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The representation of analytic multivalued functions by compact operators

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1. *Introduction*

In this paper we consider the problem of characterizing the variation of the spectrum of a holomorphic family of compact operators $f: G \to KB(X)$, where G is an open subset of $\mathbb{C}$ and X is a Banach space. The natural conjecture, which the author first heard in a lecture by Professor B. Aupetit, is that these spectra are characterized as those analytic multivalued functions which have as values null sequences. This is obviously a necessary condition, and we prove that this is also sufficient. It will be convenient to use the notation $K(\mathbb{C})$ for the set of compact non-empty subsets of the plane and $K_0(\mathbb{C})$ for the subset of $K(\mathbb{C})$ consisting of null sequences.

First we shall say a little about analytic multivalued functions. It is a folklore result in Banach-algebra theory that the spectra of a continuous family of operators vary upper semi-continuously. By this we mean that the compact set-valued function, $\mathrm{Sp}_A f(x)$, where f is a continuous function taking values in a Banach algebra A, is such that

$$\{x : \mathrm{Sp}_A f(x) \subset G\}$$

is open for all open subsets G of the complex plane. This follows easily from the fact that the invertible elements of a Banach algebra form an open set. In [7] Vesentini showed that $\log \rho(f(z))$, and hence $\rho(f(z))$, the spectral radius of a Banach-algebra valued holomorphic function, are subharmonic. This was one of the first uses of the holomorphic structure of the set of invertible elements. This theorem was the basis of much of the initial work in the area which has become the theory of analytic multivalued functions. In particular Professor Aupetit's book [1] is based mainly on this result. But this is inevitable as, by definition, analytic multivalued functions (a.m.v. functions) form the smallest class of upper semi-continuous set-valued functions which is closed under composition with analytic functions and for which Vesentini's Theorem holds. For an introduction to the elementary properties of a.m.v. functions one should consult [2]. The spectra of holomorphic families of Banach-algebra valued functions certainly satisfy these conditions. In 1981 Słodkowski proved that all analytic multivalued functions arise in this way.

THEOREM 1·1 (Słodkowski [4]). *Let* $K: G \to K(\mathbb{C})$ *be an analytic multivalued function defined on the open subset* G *of* $\mathbb{C}$, *and suppose that* $\sup_{z \in G} \rho(K(z)) < \infty$, *where* ρ *is the radius. Then there is an analytic operator-valued function* $T: G \to B(H)$, *such that* $K(z) = \mathrm{Sp}\, T(z)\, (z \in G)$, *where* $B(H)$ *is the Banach algebra of bounded operators on the separable Hilbert space* H.

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The proof of this result uses the theory of uniform algebras to produce local solutions and then these are 'glued together' using cohomology theory. The solution to the problem we consider here is similar, but as we demand that our solution be of a particular form, we must first solve the cohomology problem and then we can explicitly write out the solution. Before proceeding to the general problem of compact operators, we first look at the case of finite spectrum.

The structure of finite a.m.v. functions was first given in [2], although it was not until [4] that they were called a.m.v. functions. We give the proof of this structure theorem here for the sake of completeness, together with the simple corollary that this structure implies that all such a.m.v. functions are the spectrum of an analytic family of matrices of the obvious minimal size. This special case indicates the form of the solution for the general case, namely, a family of companion matrices. The idea is not entirely new, as such families were used in [2], even in an infinite form, but the method by which we produce compact operators of this form is new.

THEOREM 1·2 (Aupetit [1, 2]). *Let $K: G \to K(\mathbb{C})$ be an a.m.v. function, where G is a domain in $\mathbb{C}$. Suppose that $\{z \in G: \#K(z) < \infty\}$ is non-polar. Then there exists an $n \in \mathbb{N}$ such that $\#K(z)$, the number of points in $K(z)$, is less than n for all z. Moreover, there is a function F on $G \times \mathbb{C}$ of the form*

$$F(z, \lambda) = \sum_{k=0}^{n} a_k(z)\, \lambda^{n-k}, \tag{1}$$

where $a_0, \ldots, a_n$ are holomorphic functions on G and $a_0 \equiv 1$, such that

$$K(z) = \{\lambda : F(z, \lambda) = 0\}.$$

Proof. Firstly we must show that such an n exists by forming a subharmonic function of the a.m.v. function using $\delta_m(K)$, the mth diameter, which is given by the formula

$$\delta_m(K) = \sup_{\lambda_1, \ldots, \lambda_{m+1} \in K} \left(\prod_{1 \leqslant i < j \leqslant m+1} |\lambda_i - \lambda_j| \right)^{2/m(m+1)}$$

