_\alpha \leq H$ and $G_\alpha \leq K$, we have $G_\alpha KH = KH$, and so $g_1 \in KH$. Therefore $HK \subseteq KH$. By symmetry, we also get $KH \subseteq HK$, and thus $HK = KH$, as claimed. □

Corollary 4.2. *G has the OB property if and only if, for any two subgroups H and K between G_α and G, we have $HK = KH$.*

Proof. This simply means that the conditions of Theorem 4.1 hold for all G-invariant partitions (or all subgroups containing G_α). □

Subgroups H and K are said to *commute* if $HK = KH$. Thus a transitive permutation group has the OB property if any two subgroups containing a given point stabiliser commute. (Note: In the literature the term 'permute' is often used for this concept; since our subject is permutation groups, we feel that 'commute' is less confusing.)

In some cases we can describe all the orthogonal block structures arising from OB groups.

(a) If the degree n is prime, then a transitive permutation group of degree n preserves only the trivial partitions, so it is OB, with the corresponding OBS being trivial.

(b) Suppose that $n = pq$, where p and q are distinct primes. If G is OB, then it has at most one invariant partition with parts of size p, and at most one with parts of size q. Thus, if G is imprimitive, the OBS preserved by G is obtained from the trivial structures on p and q points either by crossing or by nesting in either order. Thus G is embedded either in the direct product or the wreath product (in some order) of transitive groups of degrees p and q.

(c) Suppose that $n = p^2$ for some prime p and that there are more than two nontrivial G-invariant partitions. Then each such partition has p parts of size p, and any two have meet E and join U. Thus any three of these partitions give the structure of a Latin square L to the underlying set Ω.

Let P be a Sylow p-subgroup of G. The stabiliser in P of a part of a nontrivial G-invariant partition Π_1 has index p in G by the Orbit-Stabiliser Theorem, and fixes all parts of Π_1. If Π_2 is another such partition, then the stabiliser in P of a part in each of Π_1 and Π_2 fixes all parts of $\Pi_1 \wedge \Pi_2$, that is, it is the identity. So $|P| = p^2$; and P, having more than one subgroup of index p, is the elementary abelian group. Since P induces a cyclic group C_p of order p on each of the sets of rows, columns and letters of L, we see that L is the Cayley table of C_p.

The automorphism group of this Latin square is $(C_p \times C_p) : C_{p-1}$. So G is contained in this group. But $C_p \times C_p$ has $p+1$ nontrivial invariant partitions (corresponding to its $p+1$ subgroups of order p) and C_{p-1} fixes just two of these partitions and permutes the other $p-1$ regularly. Since G fixes

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at least three partitions, we conclude that $G = C_p \times C_p$. The $p + 1$ nontrivial G-invariant partitions together with U and E form an orthogonal block structure which is not a poset block structure.

(Transitive groups of degree pq may not be OB. If $q \mid p - 1$, then the nonabelian group of order pq, acting regularly, has p invariant partitions each with p parts of size q; these do not commute. In other words, the subgroups of order q do not commute.)

4.2. Properties of OB groups

4.2.1. General results

A transitive permutation group G is *pre-primitive* (see [1]) if every G-invariant partition is the orbit partition of a subgroup of G. As explained in that paper, we may assume that this subgroup of G is normal.

Corollary 4.3. *If G is pre-primitive, then it has the OB property.*

Proof. If G is pre-primitive, then the G-invariant partitions are orbit partitions of normal subgroups of G; and normal subgroups commute, so the corresponding partitions commute. □

Both properties can be expressed in group-theoretic terms. Thus, the transitive permutation group G is pre-primitive if and only if G_α has a normal supplement in every overgroup (that is, every overgroup has the form $N_H G_\alpha$, where N_H is a normal subgroup of G). By Theorem 4.1, G is OB if and only if all the subgroups containing G_α commute. If $H = N_H G_\alpha$ and $K = N_K G_\alpha$, with N_H, N_K normal in G, then $HK = N_H G_\alpha . N_K G_\alpha = N_H N_K G_\alpha = N_K N_H G_\alpha$, so $HK = KH$.

Corollary 4.4. *Suppose that the G-invariant partitions form a chain under $\preccurlyeq$. Then G has the OB property.*

Proof. If $\Pi_1 \preccurlyeq \Pi_2$, then Π_1 and Π_2 commute. □

A transitive permutation group G is *primitive* if the only G-invariant partitions are the trivial ones (the partition E into singletons and the partition U with a single part); it is *quasiprimitive* if every nontrivial normal subgroup of G is transitive. It was observed in [1] that pre-primitivity and quasiprimitivity together are equivalent to primitivity. However, this is not the case if we replace pre-primitivity by the OB property.

For example, the transitive actions of S_5 and A_5 on 15 points (on the cosets of a Sylow 2-subgroup) are both quasiprimitive but not pre-primitive. However, there is a unique nontrivial invariant partition in each case, with 5 parts each of size 3; so, by Corollary 4.4, these groups are OB.

Another related concept is that of stratifiability, see [3, 13]. The permutation group G on Ω is *stratifiable* if the orbits of G on *unordered* pairs of points of Ω form an association scheme. Since the relations in an association scheme commute, this is equivalent to saying that the symmetric G-invariant relations commute. Since equivalence relations are symmetric, we conclude:

Proposition 4.5. *A stratifiable permutation group has the OB property.*

The paper [13] defines a related property for a transitive permutation group G, that of being *AS-friendly*: this holds if there is a unique finest association scheme which is G-invariant. It is easy to see that a stratifiable group is AS-friendly. So we could ask, is there any relation between being AS-friendly and having the OB property?

In common with many other permutation group properties, the following holds:

Proposition 4.6. *The OB property is upward-closed; that is, if G has the OB property and $G \leq H \leq \mathrm{Sym}(\Omega)$ then H has the OB property.*

Proof. The H-invariant equivalence relations form a sublattice of the lattice of G-invariant equivalence relations. □

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4.2.2. Products

We consider direct and wreath products of transitive groups.

Theorem 4.7. *Let G and H be transitive permutation groups. Then $G \wr H$ (in its imprimitive action) has the OB property if and only if G and H do.*

Proof. If G and H act on Γ and Δ respectively, then $G \wr H$ acts on $\Gamma \times \Delta$, and preserves the canonical partition Π_0 into the sets $\Gamma_\delta = \{(\gamma, \delta) : \gamma \in \Gamma\}$ for $\delta \in \Delta$. It was shown in [1] that any invariant partition for $G \wr H$ is comparable with Π_0; the partitions below Π_0 induce a G-invariant partition on each part of Π_0, while the partitions above Π_0 induce an H-invariant partition on the set of parts.

Suppose that G and H have the OB property, and let Σ_1 and Σ_2 be $G \wr H$-invariant partitions. If one is below Π_0 and the other above, then they are comparable, and so they commute. If both are below, then they commute since G has the OB property; and if both are above, then they commute since H has the OB property. So the OBS is obtained by nesting the OBS for G in that for H.

Conversely, suppose that $G \wr H$ has the OB property. Then the partitions below Π_0 commute, so G has the OB property; and the partitions above Π_0 commute, so H has the OB property. □

Corollary 4.8. *Let G and H be permutation groups. If $G \times H$ has the OB property in its product action then G and H both have the OB property.*

Proof. As in [1], $G \times H$ is a subgroup of $G \wr H$. So, if $G \times H$ has the OB property, then $G \wr H$ has the OB property by Proposition 4.6, and the result holds by Theorem 4.7. □

We will see later (after Theorem 4.14) that the converse is false. However, we have some positive results.

