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COMPUTING LIMITING LOCAL POWERS AND POWER ENVELOPES OF PANEL MA UNIT ROOT TESTS AND STATIONARITY TESTS

KATSUTO TANAKA
Gakushuin University

The present article discusses panel unit root tests for two classes of models. One is the moving average (MA) model and the other is the error components model. We conduct score type tests for these models, allowing for various types of regressors, and examine the cross-sectional effect explicitly by presenting an efficient way of computing limiting local powers. It is found that the existence of common regressors does not affect the asymptotic behavior of tests, although heterogeneous regressors do. We also derive the limiting power envelopes for some simple panel models, which shows that the score type panel tests are asymptotically efficient, unlike the time series case.

1. INTRODUCTION

We deal with the panel MA model and the panel error components model to conduct unit root tests. The former is useful for checking overdifferencing in the time series direction, whereas the latter can be used for checking stationarity of levels. For these models we consider four types of regressors, (1) a common intercept and trend, (2) heterogeneous intercepts and a common trend, (3) a common intercept and heterogeneous trends, (4) heterogeneous intercepts and trends.

For panel MA models we conduct score type tests which are locally best invariant (LBI) or LBI and unbiased (LBIU). These were earlier suggested in Tanaka (1990, 1996) in the time series context. The panel MA models were also initially discussed in Tanaka (2017). We consider two cases for the initial value that generates the noninvertible MA error. One is the conditional case that restricts the initial value at 0 and the other is the stationary case that produces a stationary MA error. It is shown that, under a suitable group of transformations, the score test in the conditional case is LBI, whereas that in the stationary case is LBIU. The two cases yield different asymptotic results unlike panel autoregressive (AR) models, where the initial value does not affect the asymptotics as far as it is of constant order.

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For panel error components models we conduct LBI tests, allowing for four types of regressors in the same way as for panel MA models. The test was earlier suggested in Hadri (2000) and Shin and Snell (2006), where only the limiting null distributions of the test statistics were computed. We derive limiting local powers of LBI tests on the basis of the results obtained in the time series context in Nabeya and Tanaka (1988) and Tanaka (1996).

For the above models we present an efficient way of computing limiting local powers of LBI and LBIU tests using the sequential limit as the time series dimension T goes to ∞ with the cross-section dimension N fixed, and then $N \to \infty$. The joint limit is desirable under the situation where both N and T are jointly large. The joint limit theory was initially developed by Phillips and Moon (1999) for panel AR models, whereas Shin and Snell (2006) used the joint limit for panel error components models. A set of sufficient conditions for the joint limit to tend to a weak limit is that the error process has higher order moments and that $N/T \to 0$ as $N \to \infty$ and $T \to \infty$. The joint limit justifies the sequential limit result obtained in this article, although the validity of our results needs to be closely investigated because we consider more general models than Shin and Snell (2006) and deal with panel models under the alternative as well. The sequential limit used here is for computational simplicity. It, however, enables us to use results already available from the time series limit theory, and to examine the cross-sectional effect of each intermediate N on the asymptotic behavior in the time series direction and the difference between common and heterogeneous regressors as N becomes large. The analysis of heterogeneous regressors is similar to that discussed in Tanaka (1996) for seasonal MA time series models with heterogeneous seasonals. The analysis of common regressors, however, is complicated, but we can use an approach carried out in Nabeya (2000) for nonstationary seasonal AR models with common seasonals. For panel MA and error components models we find that the existence of common regressors does not affect the asymptotic behavior of the tests. This fact was also partly observed in panel AR models discussed in Breitung (2000) and Moon, Perron, and Phillips (2007). We give more detailed analysis of this fact as N becomes large.

The outline of the article is as follows. In Section 2 we introduce panel MA models, where two cases are considered for the initial value of the MA error, that is, the conditional case and the stationary case. Section 3 computes limiting powers of various unit root tests associated with panel MA models. In Section 3.1 we deal with the conditional case, whereas the stationary case is discussed in Section 3.2. In Section 3.3 the limiting power envelopes are computed for some simple panel MA models in the two cases, which shows that the panel score tests are asymptotically efficient, unlike the time series case. Section 3.4 considers the situation where the error term is temporally or cross-sectionally dependent and examines its effect on the test statistics. Section 4 introduces panel error components models, for which the LBI tests are suggested. We compute limiting powers and explore asymptotic properties of LBI tests, where a close relationship with noninvertible MA models is described. It is found that LBI tests are

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asymptotically efficient, unlike the time series case. Section 5 concludes this article. The proofs of theorems are given in the Appendix.

2. PANEL MA MODELS

The time series MA unit root test was initially developed in Tanaka (1990, 1996), which we extend to the panel MA unit root test here. This test can be used to check if differencing in the time series direction is superfluous or not. For this purpose, we consider the following four models for first differences of original observations $y_{it} = z_{it} - z_{i,t-1}$:

$$\text{Model A: } y_{it} = \beta_0 + \beta_1 t + \eta_{it}, \tag{1}$$

$$\text{Model B: } y_{it} = \beta_{0i} + \beta_1 t + \eta_{it}, \tag{2}$$

$$\text{Model C: } y_{it} = \beta_0 + \beta_{1i} t + \eta_{it}, \tag{3}$$

$$\text{Model D: } y_{it} = \beta_{0i} + \beta_{1i} t + \eta_{it}, \tag{4}$$

where the error term $\{\eta_{it}\}$ is defined for all models by

$$\eta_{it} = \varepsilon_{it} - \alpha\, \varepsilon_{i,t-1}, \quad (i = 1, \ldots, N;\ t = 1, \ldots, T), \tag{5}$$

with $\{\varepsilon_{it}\}$ defined shortly. Note here that the coefficient α is common to all cross sections. This is a strong assumption, which can be relaxed so that it becomes specific to each cross section. Because the tests suggested in this article are based on the derivative of the log-likelihood with respect to a single parameter α, we need to take a different approach if the likelihood is a function of multiple parameters $\alpha_1, \ldots, \alpha_N$.

All of the above models contain intercept and trend as regressors. Model A is the most restricted model with a common intercept and trend, whereas Model B has heterogeneous intercepts and a common trend. Model C has a common intercept and heterogeneous trends, whereas Model D is the most unrestricted model with heterogeneous intercepts and trends. Note that these models coincide with each other when $N = 1$, that is, the time series case.