We consider the sets

$$\{z : \#K(z) \leqslant m\} = \{z : \log \delta_m(K(z)) = -\infty\},$$

for $m \in \mathbb{N}$. These sets are either polar or the whole of G, as they are the set where a subharmonic function is $-\infty$. Hence we either have the desired conclusion, that there is an n such that $\#K(z) \leqslant n$ for all z, or we have the non-polar set of the Theorem contained in a countable union of polar sets. This is impossible as such sets are polar! The contradiction gives us the first result. We next consider the continuous function

$$z \mapsto \prod_{\substack{i \neq j \\ \lambda_i, \lambda_j \in K(z)}} (\lambda_i - \lambda_j),$$

defined on G. Since the λ_i vary holomorphically when they are distinct, this function is holomorphic when non-zero. It follows from a theorem of Radó [5] that it is actually holomorphic and so its roots are discrete in G, i.e., the λ_i are distinct at all but a discrete set S. We now form the function

$$F(z, \lambda) = \prod_{\lambda_i \in K(z)} (\lambda - \lambda_i).$$

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This is a holomorphic function on $(G\backslash S)\times\mathbb{C}$ and is a monic polynomial in λ. The coefficients of the λ^k in this polynomial are continuous, and holomorphic except possibly on S, so these points must be removable singularities of the coefficients and so F is holomorphic in the domain. It remains only to check that the zero set is the graph of the a.m.v. function, but this is obvious from the definition of F. ∎

COROLLARY 1·3. *Let $K: G\to K(\mathbb{C})$ be a finite a.m.v. function with $\#K(z)\leqslant n$. Then there is a holomorphic function $f: G\to M_n(\mathbb{C})$ such that $K(z)=\mathrm{Sp}f(z)$.*

Proof. By Theorem 1·2, there is a function F whose zeros are the values of the a.m.v. function. Moreover this function is a monic polynomial of degree at most n in λ. Using this polynomial we define f to be the companion matrix at each point:

$$f(z)=\begin{pmatrix} -a_1(z) & -a_2(z) & \dots & -a_{n-1}(z) & -a_n(z) \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

The $a_k(z)$ are the same as those in (1) of Theorem 1·2. It is well known that the eigenvalues of this are the roots of the monic polynomial

$$m(\lambda)=\lambda^n+a_1(z)\lambda^{n-1}+\dots+a_{n-1}(z)\lambda+a_n(z),$$

and so this matrix has the required spectrum. ∎

2. *The compact operator case*

In this section we solve the problem of deciding whether an a.m.v. function, $K(z)$, which has values which consist of null sequences, can be represented on Hilbert space by a holomorphic family of compact operators. The solution to this problem uses a form of companion matrix, as above, where these matrices give a representation for finite a.m.v. functions. The problems which arise when one attempts to generalize this method are twofold. Firstly, the obvious infinite version of a companion matrix would have a shift operator on the subdiagonal, which would destroy any hopes of producing a compact operator. The companion matrix is obtained as a perturbation of this unilateral shift by a holomorphic family of rank 1 operators. In fact the spectrum of such an operator is very close to what we want, being $K(z)\cup\bar{B}(0;1)$, the value of the a.m.v. function union the closed unit disc. Secondly there is the problem of finding some analogue of the characteristic polynomial, which will be holomorphic in z. If the 'eigenvalues' were summable one might try the same trick as for the finite a.m.v. functions, but unfortunately the set of points at which repeated 'eigenvalues' occur could now be dense and in general the infinite product $\prod_{i\neq j}(z_i-z_j)$ need no longer converge. Fortunately, we can use the methods of cohomology theory to obtain a suitable function by a rather more indirect technique.

The canonical form used for the representation is exactly the same as that of the previous section for finite a.m.v. functions. However, the unilateral shift is replaced by a shift weighted in such a way as to make it have zero spectral radius and be compact. The perturbation of this Volterra operator by a rank 1 operator has a spectrum consisting entirely of eigenvalues. The eigenvalues of a compact operator in the infinite version are easy to calculate as we see from the next lemma.

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LEMMA 2·1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of complex numbers, and define $f(z)$ by*

$$f(z) = 1 - \sum_{n=1}^{\infty} a_n/z^n.$$

Suppose that $(s_n)_{n=1}^{\infty}$ is a decreasing sequence of strictly positive reals which tend to 0 and such that $(a_n/s_1 \dots s_n)_{n=2}^{\infty}$ belongs to l^2. Define $T_f: l^2 \to l^2$ by

$$T_f = \begin{pmatrix} a_1 & a_2/s_1 & \dots & a_n/(s_1 \dots s_{n-1}) & \dots \\ s_1 & & & & \\ & s_2 & & & \\ & & \ddots & & \\ & & & s_n & \\ & & & & \ddots \end{pmatrix}.$$

Then the spectrum of T_f consists of the zeros of the analytic function $f(z)$ together with zero.