First we prove some general facts about invariant partitions of direct products of an arbitrary number of groups in their product action, and slightly extend a result in [1], proving that the direct product of an arbitrary number of primitive groups in its product action is pre-primitive. This result is interesting in its own right, but it will also be used to show that a generalised wreath product of primitive groups is pre-primitive, which constitutes a part of Theorem 4.19. First we give some language to describe partitions of products.

Let G and H act transitively on Γ and Δ respectively, and let Π be a $(G \times H)$-invariant partition of $\Gamma \times \Delta$. We define two partitions of Γ in the following way:

- Let P be a part of Π. Let P_0 be the subset of Γ defined by

$$P_0 = \{\gamma \in \Gamma : (\exists \delta \in \Delta)((\gamma, \delta) \in P)\}.$$

  We claim that the sets P_0 arising in this way are pairwise disjoint. For suppose that $\gamma \in P_0 \cap Q_0$, where Q_0 is defined similarly for another part Q of Π; suppose that $(\gamma, \delta_1) \in P$ and $(\gamma, \delta_2) \in Q$. There is an element $h \in H$ mapping δ_1 to δ_2. Then $(1, h)$ maps (γ, δ_1) to (γ, δ_2), and hence maps P to Q, and P_0 to Q_0; but this element acts trivially on Γ, so $P_0 = Q_0$. It follows that the sets P_0 arising in this way form a partition of Γ, which we call the *G-projection partition* of Γ induced by Π.
- Choose a fixed $\delta \in \Delta$, and consider the intersections of the parts of Π with $\Gamma \times \{\delta\}$. These form a partition of $\Gamma \times \{\delta\}$ and so, by ignoring the second factor, we obtain a partition of Γ called the *G-fibre partition* of Γ induced by Π. Now the action of the group $\{1\} \times H$ shows that it is independent of the element $\delta \in \Delta$ chosen.

We note that the G-projection partition and the G-fibre partition are both G-invariant, and the second is a refinement of the first. In a similar way we get H-fibre and H-projection partitions of Δ, both H-invariant.

Proposition 4.9. *Let Π be a $G \times H$-invariant partition of $\Gamma \times \Delta$, where G and H act transitively on Γ and Δ respectively. Then the projection and fibre partitions of Π on Γ are equal if and only if Π is obtained by crossing a G-invariant partition of Γ with an H-invariant partition of Δ.*

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Proof. First we observe that the projection and fibre partitions on Γ agree if and only if those on Δ agree. For the pairs in a part P of Π are the edges of a bipartite graph on $A \cup B$, where A and B are parts of the projection partitions on Γ and Δ respectively; the valency of a point in A is equal to the number of points of B in a part of the fibre partition on Δ, which we will denote by a; and similarly the valencies b of the points in B. Then counting edges of the graph (that is, pairs in part P of Π), we see that $|A|a = |B|b$. Now the fibre and projection partitions on Γ agree if and only if $|A| = b$, which is equivalent to $|B| = a$.

Moreover, if this equality holds, then every pair in $A \times B$ lies in the same part of Π, so A and B are parts of both the projection and fibre partitions on the relevant sets. In this case, Π is obtained by crossing these partitions.

Conversely, it is easy to see that if Π is obtained by crossing, then the fibre and projection partitions coincide. □

Next we introduce the notion of partition orthogonality.

Definition

Let G, H be transitive permutation groups, on Γ, Δ respectively, as above. We say that G and H are *partition-orthogonal* if the only $G \times H$-invariant partitions of $\Gamma \times \Delta$ are of the form $\{\Gamma_i \times \Delta_j \mid i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}\}$ where $\{\Gamma_1, \ldots, \Gamma_m\}$ is a G-invariant partition of Γ and $\{\Delta_1, \ldots, \Delta_n\}$ is an H-invariant partition of Δ.

Lemma 4.10. *Let $G_i \leq \mathrm{Sym}(\Omega_i)$ for $i \in \{1, \ldots, m\}$ be transitive, and let $G = G_1 \times \cdots \times G_m$ act on $\Omega = \Omega_1 \times \cdots \times \Omega_m$ component-wise. If G_i and G_j are partition-orthogonal for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$, then the G-invariant partitions are precisely the products of G_i-invariant partitions for $i \in \{1, \ldots, m\}$.*

Proof. We prove the claim by induction. If $m = 2$, then the claim follows by the definition. Suppose that the claim holds for $m - 1$ factors. Let $H = G_1 \times \cdots \times G_{m-1}$ and suppose for a contradiction that there is some G-invariant partition Π which is not a direct product of partitions of the sets Ω_i. Then the H-fibre and H-projection partitions induced on $\Omega_1 \times \cdots \times \Omega_{m-1}$ by Π must differ.

By the induction hypothesis, all the H-invariant partitions are direct products of partitions, and therefore there must exist some $i \in \{1, \ldots, m-1\}$ such that the G_i-fibre and the G_i-projection partition induced on Ω_i by Π differ. However, this means that the partition induced on $G_i \times G_m$ is not a direct product of partitions of $\Omega_i \times \Omega_m$, which is a contradiction since we have assumed that G_i and G_m are partition-orthogonal.

Therefore, every G-invariant partition of Ω must be a direct product of partitions of the sets Ω_i. □

Lemma 4.11. *Let $G_1 \leq \mathrm{Sym}(\Omega_1), \ldots, G_m \leq \mathrm{Sym}(\Omega_m), H \leq \mathrm{Sym}(\Delta)$ be transitive groups. If H is partition-orthogonal to G_i for all $i \in \{1, \ldots, m\}$, then H is partition-orthogonal to $G_1 \times \cdots \times G_m$.*

Proof. Let $G = G_1 \times \cdots \times G_m \times H$ and $\Omega = \Omega_1 \times \cdots \times \Omega_m \times \Delta$. We prove the claim by induction on m.

We first prove the claim for $m = 2$. Let $(\alpha, \beta, \delta), (\alpha', \beta', \delta') \in \Omega_1 \times \Omega_2 \times \Delta$. First consider the H-fibre partition Π_Δ of Δ induced by Π. Note that by definition (α, β, δ) and (α, β, δ') are in the same part of Π if and only if δ and δ' are in the same part of Π_Δ.

Now consider the $(G_1 \times H)$-fibre partition $\Pi_{\Omega_1 \times \Delta}$ of $\Omega_1 \times \Delta$ induced by Π. Since G_1 and H are partition-orthogonal by assumption, we must have $\Pi_{\Omega_1 \times \Delta} = \Pi_{\Omega_1} \times \Pi_\Delta$, where Π_{Ω_1} denotes the G_1-fibre partition of Ω_1 induced by Π. This means that (α, β, δ) and $(\alpha', \beta, \delta')$ are in the same part of Π if and only if α and α' are in the same part of Π_{Ω_1} and δ and δ' are in the same part of Π_Δ.

An entirely similar argument shows that (α, β, δ) and $(\alpha, \beta', \delta')$ are in the same part of Π if and only if β and β' are in the same part of the G_2-fibre partition Π_{Ω_2} of Ω_2 induced by Π and δ and δ' are in the same part of Π_Δ.

It therefore follows that (α, β, δ) and $(\alpha', \beta', \delta')$ are in the same part of Π if and only if α and α' are in the same part of Π_{Ω_1}, β and β' are in the same part of Π_{Ω_2} and δ and δ' are in the same part of Π_Δ, and so

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$$\Pi = \Pi_{\Omega_1} \times \Pi_{\Omega_2} \times \Pi_{\Delta}$$

which proves the claim.