We initially assume the innovation sequence $\{\varepsilon_{it}\}$ to be independent among time series and cross sections. This is because of simplicity of presentation, but we shall discuss the effect of temporal or cross-sectional dependence in Section 3.4. We deal with two cases for the initial value ε_{i0}. One is the conditional case where $\varepsilon_{i0} \equiv 0$ and $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2)$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. The other is the stationary case where $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2)$ for $i = 1, \ldots, N$ and $t = 0, 1, \ldots, T$. For each case the error vector $\boldsymbol{\eta}_i = (\eta_{i1}, \ldots, \eta_{iT})'$ satisfies

$$\boldsymbol{\eta}_i^C = \begin{pmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} - \alpha\, \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} - \alpha\, \varepsilon_{i,T-1} \end{pmatrix} = C^{-1}(\alpha)\boldsymbol{\varepsilon}_i \ \sim\ \text{N}\Big(\mathbf{0}, \sigma^2\, \Sigma(\alpha)\Big) \ : \text{conditional case,} \tag{6}$$

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$$\boldsymbol{\eta}_i^S = \begin{pmatrix} \varepsilon_{i1} - \alpha\,\varepsilon_{i0} \\ \varepsilon_{i2} - \alpha\,\varepsilon_{i1} \\ \vdots \\ \varepsilon_{iT} - \alpha\,\varepsilon_{i,T-1} \end{pmatrix} \overset{\mathcal{D}}{=} \boldsymbol{\Omega}^{1/2}(\alpha)\boldsymbol{\varepsilon}_i \sim \mathrm{N}\left(\mathbf{0}, \sigma^2\boldsymbol{\Omega}(\alpha)\right) : \text{stationary case,} \quad \textbf{(7)}$$

where '$\overset{\mathcal{D}}{=}$' stands for the distributional equivalence, $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \ldots, \varepsilon_{iT})'$, and

$$\Sigma(\alpha) = \left(C'(\alpha)C(\alpha)\right)^{-1} = \begin{pmatrix} 1 & -\alpha & & & \\ -\alpha & 1+\alpha^2 & \cdot & & 0 \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -\alpha \\ 0 & & & -\alpha & 1+\alpha^2 \end{pmatrix}, \quad \textbf{(8)}$$

$$C(\alpha) = \begin{pmatrix} 1 & & & \\ \alpha & 1 & 0 & \\ \cdot & \cdot & \cdot & \\ \cdot & & \cdot & \cdot \\ \alpha^{T-1} & \cdots & \alpha & 1 \end{pmatrix}, \quad \textbf{(9)}$$

$$\boldsymbol{\Omega}(\alpha) = \begin{pmatrix} 1+\alpha^2 & -\alpha & & & \\ -\alpha & 1+\alpha^2 & \cdot & & 0 \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & -\alpha \\ 0 & & & -\alpha & 1+\alpha^2 \end{pmatrix}. \quad \textbf{(10)}$$

Note that $\boldsymbol{\Omega}(\alpha) - \Sigma(\alpha) = \alpha^2(1\ 0 \ldots 0)'(1\ 0 \ldots 0)$. It will be shown that this difference between the conditional and stationary cases affects the asymptotic theory discussed below.

The panel unit root test considered here is concerned with testing

$$H_0 : \alpha = 1 \quad \text{versus} \quad H_1 : \alpha < 1, \quad \textbf{(11)}$$

where we assume that the true value of α under H_1 takes the following form:

$$\alpha = 1 - \frac{c_N}{T}, \qquad c_N = \frac{c}{N^\kappa}, \quad \textbf{(12)}$$

with $c > 0$ and $0 < \kappa < 1$.

The next section conducts score type tests for the above testing problem (11) and computes limiting powers under the local alternative (12).

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3. LIMITING LOCAL POWERS OF THE PANEL MA UNIT ROOT TESTS

This section computes the limiting local powers of the MA unit root tests in the conditional and stationary cases as $T \to \infty$ for each intermediate N, and then $N \to \infty$. Section 3.1 deals with the conditional case followed by the stationary case in Section 3.2. Section 3.3 derives the power envelope of the most powerful invariant (MPI) test, concentrating on simple models. Section 3.4 briefly discusses the situation where the error term becomes temporally or cross-sectionally dependent.

3.1. Conditional Case

Given observations $\{y_{it}\}$ for $i = 1, \ldots, N$ and $t = 1, \ldots, T$, the above models (1) through (4) may be expressed as

$$\boldsymbol{y} = X\gamma + \boldsymbol{\eta}^C = X\gamma + \Big(I_N \otimes C^{-1}(\alpha)\Big)\varepsilon \sim \mathrm{N}\Big(X\gamma, \sigma^2\Big(I_N \otimes \Sigma(\alpha)\Big)\Big), \tag{13}$$

where X and γ are the regression matrix and the coefficient vector, respectively, contained in each model, I_N is the identity matrix of order N, whereas $\otimes$ is the Kronecker product and

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \vdots \\ \boldsymbol{y}_N \end{pmatrix}, \quad \boldsymbol{y}_i = \begin{pmatrix} y_{i1} \\ \vdots \\ y_{iT} \end{pmatrix}, \quad \boldsymbol{\eta}^C = \begin{pmatrix} \boldsymbol{\eta}_1^C \\ \vdots \\ \boldsymbol{\eta}_N^C \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_N \end{pmatrix}.$$

The regression matrix X is defined for each model as

$$\begin{aligned}
&\text{Model A: } X = (\boldsymbol{i}_N \otimes \boldsymbol{i}_T \quad \boldsymbol{i}_N \otimes \boldsymbol{d}_T) : NT \times 2, \\
&\text{Model B: } X = (I_N \otimes \boldsymbol{i}_T \quad \boldsymbol{i}_N \otimes \boldsymbol{d}_T) : NT \times (N+1), \\
&\text{Model C: } X = (\boldsymbol{i}_N \otimes \boldsymbol{i}_T \quad I_N \otimes \boldsymbol{d}_T) : NT \times (N+1), \\
&\text{Model D: } X = (I_N \otimes \boldsymbol{i}_T \quad I_N \otimes \boldsymbol{d}_T) : NT \times 2N,
\end{aligned}$$

where

$$\boldsymbol{i}_N = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} : N \times 1, \quad \boldsymbol{d}_T = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ T \end{pmatrix}.$$

The parameter vector γ is defined accordingly. Namely, γ is 2×1, $(N+1) \times 1$, $(N+1) \times 1$, and $2N \times 1$ for Models A, B, C, and D, respectively.