Proof. For T_f to define a bounded operator the first row taken by itself must define a rank 1 operator, which it does as $\{a_n/s_1 \dots s_n\}_{n=1}^{\infty}$ is in l^2. The subdiagonal part by itself defines a Volterra operator σ, that is a compact operator with $\{0\}$ as its spectrum. It is compact because it is the uniform limit of its truncations, which are finite rank. It has zero spectral radius because

$$\|\sigma^n\|^{1/n} = (s_1 \dots s_n)^{1/n} \tag{2}$$

and the right-hand side of (2) tends to zero because the s_n do. Since T_f is a rank 1 perturbation of a compact operator, it is also compact. The spectrum of a compact operator on infinite-dimensional Hilbert space consists of its eigenvalues together with zero. If T has an eigenvector $\xi = \{\xi_i\}_{i=1}^{\infty}$ with non-zero eigenvalue λ we deduce the following:

$$T_f \xi = \lambda \xi$$

so that

$$\sum_{n=1}^{\infty} a_n \xi_n/(s_1 \dots s_{n-1}) = \lambda \xi_1,$$

and

$$s_n \xi_n = \lambda \xi_{n+1} \quad (n \geqslant 1).$$

Thus, assuming that $\xi_1 = 1$, we deduce that

$$s_1 \dots s_n = \lambda^n \xi_{n+1}.$$

This shows that if $\xi = \{\xi_i\}_{i=1}^{\infty}$ is in l^2, then to be an eigenvector it need only satisfy the above equations. But this is just to say that λ is a root of the analytic function.

Conversely, given a root λ of f, the vector it defines using the above equations is in l^2, and so λ is an eigenvalue of T_f. ∎

To use the theorem for a given function we need to be able to choose the constants s_n to ensure that the operator is compact. This is possible for a single function by taking

$$s_n = 2 \sup_{m \geqslant n} |a_m|^{1/m}.$$

Now we must produce bounds for an analytic family of functions. But first we show how the graph of a $K_0(\mathbb{C})$-a.m.v. function determines such an analytic family of

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functions on $G \times (\mathbb{C}_\infty \backslash \{0\})$, where $\mathbb{C}_\infty$ is the Riemann sphere. It is here that the cohomology theory is used. The $K_0(\mathbb{C})$-a.m.v. function defines a Cousin II distribution on $G \times (\mathbb{C}_\infty \backslash \{0\})$. This set has trivial $H^2(X, \mathbb{Z})$ so the distribution determines a global meromorphic function and since the data is positive, i.e., defines only zeros, the solution is a holomorphic function. For details of the relevant results from several complex variables see [3].

THEOREM 2·2. *Let $K : G \to K_0(\mathbb{C})$ be an a.m.v. function. Then the graph of K determines a positive Cousin II distribution on $G \times (\mathbb{C}_\infty \backslash \{0\})$, which is zero precisely on the graph of K.*

Proof. We consider two kinds of points in $G \times (\mathbb{C}_\infty \backslash \{0\})$. If the point is not on the graph, then it has a neighbourhood which avoids the graph because the graph is closed: on this open set we take our function to be 1. If the point is on the graph, then we take a neighbourhood, U, of the coordinate in $\mathbb{C}_\infty \backslash \{0\}$ which contains only finitely many points of $K(z)$. We then consider the finite a.m.v. function which is defined locally by $U \cap K(z)$ in a neighbourhood of the other coordinate. On this set we can form the finite canonical product of its values to obtain a function which defines the graph of this finite a.m.v. function, and so also the graph of the original a.m.v. function in a product neighbourhood of the original point. To ensure these functions form Cousin II data it remains to check that the quotients are holomorphic on overlaps.

The quotient is just

$$\frac{\prod_{\lambda_i \in U \cap K(z)} (\lambda - \lambda_i)}{\prod_{\lambda_i \in V \cap K(z)} (\lambda - \lambda_i)}.$$

This is holomorphic off the graph, and continuous in the domain of definition and so by the first Riemann removable singularity theorem (see [3]) it is holomorphic on the whole domain. It is continuous because on the first region, the a.m.v. function never touches ∂U and on the second it never touches ∂V, so on the intersection it never touches $\partial(U \cap V)$. This allows us to form the canonical product of the values it takes in $U \cap V$, which is a holomorphic function which can be cancelled top and bottom. The resulting function has a denominator which is never zero in the domain and so is continuous. ∎

Now we have to see that we can solve all Cousin II problems on $X = G \times (\mathbb{C}_\infty \backslash \{0\})$. First we note that as a product of plane domains it is a domain of holomorphy, so has $H^1(X, \mathcal{O}) = 0$. So to solve all Cousin II problems all we need is that $H^2(X, \mathbb{Z}) = 0$. But this is now just ordinary topological cohomology which is homotopy invariant and so $H^2(X, \mathbb{Z}) = H^2(G, \mathbb{Z})$. Now there are a number of ways of seeing that this is zero. One could say for example that all plane domains are domains of holomorphy on which the Weierstrass theorem holds and so $H^2(G, \mathbb{Z}) = 0$ for all plane domains. This proves the following lemma.