Now suppose that the claim holds for all integers less than m. Then, it follows that H is partition-orthogonal to $G_1 \times \cdots \times G_{m-1}$. Now, since H is partition-orthogonal to both $G_1 \times \cdots \times G_{m-1}$ and G_m, using the inductive hypothesis once more gives us that H is indeed partition-orthogonal to $G_1 \times \cdots \times G_m$. □

Lemma 4.12. *Let $G \leq \mathrm{Sym}(\Gamma)$ and $H \leq \mathrm{Sym}(\Delta)$ be partition-orthogonal pre-primitive groups. Then $G \times H$ in its product action is pre-primitive.*

Proof. Let Π be a $G \times H$-invariant partition of $\Gamma \times \Delta$. Since G and H are partition-orthogonal, Π is the direct product of a G-invariant partition Π_G and an H-invariant partition Π_H. Since both G and H are pre-primitive, it follows that Π_G and Π_H are orbit partitions of some subgroups G^* and H^* of G and H respectively. It is then easy to check that Π is the orbit partition of $G^* \times H^*$, which proves the claim. □

Theorem 4.13. *Let $G_i \leq \mathrm{Sym}(\Omega_i)$ for $i \in \{1, \ldots, m\}$, and let G_i act primitively on Ω_i for all $i \in \{1, \ldots, m\}$. Then $G = G_1 \times \cdots \times G_m$ in its product action is pre-primitive.*

Proof. Abelian primitive groups are cyclic of prime order. So, by rearranging the components if necessary, we can write G as a direct product of elementary abelian groups of different prime power order and nonabelian primitive groups.

It has been shown in [1] that two primitive groups are partition-orthogonal if and only if they are not cyclic of the same prime order. Therefore, if P and Q are two elementary abelian groups of orders p^a and q^b respectively, with $p \neq q$, then it follows by Lemma 4.11 that every component of Q is partition-orthogonal to P, and then applying Lemma 4.11 again, we get that P must be partition-orthogonal to Q. Similarly, we get that any elementary abelian group and any nonabelian primitive group are partition-orthogonal. Then Lemma 4.10 gives us that G can be written as a direct product of mutually partition-orthogonal factors, and it is hence pre-primitive by Lemma 4.12. □

4.2.3. Regular groups

It follows from Corollary 4.2 that, if G is a regular permutation group, then G has the OB property if and only if any two subgroups of G commute. These groups were determined by Iwasawa [25]; we refer to Schmidt [31, Chapter 2] for all the material we require. In this section we use the term *quasi-hamiltonian*, taken from [15], for a group in which any two subgroups commute. (The term will not be used outside this section.)

We warn the reader that both Iwasawa and Schmidt consider hypotheses which are more general in two ways:

- they consider groups whose subgroup lattices are modular, which is weaker than requiring all subgroups to commute;
- they consider infinite as well as finite groups.

We have not found a reference for precisely what we want, so we give a direct proof of the first part; the second is [31, Theorem 2.3.1].

Theorem 4.14. *The following are true:*

(a) *A finite group G is quasi-hamiltonian if and only if it is the direct product of quasi-hamiltonian subgroups of prime power order.*
(b) *Suppose that p is prime, and G is a nonabelian quasi-hamiltonian p-group. Then either*
   - *$G = Q_8 \times V$, where Q_8 is the quaternion group of order 8 and V an elementary abelian 2-group; or*
   - *G has an abelian normal subgroup A with cyclic factor group and there is $b \in G$ with $G = A\langle b \rangle$ and s such that $b^{-1}ab = a^{1+p^s}$ for all $a \in A$, with $s \geq 2$ if $p = 2$.*

Here is the proof of part (a). Suppose that P_1 and P_2 are Sylow p-subgroups of the quasi-hamiltonian group G. Then P_1P_2 is a subgroup, and $|P_1P_2| = |P_1| \cdot |P_2|/|P_1 \cap P_2|$. Since P_1 and P_2 are Sylow

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subgroups, this implies that $P_1 = P_2$. So all Sylow subgroups of G are normal, and G is nilpotent. Thus it is the direct product of its Sylow subgroups. Since quasi-hamiltonicity is clearly inherited by subgroups, the result follows

Conversely, if G is nilpotent with quasi-hamiltonian Sylow subgroups, then any subgroup is nilpotent and hence a direct product of its Sylow subgroups. Factors whose orders are powers of different primes commute; factors whose orders are powers of the same prime commute by hypothesis. So any two subgroups commute.

Note that not every quasi-hamiltonian group is a Dedekind group, namely a group all of whose subgroups are normal; so the OB property lies strictly between transitivity and pre-primitivity. Note also that Q_8 (acting regularly) is quasi-hamiltonian but $Q_8 \times Q_8$ in the product action is not; so the OB property is not closed under direct product.

For groups with a regular normal subgroup, we have the following result.

Theorem 4.15. *If $G \leq \mathrm{Sym}(\Omega)$ is a transitive group containing a regular normal subgroup N, then G is OB if and only if the subgroups of N normalised by G_α commute.*

Proof. Suppose that G is OB. Since N is a regular normal subgroup of G we can write $G = NG_\alpha$ for some $\alpha \in \Omega$, where $N \cap G_\alpha = 1$ and we can identify Ω with N in such a way that G_α acts by conjugation and N acts by right multiplication.

We first show that the subgroups containing G_α are of the form HG_α for some $H \leq N$ invariant under the action of G_α. Let F be a subgroup containing G_α. Since $F \leq G = NG_\alpha$ all the elements of F are of the form ng where $n \in N$ and $g \in G_\alpha$. Then since $G_\alpha \leq F$ it follows that $n = ngg^{-1} \in F$. Hence, $F = HG_\alpha$, where $H = N \cap F \leq N$. Since both F and N are invariant under the action of G_α, so is H.

Let KG_α, LG_α be two such subgroups. Since G is OB, they commute (Corollary 4.2), and we have

$$KG_\alpha LG_\alpha = LG_\alpha KG_\alpha. \tag{4.1}$$

But $G_\alpha L = LG_\alpha$ and $G_\alpha K = KG_\alpha$ since $KG_\alpha, LG_\alpha \leq G$. Therefore, by Equation (4.1) we get

$$KLG_\alpha = LKG_\alpha$$

and intersecting both sides with N gives us $KL = LK$. Since K, L were arbitrary G_α-invariant subgroups of N the claim holds.

Conversely, suppose that all the G_α-invariant subgroups of N commute and consider subgroups $KG_\alpha, LG_\alpha \leq G$, where K and L are subgroups of N normalised by G_α. Then

$$KG_\alpha LG_\alpha = KLG_\alpha = LKG_\alpha = LG_\alpha KG_\alpha,$$

and so G is OB (again by Corollary 4.2). □

4.2.4. Modularity and distributivity

We have seen at the start of Section 4.2.3 that the subgroup lattice of a group, which clearly determines modularity, does not determine whether the subgroups commute. So we cannot expect a characterisation of the OB property in terms of the lattice of subgroups containing a given point stabiliser. But is there anything to say here?

An example of a transitive group in which the lattice of invariant equivalence relations is the pentagon (P_5 in Figure 2) is the following. Let G be the 2-dimensional affine group over a finite field F of order q, and let G act on the set of *flags* (incident point-line pairs) in the affine plane. The three nontrivial G-invariant relations are 'same line', 'parallel lines', and 'same point'. Clearly the equivalence relations 'same point' and 'same line' do not commute.

Since modularity does not imply the OB property, we could ask whether a stronger property does. We saw in Corollary 4.4 that the property of being a chain does suffice. Is there a weaker property?

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Proposition 4.16. *Let G be a finite regular permutation group. Then the lattice of G-invariant partitions is distributive if and only if G is cyclic.*

This is true because a group with distributive subgroup lattice is locally cyclic, by Ore's theorem [31, Section 1.2], and a finite locally cyclic group is cyclic. Since a cyclic group is Dedekind, it is pre-primitive and so has the OB property.

