Proof:

(1) follows directly from Proposition 3.

(2) If M, N are matrices both with period $n > 2$ then from the Theorem we can write $M = P(rD_1)P^{-1}$ and $N = Q(sD_2)Q^{-1}$, where r, s are non-zero numbers and $D_1 = D^{k_1}$, $D_2 = D^{k_2}$ with

$$D = \begin{pmatrix} \exp\left(\frac{\pi i}{n}\right) & 0 \\ 0 & \exp\left(-\frac{\pi i}{n}\right) \end{pmatrix}$$

and k_1, k_2 are integers each coprime to n.

Hence there is an integer k (coprime to n) such that $kk_2 \equiv k_1 \pmod n$, giving $D_2^k = \pm D_1$. So

$$N^k = s^k Q D_2^k Q^{-1} = \pm s^k Q D_1 Q^{-1} = \pm r^{-1} s^k Q P^{-1} M P Q^{-1} = \pm r^{-1} s^k R M R^{-1}$$

where $R = QP^{-1}$. So $M = \pm r s^{-k} R^{-1} N^k R \sim R^{-1} N^k R$, where k is coprime to n. So we can use one matrix N with period n to generate all others.

For $n \in \{3, 4, 6\}$ we can take $k = \pm 1$, so if M has period 3, 4 or 6 then $M \sim R^{-1} N R$ or $M \sim R^{-1} N^{-1} R$.

It is easy to check that $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ have periods 3, 4 and 6 respectively. So we can use these matrices for N in these cases.

For $N = \begin{pmatrix} 1 & -1 \\ 1 & d \end{pmatrix}$, let $S = \begin{pmatrix} -1 & 0 \\ d-1 & 1 \end{pmatrix}$ then $S^{-1} N S = (1+d) N^{-1}$ so $R^{-1} N^{-1} R = \frac{1}{1+d} (SR)^{-1} N (SR)$. So for $k = 1$, take $A = R$ and for $k = -1$, take $A = SR$ and then in both cases $M \sim A^{-1} N A \sim \operatorname{adj}(A) N A$.

References

1. https://en.wikipedia.org/wiki/Niven%27s_theorem
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105.29 A geometric memory

I fell in love with Euclidean geometry when I was about ten and have loved it ever since. I remember at the age of twelve doing a term examination which contained the following problem.

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Theorem 1: If two chords of a circle of equal length intersect, then the corresponding segments of those chords are equal in length.

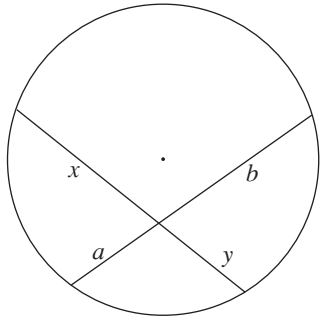


FIGURE 1

My solution was as follows. Let the segments have lengths a, b, x and y as in Figure 1. We are given that $a + b = x + y$ and by the intersecting chords theorem, $xy = ab$, so $4xy = 4ab$.

Then $a^2 + 2ab + b^2 = x^2 + 2xy + y^2$, so

$$a^2 + 2ab + b^2 - 4ab = x^2 + 2xy + y^2 - 4xy$$

i.e.
$$(a - b)^2 = (x - y)^2.$$

Thus $a - b = x - y$ or $a - b = y - x$. Combining this with $a + b = x + y$, we get either $a = x$ and $b = y$, or $a = y$ and $b = x$.

Not bad for a twelve-year-old — I wish I was as clever now as I was then!

But then I got worried; this was a geometry exam and I was using algebra, so I asked the teacher, "Are we allowed to use algebra to solve this problem, sir?"

"No," he smiled, "this is the geometry exam; the algebra exam is tomorrow, but I'm sure you can find a purely geometric proof if you try."

