SOME SELF-RECIPROCAL FUNCTIONS AND KERNELS

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Communicated by H. P. F. Swinnerton-Dyer

Received 18 March 1958

1. The object of this note is to deduce some self-reciprocal functions and kernels for Hankel transforms. We denote by R_μ the class of functions which are self-reciprocal for Hankel transforms of order μ; that is, those that satisfy

$$f(x) = \int_0^\infty J_\mu(xy)\, f(y)\, (xy)^{\frac{1}{2}}\, dy, \tag{1·1}$$

where $J_\mu(x)$ is a Bessel function of order μ. In particular we write R_s for $R_{\frac{1}{2}}$ and R_c for $R_{-\frac{1}{2}}$.

2. Meijer (7) has shown that

$$\int_0^\infty K_p(\tfrac{1}{2}v^2)\, {}_0F_3(a, b, c;\, xv^4)\, v^{2\lambda+1}\, dv$$
$$= 2^{2\lambda-1}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda-\tfrac{1}{2}p)\ \Gamma(\tfrac{1}{2}+\tfrac{1}{2}\lambda+\tfrac{1}{2}p)\, {}_2F_3\left(\begin{matrix}\tfrac{1}{2}(1+\lambda-p),\ \tfrac{1}{2}(1+\lambda+p);\\ a,\ b,\ c;\end{matrix}\ 16x\right) \tag{2·1}$$

for all values of x, where $|\Re p| - \Re(\lambda) < 1$. Putting

$$p = \tfrac{1}{2}, \quad a = 1+\mu, \quad b = 1+\nu, \quad c = 1+\mu+\nu, \quad \lambda = \tfrac{1}{2}+\mu+\nu$$

and writing $-\frac{1}{16}x^2$ for x we obtain

$$\int_0^\infty K_{\frac{1}{2}}(\tfrac{1}{2}v^2)\, {}_0F_3(1+\mu, 1+\nu, 1+\mu+\nu;\, -\tfrac{1}{16}x^2v^4)\, v^{2(1+\mu+\nu)}\, dv$$
$$= 2^{2(\mu+\nu)}\Gamma(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\, \Gamma(1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu)\, {}_2F_3\left(\begin{matrix}\tfrac{1}{2}(1+\mu+\nu),\ \tfrac{1}{2}(2+\mu+\nu);\\ 1+\mu, 1+\nu, 1+\mu+\nu;\end{matrix}\ -x^2\right), \tag{2·2}$$

where $\Re(\mu+\nu) > -1$. The author has shown (10) that

$$x^{-\frac{1}{2}}\, {}_2F_3(\tfrac{1}{2}+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu;\ 1+\mu, 1+\nu, 1+\mu+\nu;\ -x^2)$$

is in R_1, provided both $\Re\mu > -1$ and $\Re\nu > -1$. It follows from this and (2·2) that if all three conditions are satisfied then the integral

$$\int_0^\infty v^{2\mu+2\nu+3}\, K_{\frac{1}{2}}(\tfrac{1}{2}v^2)\, {}_0F_3(1+\mu, 1+\nu, 1+\mu+\nu;\, -\tfrac{1}{16}x^2v^4)\, \frac{dv}{vx^{\frac{1}{2}}}$$

is a function in R_1.

3. Sastry (9) has shown that

$$(-1)^{n-1}\, e^{-x}\, k_{2n}(x) = 2x\, {}_1F_1(n+1;\, 2;\, -2x),$$

https://doi.org/10.1017/S0305004100033697 Published online by Cambridge University Press

where $k_{2n}(x)$ is a Bateman polynomial of order $2n$. Writing $\frac{1}{4}x^2$ for x and $n-1$ for n, we deduce that

$$(-1)^n x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} k_{2n-2}(\tfrac{1}{4}x^2) = \tfrac{1}{2}x^{n-\frac{1}{2}} {}_1F_1(n; 2; -\tfrac{1}{2}x^2). \tag{3·1}$$

Gupta (4) has shown that the function

$$x^{n-\frac{1}{2}} {}_1F_1(n; \tfrac{1}{2}(n+\alpha+1); -\tfrac{1}{2}x^2)$$

belongs to R_α if $n > 0$. Using (3·1), it follows that

$$x^{n-\frac{1}{2}} e^{-\frac{1}{4}x^2} k_{2n-2}(\tfrac{1}{4}x^2)$$

belongs to R_{3-n}.

4. Macrobert (5) has shown that

$$e^{-nx} = (2\sinh x)^{-n} {}_2F_1(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}; n+1; -\operatorname{cosech}^2 x), \tag{4·1}$$

provided that $n > 0$ and $\sinh x > 1$. Writing $x^2/2n$ for x in this, we have

$$e^{-\frac{1}{2}x^2} = (2\sinh x^2/2n)^{-n} {}_2F_1(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}; n+1; -\operatorname{cosech}^2 x^2/2n). \tag{4·2}$$

Brij Mohan (2) has shown that $e^{-\frac{1}{2}x^2}$ belongs to R_c; hence so also does the right-hand side of (4·2).