However, there is no general result along these lines. Even if we assume that the lattice of G-invariant partitions is a *Boolean lattice* (isomorphic to the lattice of subsets of a finite set), the group may fail to have the OB property, as the next example shows.

Example

Let $G = \mathrm{GL}(n, q)$ acting on the set of maximal chains of nontrivial proper subspaces

$$V_1 < V_2 < \cdots < V_{n-1}$$

in the vector space $V = \mathrm{GF}(q)^n$, where $\dim(V_k) = k$ for $0 < k < n$. The stabiliser B of such a chain is a *Borel subgroup* of G; if we take V_k to be spanned by the first k basis vectors, then B is the group of upper triangular matrices with nonzero entries on the diagonal. From the theory of algebraic groups, it is known that the only subgroups of G containing B are the *parabolic subgroups*, the stabilisers of subsets of $\{V_1, \ldots, V_{n-1}\}$ (see, e.g., [24] for the theory). Hence the lattice of G-invariant partitions is isomorphic to the Boolean lattice B_{n-1} of subsets of $\{1, \ldots, n-1\}$ (the isomorphism reverses the order since the stabiliser of a smaller set of subspaces is larger).

However, the equivalence relations do not all commute. Consider the relations Π_1 and Π_2 corresponding to the subgroups fixing V_1 and V_2. Thus, two chains are in the relation Π_1 if they contain the same 1-dimensional subspace, and similarly for Π_2. Now starting from the chain $(V_1, V_2, \ldots, V_n)$, a move in a part of Π_2 fixes V_2 and moves V_1 to a subspace V_1' of V_2; then a move in Π_1 fixes V_1', so the resulting chain begins with a subspace of V_2. But if we move in a part of Π_1, we can shift V_2 to a different 2-dimensional subspace, and then a move in a part of Π_2 can take V_1 to a subspace not contained in V_2. So $\Pi_1 \circ \Pi_2 \neq \Pi_2 \circ \Pi_1$, and the lattice is not an OBS.

So G does not have the OB property, even though the lattice of G-invariant partitions is a Boolean lattice (and hence distributive).

4.3. Generalised wreath products

In this section, we prove two main results. The first describes the group-theoretic structure of a generalised wreath product, and will be needed later. The second investigates properties of the generalised wreath product of primitive groups; in particular, they are pre-primitive and hence have the OB property, and we give necessary and sufficient conditions for them to have the PB property.

4.3.1. A group-theoretic result

First we prove a result about generalised wreath products which will be needed later.

We note that, if p is a minimal element of a poset M, then $\{p\}$ is a down-set, and so corresponds to a partition Π of the domain Ω of the generalised wreath product of a family of groups over M.

Theorem 4.17. *Let G be the generalised wreath product of the groups $G(m)$ over a poset M, acting on a set Ω. Let p be a minimal element of M. Let Π be the corresponding partition of Ω, H the group induced on the set of parts by G, N the stabiliser of all parts of Π. Then*

(a) *H is isomorphic to the generalised wreath product of the groups $G(q)$ for $q \in M \setminus \{p\}$;*
(b) *N is a direct product of copies of $G(p)$, where there is an equivalence relation $\sim$ on the set of parts of Π (determined by the poset M) such that each direct factor acts in the same way on the parts in one equivalence class and fixes every point in the other parts;*
(c) *G is a semidirect product $N \rtimes H$.*

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Proof. For (a), we note that, since p is minimal, suppressing the pth coordinate of every tuple in Ω gives the generalised wreath product of the remaining groups indexed by the elements different from p.

Part (b) is proved using the definition of a generalised wreath product. The equivalence relation is defined as follows: for parts P and Q of Π, $P \sim Q$ if and only if P and Q lie in the same part of $\Pi \vee \Phi$ for all partitions Φ of the poset block structure defined by M which are incomparable to Π.

First note that since N fixes the parts of Π, it must also fix the parts of every partition lying above Π. Therefore, only parts of partitions incomparable to Π can be moved by N. Now let Φ denote a partition incomparable to Π. Note that since $\Pi \preccurlyeq \Pi \vee \Phi$, the parts of Φ contained in the same part of $\Pi \vee \Phi$ can only be permuted amongst themselves by N. Hence, if the actions of $h \in N$ on two parts of Π are equivalent, then those two parts must be contained in the same part of $\Pi \vee \Phi$ for every partition Φ of Ω incomparable to Π.

It now remains to show that if $P, Q \in \Pi$ are such that $P \sim Q$, then N acts in the same way on P and Q. Let γ, δ lie in P and Q, and moreover suppose that they are contained in the same part of Φ for every partition Φ of Ω incomparable to Π. It suffices to show that every $h \in N$ maps γ and δ to the same part of Φ for every Φ incomparable to Π.

Now h can be written as a product $\prod_\Phi h_\Phi$, where each factor h_Φ encodes the permutation induced by h of the parts of the corresponding partition Φ of Ω induced by h. Hence, it suffices to show that h_Φ maps γ and δ to the same part for an arbitrary partition Φ of Ω incomparable to Π. We may assume without loss of generality that Φ is join-indecomposable, since every element is a join of JI elements, and the distributive law implies that if a collection of JI elements are incomparable with Π then so is their join.

Let m be the element corresponding to Φ in the poset M. Using the notation established in [9], we note that γ and δ must be such that $\gamma_i = \delta_i$ for all $i \sqsupseteq m$ in M. Therefore,

$$(\gamma h_\Phi)_i = \gamma_i(\gamma\pi^i(h_\Phi)_i) = \delta_i(\delta\pi^i(h_\Phi)_i) = (\delta h_\Phi)_i$$

for all $i \sqsupseteq m$, which proves the claim.

We finally note that $\sim$ is dependent only on the poset M and not the group G.

For (c), we have to show that H normalises N and that the action of H extends to Ω. The first statement is clear since N is the subgroup fixing all parts of Π. For the second, note that H acts on the set of $(|M| - 1)$-tuples; extend each element to act on $|M|$-tuples by acting as the identity on the p-th coordinate. □

4.3.2. Generalised wreath products of primitive groups

In this section, we will use the notation for poset block structures and generalised wreath products defined in Section 2.5. Moreover, let $[N]$ denote the set $\{1, \dots, N\}$, and for every subset J of M, let X_J be the index set of J, namely $\{i \in [N] : m_i \in J\}$. We then define P_J to be the partition of Ω whose set of parts is

$$\left\{ \prod_{j \in X_J} \Omega_j \times \prod_{k \in [N] \setminus X_J} \{\alpha_k\} : \alpha_k \in \Omega_k \text{ for all } k \in [N] \setminus X_J \right\}.$$

We now prove a small lemma that will be used in the proof of Theorem 4.19.

Lemma 4.18. *Let G be the generalised wreath product of the groups $G(m_i)$ over the poset M. Let J, K be down-sets of M such that $P_K \preccurlyeq P_J$, let Γ be a part of P_J, and let Δ be the set of parts of P_K contained in Γ. Then the permutation group $G(\Delta, \Gamma)$ induced by the setwise stabiliser of Γ on Δ is isomorphic to the generalised wreath product of the groups $G(m_i)$ for $i \in X_J \setminus X_K$.*

Proof. Let $\Gamma = \prod_{j \in X_J} \Omega_j \times \prod_{i \in [N] \setminus X_J} \{\alpha_i\}$, where α_i is fixed for $i \in [N] \setminus X_J$. Note that the setwise stabiliser G_Γ in G must be equal to the generalised wreath product of the groups $H(m_i)$, where $H(m_i) = G(m_i)$ for all $i \in X_J$ and $H(m_i) = (G(m_i))_{\alpha_i}$ for $i \notin X_J$. Now since the elements of Δ are

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blocks of imprimitivity of G, they are also blocks of G_Γ, and moreover, since Γ is a block of P_J, it follows that G_Γ induces a permutation group on Δ. Let ρ denote the associated permutation representation.