Noting that $\boldsymbol{\eta}^C \sim \mathrm{N}(\boldsymbol{0}, \sigma^2 I_N \otimes (C'(\alpha)C(\alpha))^{-1})$ and $|C(\alpha)| = 1$, the log-likelihood is expressed as

$$L^C(\psi) = -\frac{NT}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}(\boldsymbol{y} - X\gamma)'\Big(I_N \otimes C'(\alpha)C(\alpha)\Big)(\boldsymbol{y} - X\gamma), \tag{14}$$

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where ψ is the parameter vector composed of γ, α, and σ^2. Let the (j,k)th element of $C'(\alpha)C(\alpha)$ be $g_{jk}(\alpha)$. Then we have, for $j \le k$,

$$g_{jk}(\alpha) = \sum_{i=0}^{T-k} \alpha^{k-j+2i},$$

$$\begin{aligned}\left.\frac{dg_{jk}(\alpha)}{d\alpha}\right|_{\alpha=1} &= \sum_{i=0}^{T-k}(k-j+2i) = (T-k+1)(T-j)\\ &= (T-j+1)(T-k+1) - \min(T-j+1, T-k+1),\end{aligned}$$

so that

$$\left.\frac{dC'(\alpha)C(\alpha)}{d\alpha}\right|_{\alpha=1} = C'\boldsymbol{i}_T\boldsymbol{i}_T'C - C'C,$$

where $C = C(1)$. Thus we have

$$\left.\frac{\partial L^C(\psi)}{\partial\alpha}\right|_{H_0} = -\frac{1}{2\tilde{\sigma}^2}\tilde{\boldsymbol{\eta}}'\Big((I_N\otimes C'\boldsymbol{i}_T)(I_N\otimes\boldsymbol{i}_T'C)\Big)\tilde{\boldsymbol{\eta}} + \frac{NT}{2},$$

where

$$\tilde{\boldsymbol{\eta}} = \boldsymbol{y} - X\tilde{\gamma} = M_X\boldsymbol{\eta}^C, \quad \tilde{\gamma} = \Big(X'(I_N\otimes C'C)X\Big)^{-1}X'(I_N\otimes C'C)\boldsymbol{y},$$

$$M_X = I_{NT} - X\Big(X'(I_N\otimes C'C)X\Big)^{-1}X'(I_N\otimes C'C), \quad \tilde{\sigma}^2 = \frac{1}{NT}\tilde{\boldsymbol{\eta}}'(I_N\otimes C'C)\tilde{\boldsymbol{\eta}}.$$

We can now conduct the LBI test for Model M (M = A, B, C, D) which rejects H_0 for large values of

$$Q_{NT}^{(M)} = \frac{1}{T\tilde{\sigma}^2}\tilde{\boldsymbol{\eta}}'\Big((I_N\otimes C'\boldsymbol{i}_T)(I_N\otimes\boldsymbol{i}_T'C)\Big)\tilde{\boldsymbol{\eta}} = \frac{1}{T\tilde{\sigma}^2}\varepsilon' A_{NT}'\,A_{NT}\varepsilon, \tag{15}$$

where

$$A_{NT} = (I_N\otimes\boldsymbol{i}_T'C)M_X(I_N\otimes C^{-1}(\alpha)).$$

The following theorem describes the asymptotic distribution of $Q_{NT}^{(M)}$ as $T\to\infty$ for each N, whose proof is given in the Appendix.

THEOREM 1. *It holds that, as $T\to\infty$ under $\alpha = 1 - c_N/T$ for each N,*

$$Q_{NT}^{(M)} \;\Rightarrow\; Q_N^{(M)} = a^{(D)}Z_1^2 + a^{(M)}\sum_{i=2}^{N} Z_i^2, \qquad (\text{M} = \text{A, B, C, D}), \tag{16}$$

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where $\{Z_i\} \sim \text{NID}(0,1)$ and

$$a^{(A)} = 1 + c_N + \frac{1}{3}c_N^2, \quad a^{(B)} = \frac{1}{4}\left(1 + \frac{1}{4}c_N + \frac{1}{30}c_N^2\right),$$

$$a^{(C)} = \frac{4}{9}\left(1 + \frac{4}{9}c_N + \frac{1}{14}c_N^2\right), \quad a^{(D)} = \frac{1}{9}\left(1 + \frac{1}{9}c_N + \frac{1}{105}c_N^2\right).$$

The distribution of $Q_N^{(M)}$ in (16) is that of a finite sum of weighted independent $\chi^2(1)$ random variables, and can be easily computed by Imhof's formula [Imhof (1961)]. To derive the asymptotic distribution of $Q_N^{(M)}$ as $N \to \infty$, rewrite $a^{(M)}$ as

$$a^{(M)} = a_0^{(M)} + a_1^{(M)} c_N + a_2^{(M)} c_N^2.$$

As N becomes large, $Q_N^{(M)}$ is distributed as $a_0^{(M)} \times \chi^2(N)$ with

$$\text{E}(Q_N^{M)}) = N\left(a_0^{(M)} + a_1^{(M)} c_N\right) + O(Nc_N^2), \quad \text{Var}(Q_N^{(M)}) = 2N\left(a_0^{(M)}\right)^2 + O(Nc_N).$$

Thus, by taking $c_N = c/\sqrt{N}$, it holds that,

$$\frac{Q_N^{(M)} - Na_0^{(M)}}{\sqrt{2N}a_0^{(M)}} \Rightarrow \text{N}\left(\frac{a_1^{(M)}}{\sqrt{2}a_0^{(M)}}c,\, 1\right).$$

Then we obtain the following theorem by using the central limit theorem (CLT).