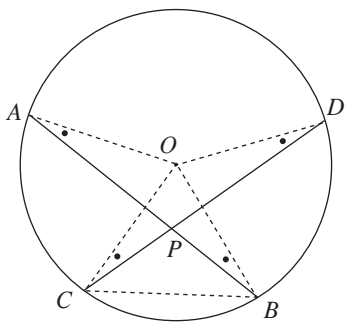


FIGURE 2

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So I had a go at coming up with a purely geometric proof and managed to do so within the allotted time. It went something like this, with the construction lines dotted:

Now $\triangle AOB \equiv \triangle COD$ (SSS) because $|OA| = |OD|$, $|OB| = |OC|$ (radii) and $|AB| = |CD|$ (given). Thus $\angle DCO = \angle ABO$ and $\angle OCB = \angle OBC$ (radii). So their differences $\angle PCB$ and $\angle PBC$ are equal, so $|PC| = |PB|$ and $|PD| = |PA|$.

Recently, trying to relive past glories, I looked at the problem again. This, by the way, is an example of a problem where a synthetic proof is natural and relatively easy, but I would imagine an analytical solution would be horrendous. It struck me that I had made the assumption that the equal chords of the circle intersected *internally*; what would happen if they intersected *externally*? Predictably, the external case also gives rise to equal segments, but I could not find any mention of this case in the literature; but geometry is so rich that I was able to come up with three different proofs.

The first proof is algebraic.

Theorem 2: If two equal chords of a circle intersect externally, then corresponding segments are equal in length.

First Proof (Algebraic):

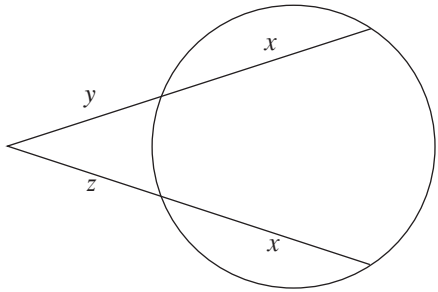


FIGURE 3

By a well-known result $y(y+x) = z(z+x)$ which gives $(y+z)(y-z) = (y-z)(-x)$. If $y \neq z$, we may cancel $(y - z)$ from both sides of this equation, and we get $y + z = -x$, which is negative, a contradiction. Thus, $y = z$ and $y + x = z + x$.

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Second Proof (Geometric):

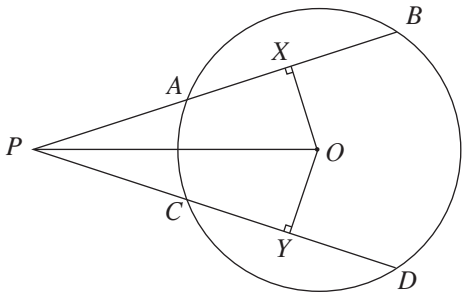


FIGURE 4

Let $|AB| = |CD|$ and let AB and CD intersect externally at P, and let O be the centre of the circle. Let OX and OY be the perpendiculars from O to AB and CD, respectively. By a well-known result $|OX| = |OY|$ so $\Delta POX \equiv \Delta POY$ (hypotenuse and side) so $|PX| = |PY|$. Thus $|PA| + |AX| = |PC| + |CY|$ or $|PA| + \frac{1}{2}|AB| = |PC| + \frac{1}{2}|CD|$. But $|AB| = |CD|$, so $|PA| = |PC|$ and $|PB| = |PD|$.

Third Proof (Geometric):

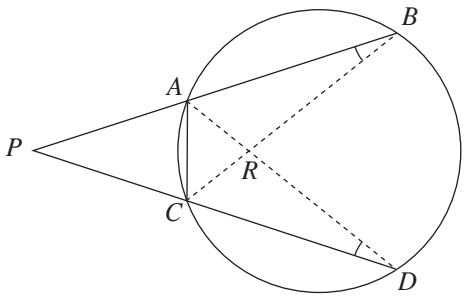


FIGURE 5

$\Delta ARB \equiv \Delta CRD$ (SAA) because $|AB| = |CD|$, $\angle ABR = \angle CDR$ (angles on same chord) and $\angle BRA = \angle CRD$. Thus $|AR| = |CR|$, so $\angle RAC = \angle RCA$; combine this with $\angle DAB = \angle BCD$ (same chord) and we get $\angle PAC = \angle PCA$. Finally, $|PA| = |PC|$ and $|PB| = |PD|$.