5. Macrobert (6) has shown that

$$K_n(\lambda) = \int_0^\infty e^{-\lambda\cosh t} \cosh nt\, dt$$

provided $\Re\lambda > 0$. Brij Mohan (1) has shown that the kernel $K_0(x)$ transforms R_c into R_c. Hence so also does the kernel

$$\int_0^\infty e^{-\lambda\cosh t}\, dt. \tag{5·1}$$

6. Ragab (8) has shown that

$$E(\alpha, \beta: \gamma: z) = \Gamma(\alpha)\,\Gamma(\beta)\, {}_2F_1(\alpha, \beta; \gamma; -z^{-1})/\Gamma(\gamma). \tag{6·1}$$

Writing $1+\frac{1}{2}n$ for α, $\frac{1}{2}+\frac{1}{2}n$ for β, $1+n$ for γ and $\sinh^2 x^2/2n$ for z, we find that

$$\begin{aligned}&(\sinh x^2/2n)^{-n} E(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}: n+1: \sinh^2 x^2/2n)\,\Gamma(n+1)\\ &\quad = \Gamma(1+\tfrac{1}{2}n)\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}n)\,(\sinh x^2/2n)^{-n} {}_2F_1(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}; n+1; -\operatorname{cosech}^2 x^2/2n).\end{aligned}$$

It follows from this and the result of § 4 that

$$(\sinh x^2/2n)^{-n} E(\tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}: n+1: \sinh^2 x^2/2n)$$

belongs to R_c, where E is Macrobert's function.

Writing x/n for x in (4·1) we find that

$$e^{-x} = (2\sinh x/n)^{-n} {}_2F_1(1+\tfrac{1}{2}n, \tfrac{1}{2}+\tfrac{1}{2}n; n+1; -\operatorname{cosech}^2 x/n). \tag{6·2}$$

Brij Mohan (1) has shown that the kernel e^{-x} transforms R_c into R_s and vice versa. It follows that the right-hand side of (6·2) has the same property; and, writing $\sinh^2 x/n$ for z in (6·1), so has

$$(\sinh x/n)^{-n} E(1+\tfrac{1}{2}n, \tfrac{1}{2}+\tfrac{1}{2}n: n+1: \sinh^2 x/n).$$

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7. In this section we obtain some self-reciprocal functions as a consequence of the familiar fact that R_s contains the function

$$x^{-\mu}(x^2-2)^{\frac{1}{2}\mu-\frac{1}{2}}\,Q^{\frac{1}{2}-\mu}_{\frac{1}{2}}(x^2-1),$$

a special case of a function in R_ν given by Erdelyi ((3), chapter VIII). If $z>1$ is real we have

$$\Gamma(n+m+1)\,Q^{-m}_n(z)=\Gamma(n-m+1)\,Q^m_n(z). \tag{7·1}$$

Putting $n=\frac{1}{2}$ and $m=\frac{1}{2}-\mu$, we deduce that

$$x^{-\mu}(x^2-2)^{\frac{1}{2}\mu-\frac{1}{2}}\,Q^{\mu-\frac{1}{2}}_{\frac{1}{2}}(x^2-1)$$

belongs to R_s if $\mu<2$. In particular, $x^{-1}\,Q^{-\frac{1}{2}}_{\frac{1}{2}}(x^2-1)$ belongs to R_s.

8. Again, Macrobert (5) has shown that

$$Q^m_n(z)=\frac{\Gamma(n+m+1)\,\Gamma(\frac{1}{2})\,(z^2-1)^{\frac{1}{2}m}}{2^{n+1}\,\Gamma(n+\frac{3}{2})\,z^{m+n+1}}\;{}_2F_1\left(\begin{matrix}\frac{1}{2}(m+n+2),\,\frac{1}{2}(m+n+1);\\ n+\frac{3}{2}\end{matrix}\;z^{-2}\right). \tag{8·1}$$

Writing $\frac{1}{2}$ for n, $\frac{1}{2}-\mu$ for m and x^2-1 for z, we obtain

$$x^{-\mu}(x^2-2)^{\frac{1}{2}\mu-\frac{1}{2}}\,Q^{\frac{1}{2}-\mu}_{\frac{1}{2}}(x^2-1)=\frac{\Gamma(2-\mu)\,\Gamma(\frac{1}{2})}{2\sqrt{2}}\left(\frac{x^2-1}{x^2}\right)^{\mu-2}x^{-\frac{1}{2}}(x^2-2)^{-\frac{1}{4}}$$
$$\times\,{}_2F_1\left(\begin{matrix}\frac{3}{2}-\frac{1}{2}\mu,\,1-\frac{1}{2}\mu;\\ 2;\end{matrix}\;(x^2-1)^{-2}\right).$$