THEOREM 2. *Suppose that we conduct the LBI tests based on* $Q_N^{(M)}$ *in (16) at the* $100\gamma\,\%$ *level. Then the limiting powers as* $N \to \infty$ *under* $c_N = c/\sqrt{N}$ *are given as follows*:

$$P\left(\frac{1}{\sqrt{2N}}\left(Q_N^{(A)} - N\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{c}{\sqrt{2}}\right) = \Phi\left(z_\gamma + 0.707\,c\right),$$

$$P\left(\frac{4}{\sqrt{2N}}\left(Q_N^{(B)} - \frac{N}{4}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{c}{4\sqrt{2}}\right) = \Phi\left(z_\gamma + 0.177\,c\right),$$

$$P\left(\frac{9}{4\sqrt{2N}}\left(Q_N^{(C)} - \frac{4N}{9}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{4c}{9\sqrt{2}}\right) = \Phi\left(z_\gamma + 0.314\,c\right),$$

$$P\left(\frac{9}{\sqrt{2N}}\left(Q_N^{(D)} - \frac{N}{9}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{c}{9\sqrt{2}}\right) = \Phi\left(z_\gamma + 0.0786\,c\right),$$

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where $\Phi(\cdot)$ is the distribution function of N(0, 1) and z_γ is the $100\gamma\,\%$ point of N(0, 1).

Note that the LBI tests have nontrivial powers in a $N^{-1/2}T^{-1}$ neighborhood of unity for all models considered here. This contrasts with the unit root tests in panel AR models [see Moon et al., 2007 and Tanaka, 2017]. It is also noticed that the limiting power decreases in the order of A, C, B, and D.

To see the cross-sectional effect of N, the distributions of $Q_N^{(M)}/N$ are drawn in Figure 1 and 2. Figure 1 presents the probability densities of $Q_N^{(B)}/N$ under H_0 for $N = 1, 10, 100$. It is seen that the distribution converges to $1/4$, as is anticipated from Theorem 2. Figure 2 presents similar graphs for $Q_N^{(D)}/N$. The distribution converges to $1/9$ in the present case. Note that $Q_N^{(B)}/N$ and $Q_N^{(D)}/N$ are the same when $N = 1$. Although the axis scales of the two figures are different, it is seen that $Q_N^{(D)}/N$ converges more rapidly than $Q_N^{(B)}/N$. This is because the distribution of $Q_N^{(B)}$ gets away from $Q_1^{(D)}$, whereas $Q_N^{(D)}$ is just the convolution. Similar performances are observed for M=A and C, although not presented here. Figure 3 presents powers of the LBI $Q_N^{(B)}$-test against c with $c_N = c/\sqrt{N}$ when $N = 1, 10, 100, \infty$. It is seen that the powers for $N = 100$ are well approximated by the limiting power, although those for $N = 10$ are not. This means that the powers eventually depend on c only as N becomes large.

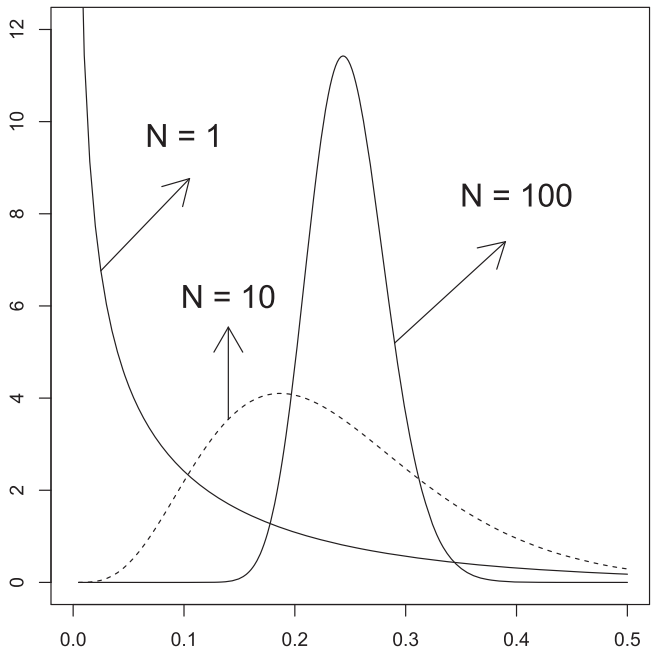


FIGURE 1. Densities of $Q_N^{(B)}/N$ under H_0 for $N = 1, 10, 100$.

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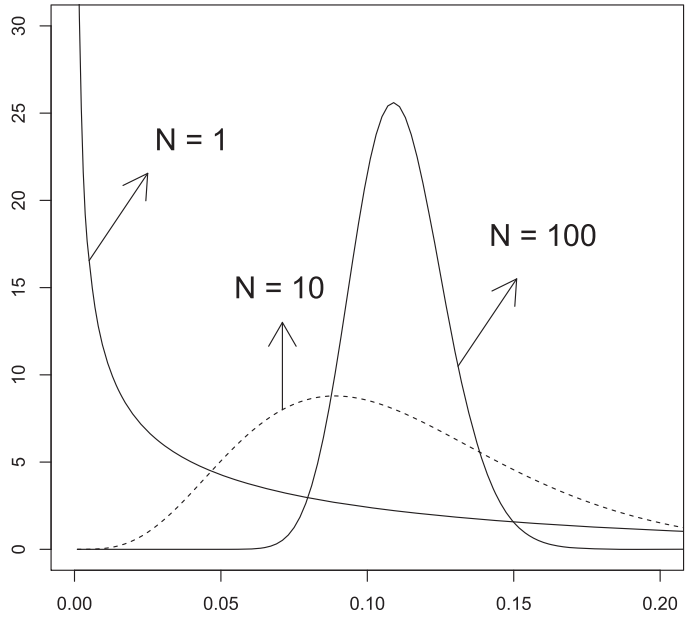


FIGURE 2. Densities of $Q_N^{(D)}/N$ under H_0 for $N = 1, 10, 100$.