The following result can now easily be deduced:

Theorem 3: If $ABCD$ is a cyclic quadrilateral with $|AB| = |CD|$, then AC is parallel to BD, and the figure is a trapezium with diagonals of equal length.

For completeness, we should also consider the case where the chords of equal length intersect on the circumference of the circle. No great surprises here, but we have:

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Theorem 4: If two chords of a circle of equal length intersect on the circumference of the circle, then the line joining the point of intersection to the centre of the circle bisects the angle between the chords.

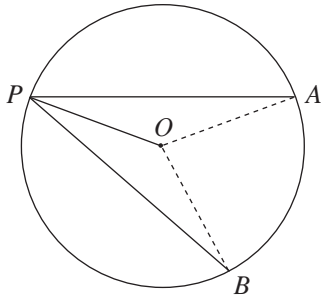


FIGURE 6

Proof: $\Delta POA \equiv \Delta POB$ (SSS), so $\angle APO = \angle BPO$.

Theorem 5: If two equal chords of a circle intersect on the circumference of the circle, then the bisector of the angle between them passes through the centre of the circle.

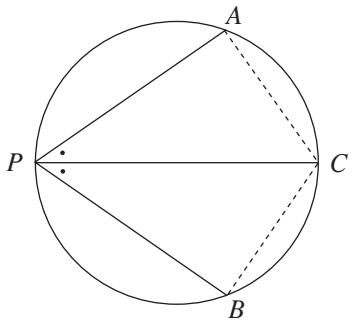


FIGURE 7

Proof: Clearly, $\Delta PAC \equiv \Delta PBC$ (SAS), so $\angle PAC = \angle PBC$. But $\angle PAC + \angle PBC = 180°$, so $\angle PAC = 90° = \angle PBC$, so PC is a diameter of the circle.

Of course the analogues of Theorems 4 and 5 hold in the internal and external cases also, and in traditional fashion we leave these as easy exercises for the reader.

Incidentally, all three triangle congruence theorems are used in the above proofs. We note that synthetic geometric proofs seem to require a good deal more work and therefore are probably more satisfying, whereas analytical geometric proofs and algebraic proofs often let symbol manipulation and calculation achieve the result.

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So it appears that I did not deserve full marks in my school geometry exam, but they cannot take them back from me now. Aristotle asserted that a good education is the best insurance for old age, but Euclidean geometry is quite enough for me!

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105.30 A Cinderella theorem in circle geometry

The 'circle theorems' have an established place in the English school geometry curriculum, although, as their converses are not in the syllabus, it turns out that there is not much one can do with them apart from chasing angles around a diagram. However, there is a neglected circle theorem which is rarely mentioned and of which, I suspect, many teachers are not even aware. It turns out, that this 'poor relation' of the circle theorems is really quite useful. I was reminded of it when I processed Des MacHale's paper [1]. I will begin by explaining what the theorem says and then prove it.

In the circle AB and CD are two chords and X and Y are two points on the circumference. Then $AB = CD$ if, and only if, $\angle BXA = \angle DYC$.

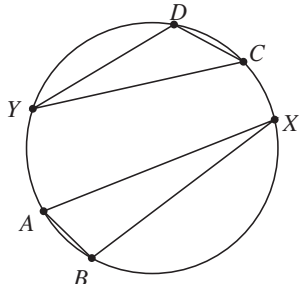


FIGURE 1

At first sight this looks very much like 'angles in the same segment' but the important difference is that the chords are distinct; they can even overlap. In plain language, what is being said is that two chords are equal in length if, and only if, they subtend the same angles at the circumference.

Of course, we really ought to be more careful in the exposition of this result. Any chord which is not a diameter actually subtends two angles at the circumference – one in the major segment which is acute and one in the minor segment which is obtuse. So obviously we must be careful to select the correct pair of angles when working with this theorem. For convenience, the diagrams will denote the acute case.

Usually this theorem is stated in the form 'equal chords subtend equal angles', which makes the point that 'angle size' and 'chord length' are intimately related. One can even bring the concept of arc length into play,

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