Note that every element of Δ is of the form $\prod_{i\in X_K}\Omega_i \times \prod_{i\in[N]\setminus X_K}\{\alpha_i\}$, where α_i is fixed for $i \in X_J \setminus X_K$. Therefore, $\ker\rho$ must fix all elements of Ω_i for $i \in X_J \setminus X_K$, must fix α_i for $i \in [N] \setminus X_J$, and can permute the elements of Ω_i for $i \in X_K$ in any way G_Γ allows. Hence, $\ker\rho$ is equal to the generalised wreath product of $L(m_i)$, where $L(m_i) = G(m_i)$ for $i \in X_K$, $L(m_i) = 1$ for $i \in X_J \setminus X_K$, and $L(m_i) = G(m_i)_{\alpha_i}$ for $i \in [N] \setminus X_J$. We then deduce that

$$G(\Delta, \Gamma) \cong G_\Gamma/\ker\rho,$$

the generalised wreath product of $G(m_i)$ for $i \in X_J \setminus X_K$, as claimed. □

We are now in a position to state and prove the main theorem of this section.

Theorem 4.19. *If $G(m_i)$ is primitive for every $i \in [N]$, then the following hold for their generalised wreath product G:*

(a) *G is pre-primitive, and hence has the OB property;*
(b) *the following are equivalent:*
  (i) *G has the PB property;*
  (ii) *the only G-invariant partitions are the ones corresponding to down-sets in M;*
  (iii) *there do not exist incomparable elements $m_i, m_j \in M$ such that $G(m_i)$ and $G(m_j)$ are cyclic groups of the same prime order.*

Proof. Let K denote the direct product of the groups $G(m_i)$ (for $i \in [N]$) in its product action and P denote the lattice of partitions corresponding to down-sets in M.

(a) Since pre-primitivity is upward-closed, it suffices to show that K can be embedded in G. Then the claim will follow by Theorem 4.13. Let H be the set of all functions $f = \prod_{i\in[N]} f_i \in G$ such that f_i sends all elements of Ω^i to the same element of $G(m_i)$ for all $i \in [N]$. We will show that H is permutation isomorphic to K.

We first start by showing that $H \leq G$. To prove closure, it suffices to show that for $f, h \in H$, then $(fh)_i$ sends all elements of Ω^i to the same element of $G(m_i)$ for every $i \in [N]$. We can do this by showing that if $\gamma, \delta \in \Omega$, then fh acts on both γ and δ with the same group element on each coordinate. We will slightly abuse notation and for $f = \prod_{i\in[N]} f_i \in H$, we will write $\mathrm{im}(f_i)$ for the element that f_i maps all the elements of Ω^i to, instead of the set containing just this element. Now let $g = \mathrm{im}(f_i)$ and $g' = \mathrm{im}(h_i)$, then

$$(\gamma fh)_i = (\gamma f)_i(\gamma f\pi^i h_i) = \gamma_i(\gamma\pi^i f_i)(\gamma f\pi^i h_i) = \gamma_i g g'$$

for all $i \in [N]$. Similarly,

$$(\delta fh)_i = (\delta f)_i(\delta f\pi^i h_i) = \delta_i(\delta\pi^i f_i)(\delta f\pi^i h_i) = \delta_i g g'$$

for all $i \in [N]$, and therefore, $fh \in H$. For i in $[N]$, let z_i be the function which maps all the elements of Ω^i to the inverse of $\mathrm{im}(f_i)$. Put $z = \prod_{i\in[N]} z_i$. Then

$$(\gamma fz)_i = (\gamma f)_i(\gamma f\pi^i z_i) = \gamma_i(\gamma\pi^i f_i)(\gamma f\pi^i z_i) = \gamma_i g g^{-1} = \gamma_i$$

for all $i \in [N]$, and thus $z = f^{-1} \in H$.

We now need to show that H is permutation isomorphic to K in its product action on Ω. Let $\phi : G \to K$ be the function defined by the formula $(\prod_{i\in[N]} f_i)\phi = \prod_{i\in[N]} \mathrm{im}(f_i)$, and let id denote the identity function. Note that ϕ is clearly a bijection by construction, and also

$$(\delta f)_i = \delta_i g_i$$

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for all $i \in [N]$, where $g_i = \mathrm{im}(f_i)$, and hence

$$(\delta f) = \delta\left(\prod_{i\in[N]} \mathrm{im}(f_i)\right) = \delta(f\phi) = (\delta\,\mathrm{id})(f\phi),$$

which completes the proof of (a).

We now prove (b).

(ii) $\Rightarrow$ (i). Note that if the only partitions preserved by G are the ones in P, then clearly G is PB.

(iii) $\Rightarrow$ (ii). We prove the contrapositive. Suppose that there are further partitions fixed by G other than the ones corresponding to down-sets in M, and let Π be such a partition. Since $K \leq G$, it follows that Π is also preserved by K. Therefore, by the arguments in the proof of Theorem 4.13 we deduce that since all of the $G(m_i)$s are primitive, one of the following must hold:

- All the $G(m_i)$s are mutually partition-orthogonal, and so Π is of the form

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \sim_J (\beta_1, \beta_2, \ldots, \beta_n) \iff (\forall i \notin X_J)(\alpha_i = \beta_i),$$

  where J is not a down-set of M;
- at least two of the $G(m_i)$s, say $G(m_i)$ and $G(m_j)$, are cyclic of the same prime order, and Π is a partition whose corresponding $G(m_i)$ and $G(m_j)$-fibre partitions are different from the $G(m_i)$ and $G(m_j)$-projection partitions respectively. Since the degree of $G(m_i)$ and $G(m_j)$ is prime, this can only happen if the fibre partitions are the partitions into singletons and the projection partitions are the partitions into a single part.

If Π is of the first type, then there exist some $i, j \in [N]$ such that $m_i \sqsubset m_j$ and $m_j \in J$, but $m_i \notin J$. Since m_i and m_j are comparable, there exists a chain $(m_i = a_0, a_1, \ldots, a_k = m_j)$ in M. Thus, Π must be preserved by the wreath product $G(a_0) \wr G(a_1) \wr \ldots \wr G(a_k)$. However, we know that an imprimitive iterated wreath product cannot preserve partitions of equivalence relations where

$$(\alpha_{a_0}, \ldots, \alpha_{a_k}) \sim (\beta_{a_0}, \ldots, \beta_{a_k}),$$

with $\alpha_{a_s} = \beta_{a_s}$ but $\alpha_{a_t} \neq \beta_{a_t}$ for some s, t such that $s < t$, because for every $l \in \{1, \ldots, k\}$, the group $G(a_l)$ permutes whole copies of Ω_r for each $r \in \{0, \ldots, l-1\}$.

Hence, Π must be of the second type and thus there exist $G(m_i)$ and $G(m_j)$ cyclic of the same prime order and Π is a partition whose corresponding $G(m_i)$ and $G(m_j)$-fibre partitions are the partitions into singletons and the $G(m_i)$ and $G(m_j)$-projection partitions are the partitions into a single part. If m_i and m_j are related, say $m_i \sqsubset m_j$ then, as above, Π must be preserved by the iterated wreath product $G(a_0) \wr G(a_1) \wr \ldots \wr G(a_k)$. However, knowing what partitions imprimitive iterated wreath products preserve, we deduce that $G(a_0) \wr G(a_1) \wr \ldots \wr G(a_k)$ cannot preserve Π and therefore m_i and m_j must be incomparable.