In passing we point out that we can deal with some other variations of the above models and consider the LBI statistics $Q_{NT}^{(\cdot)}$. It is easily shown that, as $T \to \infty$,

$$\text{Model A1: } y_{it} = \eta_{it}, \quad Q_{NT}^{(A1)} \Rightarrow a^{(A)} \sum_{i=1}^{N} Z_i^2,$$

$$\text{Model A2: } y_{it} = \beta_0 + \eta_{it}, \quad Q_{NT}^{(A2)} \Rightarrow a^{(B)} Z_1^2 + a^{(A)} \sum_{i=2}^{N} Z_i^2,$$

$$\text{Model A3: } y_{it} = \beta_1 t + \eta_{it}, \quad Q_{NT}^{(A3)} \Rightarrow a^{(C)} Z_1^2 + a^{(A)} \sum_{i=2}^{N} Z_i^2,$$

$$\text{Model B1: } y_{it} = \beta_{0i} + \eta_{it}, \quad Q_{NT}^{(B1)} \Rightarrow a^{(B)} \sum_{i=1}^{N} Z_i^2,$$

$$\text{Model C1: } y_{it} = \beta_{1i} t + \eta_{it}, \quad Q_{NT}^{(C1)} \Rightarrow a^{(C)} \sum_{i=1}^{N} Z_i^2.$$

Thus it is seen that the existence of common regressors does not affect the asymptotic behavior of the LBI tests as $N \to \infty$, although that of heterogeneous regressors does. The following corollary summarizes the above arguments.

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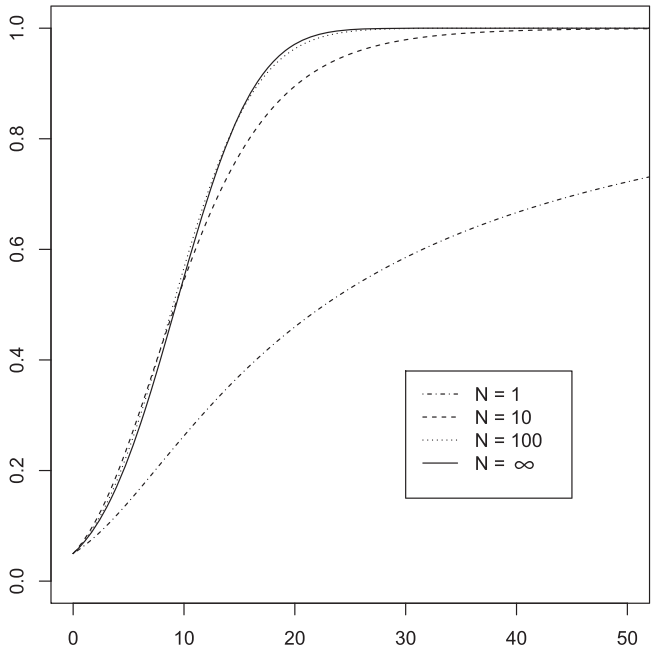


FIGURE 3. Local powers of the LBI $Q_N^{(B)}$-test ($N = 1, 10, 100, \infty$).

COROLLARY 1. *The LBI statistics for the following models in each group have the same asymptotic property.*

Group A: $y_{it} = \eta_{it}, \quad y_{it} = \beta_0 + \eta_{it}, \quad y_{it} = \beta_1 t + \eta_{it}, \quad y_{it} = \beta_0 + \beta_1 t + \eta_{it},$

Group B: $y_{it} = \beta_{0i} + \eta_{it}, \quad y_{it} = \beta_{0i} + \beta_1 t + \eta_{it},$

Group C: $y_{it} = \beta_{1i} t + \eta_{it}, \quad y_{it} = \beta_0 + \beta_{1i} t + \eta_{it}.$

3.2. Stationary Case

The present model may be expressed as

$$\boldsymbol{y} = X\boldsymbol{\gamma} + \boldsymbol{\eta}^S \overset{\mathcal{D}}{=} X\boldsymbol{\gamma} + \left(I_N \otimes \Omega^{1/2}(\alpha)\right)\boldsymbol{\varepsilon} \sim \mathrm{N}\left(X\boldsymbol{\gamma}, \sigma^2\left(I_N \otimes \Omega(\alpha)\right)\right), \tag{17}$$

where $\Omega(\alpha)$ is defined in (10). Then the log-likelihood is given by

$$L^S(\boldsymbol{\psi}) = -\frac{NT}{2}\log 2\pi\sigma^2 - \frac{1}{2}\log|I_N \otimes \Omega(\alpha)| - \frac{1}{2\sigma^2}(\boldsymbol{y} - X\boldsymbol{\gamma})'\left(I_N \otimes \Omega^{-1}(\alpha)\right)(\boldsymbol{y} - X\boldsymbol{\gamma}). \tag{18}$$

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Unlike the conditional case, it holds that $\partial L^S(\psi)/\partial\alpha|_{H_0} = 0$. Taking the second derivative evaluated under H_0 yields the LBIU test for Model M which rejects H_0 for large values of

$$R_{NT}^{(M)} = \frac{1}{T^2}\left(\left.\frac{\partial^2 L^S(\psi)}{\partial\alpha^2}\right|_{H_0} + \frac{NT(T+5)}{6}\right) = \frac{1}{T^2\tilde{\sigma}^2}\tilde{\eta}'\left(I_N \otimes \Omega^{-2}\right)\tilde{\eta}, \qquad (19)$$

where $\Omega = \Omega(1)$ and

$$\tilde{\eta} = y - X\tilde{\gamma} = N_X\eta^S, \quad \tilde{\gamma} = \left(X'(I_N \otimes \Omega^{-1})X\right)^{-1}X'(I_N \otimes \Omega^{-1})y,$$

$$N_X = I_{NT} - X\left(X'(I_N \otimes \Omega^{-1})X\right)^{-1}X'(I_N \otimes \Omega^{-1}), \quad \tilde{\sigma}^2 = \frac{1}{NT}\tilde{\eta}'(I_N \otimes \Omega^{-1})\tilde{\eta}.$$