By the result of § 7, we find that

$$x^{-\frac{1}{2}}(x^2-2)^{-\frac{1}{4}}\left(\frac{x^2-1}{x^2}\right)^{\mu-2}{}_2F_1\left(\begin{matrix}\frac{3}{2}-\frac{1}{2}\mu,\,1-\frac{1}{2}\mu;\\ 2;\end{matrix}\;(x^2-1)^{-2}\right)$$

is in R_s; and in particular, putting $\mu=-1$,

$$x^{\frac{5}{2}}\,(x^2-1)^{-3}\,(x^2-2)^{-\frac{1}{4}}\,{}_1F_0(\tfrac{3}{2};\,(x^2-1)^{-2})$$

belongs to R_s.

Again, if we write $\frac{1}{2}$ for n, $\mu-\frac{1}{2}$ for m and x^2-1 for z in (8·1), we find

$$Q^{\mu-\frac{1}{2}}_{\frac{1}{2}}(x^2-1)=\frac{\Gamma(\mu+1)\,\Gamma(\frac{1}{2})\,x^{\mu-\frac{1}{2}}(x^2-2)^{\frac{1}{2}\mu-\frac{1}{4}}}{2\sqrt{2}\,(x^2-1)^{\mu+1}}\;{}_2F_1\left(\begin{matrix}\frac{1}{2}\mu+1,\,\frac{1}{2}\mu+\frac{1}{2};\\ 2;\end{matrix}\;(x^2-1)^{-2}\right);$$

and it follows from this and the result of § 7 that

$$x^{-\frac{1}{2}}(x^2-1)^{-\mu-1}\,(x^2-2)^{\mu-\frac{3}{4}}\,{}_2F_1(\tfrac{1}{2}\mu+1,\,\tfrac{1}{2}\mu+\tfrac{1}{2};\,2;\,(x^2-1)^{-2})$$

belongs to R_s if $\mu<2$.

9. Macrobert (5) has shown that

$$(\tfrac{1}{2}\pi\,\mathrm{cosech}\,x)^{\frac{1}{2}}\,e^{-nx}=Q^{\frac{1}{2}}_{n-\frac{1}{2}}(\cosh x). \tag{9·1}$$

Putting x/n for x in this and using the result already quoted from (1) we find that the kernel

$$(\sinh x/n)^{\frac{1}{2}}\,Q^{\frac{1}{2}}_{n-\frac{1}{2}}(\cosh x/n)$$

transforms R_s into R_c and vice versa.

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Again, putting $x^2/2n$ for x in (9·1) and using the result of § 4, we find that

$$(\sinh x^2/2n)^{\frac{1}{2}} Q^{\frac{1}{2}}_{n-\frac{1}{2}}(\cosh x^2/2n)$$

is in R_c.

10. Writing $-\frac{1}{2}$ for m, $\nu-\frac{1}{2}$ for n and $\cosh x^2/2n$ for z in (8·1) we obtain

$$(\sinh x^2/2n)^{\frac{1}{2}} Q^{-\frac{1}{2}}_{\nu-\frac{1}{2}}(\cosh x^2/2n) = \frac{\Gamma(\frac{1}{2})\,(\operatorname{sech} x^2/2n)^{\nu}}{\nu \,.\, 2^{\nu+\frac{1}{2}}}\, {}_2F_1\left(\begin{matrix}\frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}\nu;\ \operatorname{sech}^2 x^2/2n \\ \nu+1\end{matrix}\right). \quad (10{\cdot}1)$$

Again, putting $m = \frac{1}{2}$ and $n = \nu-\frac{1}{2}$ in (7·1) we obtain

$$Q^{\frac{1}{2}}_{\nu-\frac{1}{2}}(z) = \nu Q^{\nu}_{-\frac{1}{2}}(z)$$

if $\nu > 0$. From these two equations and the last result of § 9 we obtain the result that

$$(\cosh x^2/2n)^{-\nu}\, {}_2F_1(\tfrac{1}{2}\nu+\tfrac{1}{2},\ \tfrac{1}{2}\nu;\ \nu+1;\operatorname{sech}^2 x^2/2n)$$

belongs to R_c. Again, writing x for $\frac{1}{2}x^2$ in (10·1) and using the second result of § 9, we deduce that the kernel

$$(\cosh x/n)^{-\nu}\, {}_2F_1(\tfrac{1}{2}\nu+\tfrac{1}{2},\ \tfrac{1}{2}\nu;\ \nu+1;\ \operatorname{sech}^2 x/n)$$

transforms R_c into R_s and vice versa.

I thank Dr Brij Mohan for his help and guidance in the preparation of this note.

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https://doi.org/10.1017/S0305004100033697 Published online by Cambridge University Press