(i) $\Rightarrow$ (iii). We again prove the contrapositive. So suppose that there are two incomparable elements in M, say m_1 and m_2, such that the corresponding groups are cyclic of the same prime order p. As defined in Section 2.5,

$$D(i) = \{m \in M : m \sqsubset m_i\}$$

for $i = 1, 2$. Set

$$S = D(1) \cup D(2), \quad Q = S \setminus \{m_1\}, \quad R = S \setminus \{m_2\}, \text{ and } T = Q \cap R.$$

These four sets are all down-sets, and the interval between T and S has the group $G(m_1) \times G(m_2)$ acting, and so we can find partitions fixed by the group, other than the ones corresponding to down-sets in M. More precisely, there are $p+1$ partitions corresponding to orbit partitions of the diagonal subgroups of $G(m_1) \times G(m_2)$, and thus preserved by $G(m_1) \times G(m_2)$. If Y is one of those, then S, T, Q, R, Y form a

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N_3 sublattice (Figure 2) of the invariant partition lattice of G, and hence G fails the PB property. This proves the claim. □

4.4. The embedding theorem

The Krasner–Kaloujnine theorem [27] says that, if G is a transitive but imprimitive permutation group on Ω, then G is embeddable in the wreath product of two groups which can be extracted from G (the stabiliser of a block acting on the block, and G acting on the set of blocks).

In this section, we extend this result to transitive groups which preserve a poset block structure (a distributive lattice of commuting equivalence relations). In particular, our result holds for groups with the PB property. As explained in Subsection 2.5, such a lattice Λ is associated with a poset M (so that M consists of the non-E join-indecomposable elements of Λ, and Λ consists of the down-sets in M). We want to associate a group with each element $m \in M$ such that G is embedded in the generalised wreath product of these groups over the poset M.

Our first attempt was as follows. Take $m \in M$; it corresponds to a join-indecomposable partition $\Pi \in \Lambda$. The join-indecomposability of Π implies that there is a unique partition Π^- in Λ which is maximal with respect to being below Π. Then let $G(m)$ be the permutation group induced by the stabiliser of a part of Π acting on the set of parts of Π^- it contains.

However, this does not work. Take G to be the symmetric group S_6. This group has an outer automorphism, and so has two different actions on sets of size 6. Take Ω to be the Cartesian product of these two sets. The invariant partitions for G are E and U together with the rows R and columns C of the square. Then $\{E, R, C, U\}$ is a poset block structure. Both R and C are join-indecomposable, and $R^- = C^- = E$. Thus M is a 2-element antichain $\{r, c\}$, and $G(r)$ is the stabiliser of a row acting on the points of the row, which is the group $\mathrm{PGL}(2, 5)$, and similarly $G(c)$. However, S_6 is clearly not embeddable in $\mathrm{PGL}(2, 5) \times \mathrm{PGL}(2, 5)$.

So we use a more complicated construction. Given Π and Π^- as above, where Π corresponds to $m \in M$, let $\mathcal{G}(m)$ be the set of partitions $\Phi \in \Lambda$ satisfying $\Phi \wedge \Pi = \Pi^-$. For $\Phi \in \mathcal{G}(m)$, let $G_\Phi(m)$ be the group induced on the set of parts of Φ contained in a given part of $\Phi \vee \Pi$.

Lemma 4.20. *The following are true:*

(a) $\mathcal{G}(m)$ *is closed under join.*
(b) *If* $\Phi_1, \Phi_2 \in \mathcal{G}(m)$ *with* $\Phi_1 \preccurlyeq \Phi_2$*, then there is a canonical embedding of* $G_{\Phi_1}(m)$ *into* $G_{\Phi_2}(m)$.

Proof. The first part is immediate from the distributive law: if $\Phi_1, \Phi_2 \in \mathcal{G}(m)$, then

$$(\Phi_1 \vee \Phi_2) \wedge \Pi = (\Phi_1 \wedge \Pi) \vee (\Phi_2 \wedge \Pi) = \Pi^- \vee \Pi^- = \Pi^-.$$

For the second part, we use the fact that, for a given point $\alpha \in \Omega$, there is a natural correspondence between partitions and certain subgroups of G containing G_α, where the partition Π corresponds to the setwise stabiliser of the part of Π containing α; meet and join correspond to intersection and product of subgroups. Let H_1, H_2, P, P^- be the subgroups corresponding to $\Phi_1, \Phi_2, \Pi, \Pi^-$. Then the definition of $\mathcal{G}(m)$ shows that $H_i \cap P = P^-$ for $i = 1, 2$, while the partitions $\Phi_i \vee \Pi$ correspond to the subgroups $H_i P$. The actions we are interested in are thus $H_i P$ on the cosets of H_i. We have

$$|H_i P : H_i| = |P : H_i \cap P| = |P : P^-|$$

for $i = 1, 2$; so coset representatives of P^- in P are also coset representatives for H_i in $H_i P$. Thus we have a natural correspondence between these sets. Since $H_1 \le H_2$, we have $H_1 P \le H_2 P$, and the result holds. □

Hence if Ψ is the (unique) maximal element of $\mathcal{G}(m)$, then the group $G_\Psi(m)$, which we will denote by $G^*(m)$, embeds all the groups $G_\Phi(m)$ for $\Phi \in \mathcal{G}(m)$.

Now we can state the embedding theorem.

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Theorem 4.21. *Let G be a transitive permutation group which preserves a poset block structure Λ, and let M be the associated poset. Define the groups $G^*(m)$ for $m \in M$ as above. Then G is embedded in the generalised wreath product of the groups $G^*(m)$ over $m \in M$.*

We remark that this theorem generalises the theorem of Krasner and Kaloujnine. If Π is a nontrivial G-invariant partition, then $\{E, \Pi, U\}$ is a poset block structure; the corresponding poset is $M = \{m_1, m_2\}$, with m_1 and m_2 corresponding to the partitions Π and U; this $G^*(m_1)$ is the group induced by the stabiliser of a part of Π on its points, and $G^*(m_2)$ the group induced by G on the parts of Π, as required.

The proof uses properties of distributive lattices: we deal with some of these first. Since these lemmas are not specifically about lattices of partitions, we depart from our usual convention and use lower-case italic letters for elements of a lattice, and 0 and 1 for the least and greatest elements respectively.

Lemma 4.22. *Let L be a distributive lattice. If $a, x, y \in L$ satisfy*

$$a \wedge x = a \wedge y \text{ and } a \vee x = a \vee y,$$

then $x = y$.

Proof. Suppose first that $x \leq y$. Then

$$\begin{aligned} y &= y \vee (a \wedge x) \\ &= (y \vee a) \wedge (y \vee x) \\ &= (x \vee a) \wedge (x \vee y) \\ &= x \vee (a \wedge y) \\ &= x \vee (a \wedge x) \\ &= x. \end{aligned}$$

Now let x and y be arbitrary, and put $z = x \wedge y$. Then $z \leq x$ and

$$\begin{aligned} a \wedge z &= (a \wedge x) \wedge (a \wedge y) = a \wedge x \\ a \vee z &= (a \vee x) \wedge (a \vee y) = a \vee x. \end{aligned}$$

By the first part, $z = x$. Similarly $z = y$, so $x = y$. □

Remark. The identity in Lemma 4.22 is commonly known as the *cancellation property for distributive lattices*, which appears in [16] as Exercise 6.6. Note that distributivity and the cancellation property are in fact equivalent lattice properties, but we only use one direction here, so we only prove the direction we use. The converse can be proved using Theorem 2.5. In particular, it is clear that P_5 and N_3 do not admit cancellation.