To derive the asymptotic distribution of $R_{NT}^{(M)}$, the following relation is useful:

$$\begin{aligned}\tilde{\eta}'\left(I_N \otimes \Omega^{-2}\right)\tilde{\eta} &\overset{\mathcal{D}}{=} \varepsilon'(I_N \otimes \Omega^{1/2}(\alpha))N_X'(I_N \otimes \Omega^{-2})N_X(I_N \otimes \Omega^{1/2}(\alpha))\varepsilon \\ &\overset{\mathcal{D}}{=} \varepsilon'(I_N \otimes \Omega^{-1})N_X(I_N \otimes \Omega(\alpha))N_X'(I_N \otimes \Omega^{-1})\varepsilon \\ &= \varepsilon'(I_N \otimes \Omega^{-1})N_X\left[I_N \otimes \left(\alpha\Omega + (1-\alpha)^2 I_T\right)\right]N_X'(I_N \otimes \Omega^{-1})\varepsilon,\end{aligned}$$

where it can be shown that

$$(I_N \otimes \Omega^{-1})N_X = N_X'(I_N \otimes \Omega^{-1}) = (I_N \otimes \Omega^{-1})N_X(I_N \otimes \Omega)N_X'(I_N \otimes \Omega^{-1}).$$

Thus we have, for $\alpha = 1 - c_N/T$,

$$\tilde{\eta}'\left(I_N \otimes \Omega^{-2}\right)\tilde{\eta} \overset{\mathcal{D}}{=} \varepsilon'\left(\left(1 - \frac{c_N}{T}\right)B_{NT} + \frac{c_N^2}{T^2}B_{NT}^2\right)\varepsilon, \quad B_{NT} = (I_N \otimes \Omega^{-1})N_X.$$

Using the above relations, the asymptotic distribution of $R_{NT}^{(M)}$ for Model M can be derived, which is described in the following theorem.

THEOREM 3. *It holds that, as $T \to \infty$ under $\alpha = 1 - c_N/T$ for each N,*

$$R_{NT}^{(M)} \Rightarrow R_N^{(M)} = U_1^{(D)} + \sum_{i=2}^{N} U_i^{(M)}, \qquad \text{(M = A, B, C, D)}, \qquad (20)$$

where

$$U_i^{(M)} = \int_0^1\int_0^1 \left[K^{(M)}(r,s) + c_N^2 K_{(2)}^{(M)}(r,s)\right]dW_i(r)\,dW_i(s). \qquad (21)$$

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Here $\boldsymbol{W}(r) = (W_1(r), \ldots, W_N(r))'$ is the N-dimensional standard Brownian motion and $K^{(M)}(r,s)$ is the kernel associated with Model M defined by

$$K^{(A)}(r,s) = \min(r,s) - rs,$$
$$K^{(B)}(r,s) = \min(r,s) - rs - 3rs(1-r)(1-s),$$
$$K^{(C)}(r,s) = \min(r,s) - rs - \frac{5}{4}rs(1-r^2)(1-s^2),$$
$$K^{(D)}(r,s) = \min(r,s) - rs - 2rs(1-r)(1-s)(4-5r-5s+10rs),$$

whereas $K^{(M)}_{(2)}(r,s)$ is the iterated kernel of $K^{(M)}(r,s)$ defined by

$$K^{(M)}_{(2)}(r,s) = \int_0^1 \int_0^1 K^{(M)}(r,u)K^{(M)}(u,s)\,du.$$

Some remarks follow.

(a) The expression for $R_N^{(M)}$ in (20) is complicated, but its distributional nature is similar to $Q_N^{(M)}$ in (16) because $Q_N^{(M)}$ can be expressed as

$$Q_N^{(M)} = a^{(D)} \int_0^1 \int_0^1 dW_1(r)\,dW_1(s) + a^{(M)} \sum_{i=2}^{N} \int_0^1 \int_0^1 dW_i(r)\,dW_i(s).$$

(b) The distribution of $R_N^{(M)}$ can be computed from the inversion formula

$$P\left(R_N^{(M)} < x\right) = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{1-e^{-i\theta x}}{i\theta}\phi_N^{(M)}(\theta)\right] d\theta, \tag{22}$$

where $\phi_N^{(M)}(\theta)$ is the characteristic function (cf.) of $R_N^{(M)}$ defined by

$$\phi_N^{(M)}(\theta) = \mathrm{E}(e^{i\theta R_N^{(M)}}) = \psi^{(D)}(\theta)\left[\psi^{(M)}(\theta)\right]^{N-1} \tag{23}$$

with $\psi^{(M)}(\theta)$ being the cf. of $U_i^{(M)}$. It follows from Tanaka (1996, Chap. 10) that

$$\psi^{(M)}(\theta) = \left[D^{(M)}\left(i\theta + \sqrt{-\theta^2 + 2ic_N^2\theta}\right) D^{(M)}\left(i\theta - \sqrt{-\theta^2 + 2ic_N^2\theta}\right)\right]^{-1/2},$$

where $D^{(M)}(\lambda)$ is the Fredholm determinant (FD) of the kernel $K^{(M)}(r,s)$. More specifically, we have

$$D^{(A)}(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$
$$D^{(B)}(\lambda) = \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right),$$
$$D^{(C)}(\lambda) = \frac{45}{\lambda^3}\left(\sqrt{\lambda}\left(1 - \frac{\lambda}{3}\right)\sin\sqrt{\lambda} - \lambda\cos\sqrt{\lambda}\right),$$
$$D^{(D)}(\lambda) = \frac{8640}{\lambda^4}\left(2 + \frac{\lambda}{3} - \sqrt{\lambda}\left(2 - \frac{\lambda}{12}\right)\sin\sqrt{\lambda} - \left(2 - \frac{2\lambda}{3}\right)\cos\sqrt{\lambda}\right).$$

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(c) $R_N^{(M)}$ is distributed as an infinite sum of weighted $\chi^2(1)$. In fact, we have

$$R_N^{(M)} \stackrel{\mathcal{D}}{=} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n^{(D)}}+\frac{c_N^2}{\{\lambda_n^{(D)}\}^2}\right)Z_{1n}^2+\sum_{i=2}^{N}\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n^{(M)}}+\frac{c_N^2}{\{\lambda_n^{(M)}\}^2}\right)Z_{in}^2,$$

where $\{Z_{in}\} \sim \mathrm{NID}(0,1)$ and $\lambda_n^{(M)}$ is the eigenvalue of $K^{(M)}(r,s)$ in the integral equation

$$f_n^{(M)}(r) = \lambda_n^{(M)} \int_0^1 K^{(M)}(r,s)\, f_n^{(M)}(s)\, ds,$$

with $f_n^{(M)}(r)$ being the eigenfunction corresponding to $\lambda_n^{(M)}$. This contrasts with the conditional case, where the distribution is that of a finite sum of weighted $\chi^2(1)$.