LEMMA 2·3. *Let $K : G \to K_0(\mathbb{C})$ be an a.m.v. function. Then there is a holomorphic function on $G \times (\mathbb{C}_\infty \backslash \{0\})$ whose zero set defines the graph of the a.m.v. function.*

Proof. As we have seen, the graph defines a Cousin II distribution and by the above we can find a holomorphic function which solves it. This function has the graph as its zero set. ∎

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To deduce the representation as a family of compact operators it remains only to produce a uniform bound of the right size on the coefficients of the analytic family of functions. This is a simple consequence of the Cauchy integral formula. First note that given any solution f of the Cousin problem $f(z, \infty) \neq 0$ for any $z \in G$ and so by considering $f(z, \lambda)/f(z, \infty)$, which also solves the Cousin problem, we may assume that $f(z, \infty) = 1$ for $z \in G$.

LEMMA 2·4. *Let $f: G \times (\mathbb{C}_\infty \backslash \{0\}) \to \mathbb{C}$ define the graph of an a.m.v. function $K: G \to K_0(\mathbb{C})$ and suppose that $f(z, \infty) = 1$. Then the coefficients $a_n(z)$ of the Laurent series*

$$f(z, \lambda) = 1 - \sum_{n=1}^{\infty} a_n(z)/\lambda^n,$$

satisfy

$$\limsup_{n \to \infty} |a_n(z)|^{1/n} = 0.$$

This limit is uniform on compact sets. Hence $|a_n(z)|^{1/n}$ is uniformly dominated on each compact set by a null sequence.

Proof. By Cauchy's integral formula applied to $|\lambda| = r$ for each $r > 0$ we have

$$|a_n(z)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} |f(z, \lambda)\, \lambda^n|\, d\theta \leqslant M_f(z, r)\, r^n$$

where $M_f(z, r) = \sup_{|\lambda| = r} |f(z, \lambda)|$. Now set $M(r) = \sup_{z \in U} M_f(z, r) < \infty$ for any set $U \subset\subset G$. Thus

$$\limsup_{n \to \infty} |a_n(z)|^{1/n} \leqslant \limsup_{n \to \infty} M(r)^{1/n} r = r,$$

which is just to say that $|a_n(z)|^{1/n}$ tends to zero uniformly on the arbitrary relatively compact set U, i.e. it is dominated on compact sets by a null sequence. ∎

THEOREM 2·5. *Let $K: G \to K_0(\mathbb{C})$ be an a.m.v. function. Then there is a holomorphic family of compact operators such that $K(z) = \mathrm{Sp}\,(f(z))$ for each $z \in G$.*

Proof. To begin with we must deal with a dichotomy. It may be that the value of the a.m.v. function has finite cardinality on a non-polar set. In this case Theorem 1·2 and Corollary 1·3 provide the solution in the form of a holomorphic family of finite rank operators. Otherwise the set where the a.m.v. function has finite cardinality is polar. In this case the complement is dense in G and for these points, and hence all of G, 0 is in $K(z)$. Now we are in a position to apply Lemma 2·3 to obtain a holomorphic function which determines the graph of the a.m.v. function. We may assume this function is 1 at infinity. Now we must choose a null sequence with which to apply Theorem 2·1. We know that on any compact subset of G the sequence $|a_n(z)|^{1/n}$ is uniformly bounded by a null sequence by Lemma 2·4. We could now apply Theorem 2·1 and have the result for the compact set. However it is possible to choose a null sequence which works for a countable number of compact sets, and so by taking a compact exhaustion of G it works for the whole of G. So we fix a compact exhaustion, K_m, of G. Then by Lemma 2·4 there are null sequences $s_n^{(m)}$ for each set K_m so that

$$|a_n(z)| \leqslant 2^{-n} s_1^{(m)} \ldots s_n^{(m)} \quad \text{for } z \in K_m.$$

Now we pick another null sequence which dominates the tail of each of these null

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sequences. There is an n_i so that for $n \geqslant n_i$ and $r \leqslant i$ we have $s_n^{(r)} \leqslant 1/i$ as each of the sequences tends to zero. Now we set $t_n = 1/i$ for $n_i \leqslant n < n_{i+1}$ and $t_n = 1$ for $n < n_1$. It follows that

$$|a_n(z)| \leqslant 2^{-n} C_m t_1^{(m)} \dots t_n^{(m)} \quad \text{for } z \in K_m,$$

where the constant C_m depends only on m. We now apply Theorem 2·1 and obtain a holomorphic family of compact operators with $K(z) = \mathrm{Sp} f(z)$ on G. ∎

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