Lemma 4.23. *Suppose that L is the lattice of down-sets in a poset M. Let p be a minimal element of M (so that $\{p\}$ is a down-set). Then the interval $[\{p\}, 1]$ in L is isomorphic to the lattice of down-sets in $M \setminus \{p\}$.*

Proof. Let $z = \{p\}$. Let $\mathrm{JI}(L)$ be the set of join-indecomposables in L. Since lattices are generated by their join-indecomposable elements, it suffices to construct an order-isomorphism F from $\mathrm{JI}(L) \setminus z$ to $\mathrm{JI}([z, 1])$. The map F is defined by

$$F(a) = a \vee z$$

for $a \in \mathrm{JI}(L) \setminus \{z\}$. We have to show that it is a bijection and preserves order. First we show that its image is contained in $\mathrm{JI}([z, 1])$.

Take $a \in \mathrm{JI}(L)$, $a \neq z$. If $z \leq a$, then $a \vee z = a$ and this is join-indecomposable in $[z, 1]$. Suppose that $z \not\leq a$. If $a \vee z$ is not JI in $[z, 1]$, then there exist $b, c \in [z, 1]$ with $b, c \neq a \vee z$ and $b \vee c = a \vee z$. Then

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$$(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c) = a \wedge (a \vee z) = a.$$

Since a is join-indecomposable, we have, without loss of generality, $a \wedge b = a$, so $a \leq b$. Since we also have $z \leq b$, it follows that $a \vee z \leq b$, and so $a \vee z = b$, a contradiction.

We show that the map is onto. Let $a \in \mathrm{JI}([z,1])$. If $a \in \mathrm{JI}(L)$ then $a = F(a)$; so suppose not. Then $a = b \vee c$ for some $b, c \in L$. Then

$$z = a \wedge z = (b \vee c) \wedge z = (b \wedge z) \vee (c \wedge z),$$

so at least one of b and c (but not both) is in $[z,1]$, say $b \in [z,1]$. Then $a = b \vee (c \vee z)$. Since $a \in \mathrm{JI}([z,1])$ and $a \neq b$, we must have $c \vee z = a$. We claim that c is join-indecomposable. For if $c = d \vee e$, then

$$a = c \vee z = (d \vee z) \vee (e \vee z).$$

If $d \vee z = a = c \vee z$, then $c = d$ (since $d \wedge z = 0 = c \wedge z$), a contradiction. The other case leads to a similar contradiction.

Next we show that F is one-to-one. Suppose that $F(a_1) = F(a_2)$. If $a_1, a_2 \in [z,1]$, then $a_1 = a_2$. If $a_1, a_2 \notin [z,1]$, then $a_1 \vee z = F(a_1) = F(a_2) = a_2 \vee z$; also $a_1 \wedge z = 0 = a_2 \wedge z$. By Lemma 4.22, $a_1 = a_2$. So suppose that $a_1 \in [z,1]$, $a_2 \notin [z,1]$. Then $a_1 = F(a_1) = F(a_2) = a_2 \vee z$, contradicting the fact that a_1 is join-indecomposable.

Finally we show that F is order-preserving. Suppose that $a_1 \leq a_2$. If $a_1, a_2 \in [z,1]$, then $F(a_1) = a_1 \leq a_2 = F(a_2)$. If $a_1, a_2 \notin [z,1]$, then $F(a_1) = a_1 \vee z \leq a_2 \vee z = F(a_2)$. We cannot have $a_1 \in [z,1]$ and $a_2 \notin [z,1]$, since then $z \leq a_1 \leq a_2$ but $z \not\leq a_2$. Finally suppose that $a_1 \notin [z,1]$ but $a_2 \in [z,1]$, so that $z \leq a_2$ and $a_1 \leq a_2$, then $F(a_1) = a_1 \vee z \leq a_2 = F(a_2)$. □

Now we turn to the proof of Theorem 4.21. The proof is by induction on the number of elements in M. We take Π_0 to be a minimal non-E partition, corresponding to a minimal element $p \in M$. We decorate things computed in the interval $[\Pi_0, U]$ with bars; for example, $\bar{G}^*(q)$ corresponds to the group associated in this lattice with the element $q \neq p$ (which is not in general the same as $G^*(q)$). Thus $\bar{G}$ is the group induced by G on the set of parts of Π_0, which is a PB group with associated poset $M \setminus \{p\}$; our induction hypothesis will imply that the group $\bar{G}$ is embedded in the generalised wreath product of the groups $\bar{G}^*(q)$ for $q \in M \setminus \{p\}$.

Let Π be a join-indecomposable partition in $[\Pi_0, U]$, corresponding to the element $q \in M \setminus \{p\}$. As we saw in the proof of Lemma 4.23, there are two possibilities:

- *Case 1:* Π is join-indecomposable in the lattice L of downsets of M. Then Π^- is above Π_0, and so the group $\bar{G}^*(q)$ is the same as $G^*(q)$.
- *Case 2:* $\Pi = \Pi_0 \vee \Psi$, where Ψ is join-indecomposable in L and Π_0 is not below Ψ. Consider the set $\bar{\mathcal{G}}(q)$, where the bar denotes that it is computed in the lattice $[\Pi_0, U]$. A partition Φ belongs to this set if it is above Π_0 and satisfies $\Phi \wedge \Pi = \bar{\Pi}^-$, where again the bar denotes the unique maximal element below Π in $[\Pi_0, U]$. An easy exercise shows that $\bar{\Pi}^- \wedge \Psi = \Psi^-$; hence

  $$\Phi \wedge \Psi = \Psi^-,$$

  and so Φ belongs to $\mathcal{G}(q)$. In other words, we have shown that

  $$\bar{\mathcal{G}}(q) \subseteq \mathcal{G}(q).$$

  By Lemma 4.20, $\bar{G}^*(q)$ is canonically embedded in $G^*(q)$.

In other words, $\bar{G}^*(q) \leq G^*(q)$ for all $q \in M \setminus \{p\}$. Now, using the induction hypothesis, the group $\bar{G}$ induced by G on the parts of Π_0 is embedded in the generalised wreath product of the groups $G^*(q)$ over $q \in M \setminus \{p\}$.

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Next, consider the normal subgroup N_0 of G which fixes every part of Π_0. Because G preserves the poset block structure, N_0 is contained in the automorphism group of this structure, which is a generalised wreath product of symmetric groups, by Proposition 2.7. Hence there is an equivalence relation on the set of parts of Π_0 as described in Theorem 4.17; the subgroup of the generalised wreath product fixing all parts of Π_0 is a direct product of symmetric groups. Since the stabiliser in G of a part of Π_0 induces the group $G(p)$ on it, we see that N_0 is actually contained in the direct product of copies of $G(p)$, where the conditions of Theorem 4.17 apply to this product. Since $G(p) \leq G^*(p)$, we have that N_0 is contained in the stabiliser of the parts of Π_0 in the generalised wreath product of the groups $G^*(q)$. We call this stabiliser N^*.

In Theorem 4.17, we saw that the generalised wreath product G^* of the groups $G^*(q)$ is the semidirect product $N^* \rtimes H^*$ of this normal subgroup by the generalised wreath product H^* of the groups $G^*(q)$ for $q \neq p$. Now G has a normal subgroup which is contained in N^*, and a complement which is contained in H^*; so G is contained in G^*. This completes the proof of Theorem 4.21.

4.5. Intersections of posets

If G_1 and G_2 are permutation groups on Ω_1 and Ω_2 respectively, then $G_1 \times G_2$ is a subgroup of $G_1 \wr G_2$; indeed, $G_1 \times G_2$ is the intersection of $G_1 \wr G_2$ and $G_2 \wr G_1$. We are going to extend this to arbitrary generalised wreath products.