(d) As N becomes large, it holds that

$$\begin{aligned}
\mathrm{E}(R_N^{(M)}) &= N\int_0^1 \left[K^{(M)}(r,r)+c_N^2 K_{(2)}^{(M)}(r,r)\right]dr + O(1)\\
&= Na_0^{(M)} + Nc_N^2 a_1^{(M)} + O(1),\\
\mathrm{Var}(R_N^{(M)}) &= 2N\int_0^1\int_0^1 \left\{K^{(M)}(r,s)\right\}^2 dr\, ds + O(Nc_N^2)\\
&= 2Nb_0^{(M)} + O(Nc_N^2),
\end{aligned}$$

where

$$\begin{aligned}
a_0^{(M)} &= \int_0^1 K^{(M)}(r,r)\, dr, \quad a_1^{(M)} = \int_0^1 K_{(2)}^{(M)}(r,r)\, dr,\\
b_0^{(M)} &= \int_0^1\int_0^1 \left\{K^{(M)}(r,s)\right\}^2 dr\, ds.
\end{aligned}$$

The asymptotic distribution of $R_N^{(M)}$ as $N \to \infty$ can be derived easily. It follows from Theorem 3 and Remark (d) above that, as $N \to \infty$ under $c_N = c/N^{1/4}$,

$$\frac{R_N^{(M)} - Na_0^{(M)}}{\sqrt{2Nb_0^{(M)}}} \Rightarrow \mathrm{N}\left(\frac{a_1^{(M)}}{\sqrt{2b_0^{(M)}}}c,\ 1\right),$$

which yields the following theorem.

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THEOREM 4. *Suppose that we conduct the LBIU tests based on* $R_N^{(M)}$ *in* (20) *at the* $100\gamma\,\%$ *level. Then the limiting powers as* $N \to \infty$ *under* $c_N = c/N^{1/4}$ *are given as follows*:

$$P\left(\sqrt{\frac{45}{N}}\left(R_N^{(A)} - \frac{N}{6}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{5}c^2}{30}\right) = \Phi\left(z_\gamma + 0.0745\,c^2\right),$$

$$P\left(\sqrt{\frac{6300}{11N}}\left(R_N^{(B)} - \frac{N}{15}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{77}c^2}{420}\right) = \Phi\left(z_\gamma + 0.0209\,c^2\right),$$

$$P\left(\frac{21}{\sqrt{N}}\left(R_N^{(C)} - \frac{N}{14}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{c^2}{42}\right) = \Phi\left(z_\gamma + 0.0238\,c^2\right),$$

$$P\left(\sqrt{\frac{22050}{11N}}\left(R_N^{(D)} - \frac{3N}{70}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{22}c^2}{420}\right) = \Phi\left(z_\gamma + 0.0112\,c^2\right).$$

Some remarks follow.

(a) The LBIU tests have nontrivial powers in a $N^{-1/4}T^{-1}$ neighborhood of unity for all models considered here, unlike $O(N^{-1/2}T^{-1})$ in the LBI tests. This means that the LBIU tests cannot differentiate between the null and the alternative so closely as the LBI tests in the cross-sectional direction.

(b) Limiting powers of the LBIU tests decrease in the order of A, C, B, and D, which is the same as the LBI tests.

(c) The initial condition on ε_{i0} does affect the asymptotic behavior of the tests.

Figure 4 presents the probability densities of $R_N^{(A)}/N$ under H_0 for $N = 1, 10, 50$, whereas Figure 5 presents similar graphs for $R_N^{(D)}/N$. The former converges quite slowly to $1/6$ and the latter rapidly to $3/70$. Note that $R_N^{(A)}/N$ and $R_N^{(D)}/N$ are the same when $N = 1$. These densities were computed by numerical integration from (22) by replacing x by Nx. Figure 6 presents powers of the LBIU $R_N^{(A)}$-test when $N = 1, 10, 100, \infty$. The approximation seems not good even for $N = 100$, unlike the conditional case. To compute the powers for finite N we used the relation

$$P\left(R_N^{(M)} > Nx_{\gamma,N}\right) = 1 - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{1 - e^{-i\theta N x_{\gamma,N}}}{i\theta}\phi_N^{(M)}(\theta)\right]d\theta,$$

where $x_{\gamma,N}$ is the upper $100\gamma\,\%$ point of the null distribution of $R_N^{(M)}/N$, whereas $\phi_N^{(M)}(\theta)$ is the cf. of $R_N^{(M)}$ under H_1 defined by (22), where $\psi^{(M)}(\theta)$ is the cf. of $U_i^{(M)}$ in (21).

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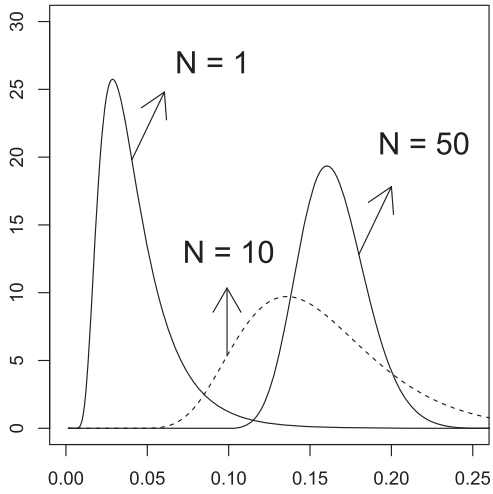


FIGURE 4. Densities of $R_N^{(A)}/N$ under H_0 for $N = 1, 10, 50$.