Given a family $(G(i) \leq \mathrm{Sym}(\Omega_i) : m_i \in M)$ of transitive permutation groups indexed by a set M, any partial order on M gives rise to a generalised wreath product of the groups. So we have a map from partial orders on M to generalised wreath products of the groups $G(i)$. In this section, we prove that this map preserves order and intersections. To explain the terminology, inclusions and intersections of partial orders on the same sets are given by inclusions and intersections of the sets of ordered pairs comprising the order relations. It is easy to show that the intersection of partial orders is a partial order.

Theorem 4.24. *Let $(G(i) \leq \mathrm{Sym}(\Omega_i) : m_i \in M)$ be a family of transitive permutation groups indexed by a set M, and let $\mathcal{M}_1 = (M, \sqsubseteq_1)$ and $\mathcal{M}_2 = (M, \sqsubseteq_2)$ be two posets based on M. Then*

(a) *the intersection of the generalised wreath products of the groups over $\mathcal{M}_1$ and $\mathcal{M}_2$ is the generalised wreath product over the intersection of $\mathcal{M}_1$ and $\mathcal{M}_2$;*
(b) *if $\mathcal{M}_1$ is included in $\mathcal{M}_2$, then the generalised wreath product over $\mathcal{M}_1$ is a subgroup of the generalised wreath product over $\mathcal{M}_2$.*

Proof. (a) We first introduce some notation. Let $\mathcal{M}_3 = (M, \sqsubseteq_3)$ be the intersection of the two given posets. For $t = 1, 2, 3$, and $m_i \in M$, let $A_t(i)$ denote the ancestral set in the poset $(M, \sqsubseteq_t)$ corresponding to $m_i \in M$: thus $A_t(i) = \{m_j : m_i \sqsubset_t m_j\}$. Let $\Omega^{t,i}$ be the product of the sets Ω_j for $j \in A_t(i)$.

We have permutation groups $G(i)$, acting on sets Ω_i, associated with the points $m_i \in M$. Our products will act on the set Ω, the Cartesian product of the sets Ω_i for $m_i \in M$.

As we have seen, the generalised wreath product over $\mathcal{M}_i$ is a product of components, where the ith component $F_t(i)$ consists of all functions from $\Omega^{t,i}$ to G_i. Since these functions have different domains, we cannot directly compare them. So we extend the functions in $F_t(i)$ so that their domain is the whole of Ω, with the proviso that they do not depend on coordinates outside $\Omega^{t,i}$.

Now we have

$$F_1(i) \cap F_2(i) = F_3(i).$$

For functions in this intersection do not depend on coordinates outside $\Omega^{1,i}$ or on coordinates outside $\Omega^{2,i}$, and so do not depend on coordinates outside $\Omega^{1,i} \cap \Omega^{2,i}$. But, from the definition of the intersection of posets, we have

$$\Omega^{1,i} \cap \Omega^{2,i} = \Omega^{3,i},$$

so $F_1(i) \cap F_2(i)$ is identified with the set of functions from $\Omega^{3,i}$ to $G(i)$, and the result follows.

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Taking the product over all i shows (a).

(b) If $(M, \sqsubseteq_1)$ is included in $(M, \sqsubseteq_2)$, then the intersection of these two posets is just the first, and so the same relation holds for the generalised wreath products, whence the first is a subgroup of the second. □

A *linear extension* of a poset M is a total order which includes the poset. It is a standard result that a poset is the intersection of all its linear extensions. (If i is below j in the poset, then i is below j in every linear extension. Conversely, if i and j are incomparable, there is a linear extension in which i is below j, and one in which j is below i.)

If G_i is a permutation group on Ω_i for $i = 1, 2, \ldots, N$, then the *iterated wreath product* of these groups is

$$(\cdots(G_1 \wr G_2) \wr \cdots \wr G_N).$$

Thus, it is the generalised wreath product of the groups over the standard linear order on $\{1, 2, \ldots, N\}$. (In fact the brackets are not necessary since the wreath product is associative.)

Corollary 4.25. *A generalised wreath product of a family of groups over a poset $(M, \sqsubseteq)$ is equal to the intersection of the iterated wreath products over all the linear extensions of $(M, \sqsubseteq)$.*

This is immediate from Theorem 4.24 and the comments before the corollary.

5. Miscellanea

5.1. Computing questions

As we did for pre-primitivity in [1], it would be good to go through the list of small transitive groups to see how many have the OB property. Here are some thoughts.

A permutation group G on Ω is 2-*closed* if every permutation which preserves every G-orbit on 2-sets belongs to G. The 2-*closure* is the smallest 2-closed group containing G, and consists of all permutations of Ω which preserve all G-orbits on Ω^2.

Proposition 5.1. *A transitive permutation group has the OB property if and only if its* 2*-closure does.*

For the group and its 2-closure preserve the same binary relations, and in particular the same equivalence relations.

So we can simplify the computation by first filtering out the 2-closed groups and testing these. The computer algebra system GAP [20] has a `TwoClosure` function.

Also, GAP has a function `AllBlocks`. Using this we can compute representatives of the blocks of imprimitivity and test the permuting property. We find, for example, that only one of the transitive groups of degree 8 (the dihedral group acting regularly) fails the OB property.

Table 1 is a table corresponding to the one in [1]. This gives the numbers of transitive groups of degree n and the numbers with the OB and PP properties (where PP is pre-primitivity). In the cases where OB holds we should determine which ones give rise to isomorphic orthogonal block structures.

Here is another approach. Taking both approaches would be a useful check on the correctness of the computations. This uses the fact that an orthogonal block structure gives rise to an association scheme.

Hanaki and Miyamoto [21] have a web page listing the association schemes on small numbers of points. (By 'association scheme' they mean a homogeneous coherent configuration, which is more general than the definition in [5].) Now we should check which association schemes come from orthogonal block structures, and which of these have transitive automorphism groups.

In fact, there is a GAP package [10] by Bamberg, Hanaki and Lansdown which can be used to check isomorphism. Using this package, we could add a column to the above table giving the number of different association schemes which result (and identifying them in the Hanaki–Miyamoto tables).

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Table 1. *Numbers of transitive, OB, and pre-primitive groups.*

n	Trans	OB	PP
10	45	44	42
11	8	8	8
12	301	285	276
13	9	9	9
14	63	62	59
15	104	104	102
16	1954	1886	1833
17	10	10	10
18	983	922	900
19	8	8	8
20	1117	1100	1019

5.2. Some problems

1. Under what conditions is the generalised wreath product of OB groups OB?

2. In [7], the *diagonal group* $D(G, n)$ is defined for any group G and positive integer n, and the conditions for this group to be primitive are determined. For which G and n does $D(G, n)$ have the OB property? The PB property?

3. In [1], the set of natural numbers n for which every transitive group of degree n is pre-primitive was considered. We can ask the analogous question for the OB property. As we saw, there are examples of products of two primes which are in the second set but not the first, such as 15.

Conjecture
If p and q are primes with $p > q$ and $q \nmid p - 1$, then every transitive group of degree pq has the OB property.

As well as 15, this is true for degrees 33 and 35.

4. In [5] it is explained how, given an orthogonal block structure on Ω, the vector space $\mathbb{R}^\Omega$ can be decomposed into pairwise orthogonal subspaces (called *strata* in the statistical literature). If the group G has the OB property, it preserves the subspaces in this decomposition. When does it happen that some or all of the subspaces are irreducible as G-modules?

More generally, what information does the permutation character give about groups with the OB property?

5. A topic worth considering is the extensions of the groups considered in this paper by groups of lattice automorphisms, as suggested at the end of Section 3.4.

6. It would be interesting to know more about transitive groups which do not have the OB property. How common are they? Are similar techniques useful in their study?

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