3.3. Power Envelope

To evaluate the performance of the LBI and LBIU tests suggested in the last two subsections, we compute the limiting power envelopes in both the conditional and

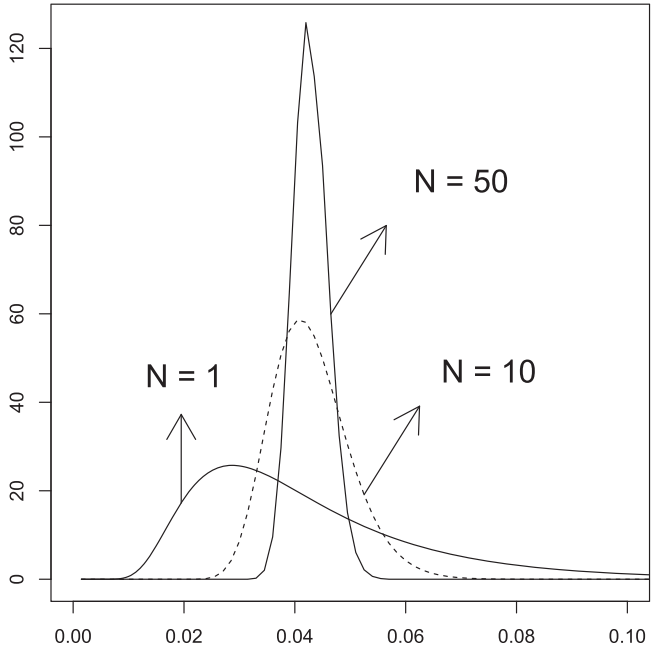


FIGURE 5. Densities of $R_N^{(D)}/N$ under H_0 for $N = 1, 10, 50$.

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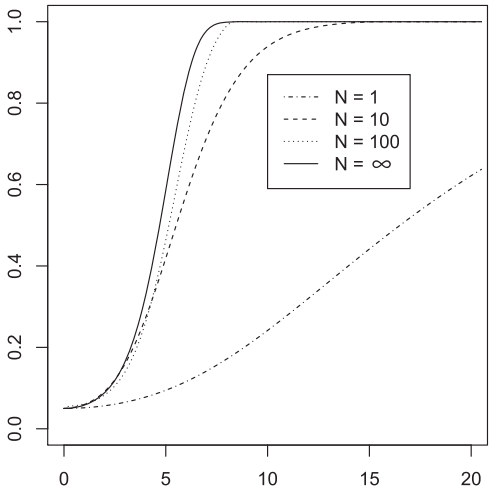


FIGURE 6. Local powers of the LBIU $R_N^{(A)}$-test ($N = 1, 10, 100, \infty$).

stationary cases. The computation, however, turns out to be quite complicated to deal with all the models. Here we first deal with the simplest model

$$\text{Model A1: } y_{it} = \varepsilon_{it} - \alpha\, \varepsilon_{i,t-1}, \tag{24}$$

where $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2)$ for $i = 1, \ldots, N;\ t = 1, \ldots, T$ with $\varepsilon_{i0} = 0$ in the conditional case, whereas $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma^2)$ for $i = 1, \ldots, N;\ t = 0, 1, \ldots, T$ in the stationary case. Note that the asymptotic properties of the tests for Model A1 are the same as those for Model A in (1). Thus the subsequent results obtained for Model A1 also apply to Model A.

Let us consider the testing problem

$$H_0 : \alpha = 1 \quad \text{versus} \quad H_1 : \alpha = 1 - \frac{\theta_N}{T} = \alpha(\theta),$$

where $\theta_N = \theta / N^\kappa$ with θ being a known given positive constant. We assume that the true value of α is given by $\alpha(c) = 1 - c/(N^\kappa T)$.

For Model A1 in the conditional case, the Neyman-Pearson lemma tells us that the test that rejects H_0 when

$$Q_{NT}(\theta) = T \frac{\sum_{i=1}^N \varepsilon_i'(C^{-1}(\alpha(c)))' \Big[C'C - C'(\alpha(\theta))C(\alpha(\theta))\Big] C^{-1}(\alpha(c))\varepsilon_i}{\sum_{i=1}^N \varepsilon_i'(C^{-1}(\alpha(c)))' C'C C^{-1}(\alpha(c))\varepsilon_i} \tag{25}$$

takes large values is MPI.

On the other hand, for Model A1 in the stationary case, the test that rejects H_0 when

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$$R_{NT}(\theta) = T\frac{\sum_{i=1}^{N}\varepsilon_i'\Omega^{1/2}(\alpha(c))\left[\Omega^{-1}-\Omega^{-1}(\alpha(\theta))\right]\Omega^{1/2}(\alpha(c))\varepsilon_i}{\sum_{i=1}^{N}\varepsilon_i'\Omega^{1/2}(\alpha(c))\Omega^{-1}\Omega^{1/2}(\alpha(c))\varepsilon_i} \tag{26}$$

takes large values is MPI.

The following theorem gives the power envelopes of the MPI tests.

THEOREM 5. *For Model A1 in the conditional case, suppose that* $\alpha(c) = 1 - c/(\sqrt{N}T)$ *and* $\alpha(\theta) = 1 - \theta_N/T$ *with* $\theta_N = \theta/\sqrt{N}$. *Then the limiting power envelope of the MPI test at the* $100\gamma\,\%$ *level is given by*

$$P\left(\sqrt{\frac{N}{2}}\left(\frac{Q_{NT}(\theta)}{\theta_N}+\frac{\theta_N}{2}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + 0.707\,c\right). \tag{27}$$

For Model A1 in the stationary case, suppose that $\alpha(c) = 1 - c/(N^{1/4}T)$ and $\alpha(\theta) = 1 - \theta_N/T$ with $\theta_N = \theta/N^{1/4}$. Then the limiting power envelope of the MPI test at the $100\gamma\,\%$ level is given by

$$P\left(\sqrt{45N}\left(\frac{R_{NT}(\theta)+\theta_N}{\theta_N^2}-\frac{1}{6}+\frac{\theta_N^2}{90}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + 0.0745\,c^2\right). \tag{28}$$

Some remarks follow.

(a) The power envelope of the MPI test in the conditional case is nontrivial in a $N^{-1/2}T^{-1}$ neighborhood of unity, whereas that of the MPI test in the stationary case is nontrivial in a $N^{-1/4}T^{-1}$ neighborhood of unity.

(b) It follows from Theorems 2 and 4 that the limiting powers of the LBI and LBIU tests for Model A coincide with the power envelopes. Thus these tests are asymptotically efficient. The fact that the panel LBI and LBIU tests are asymptotically efficient contrasts with the time series case ($N = 1$), where these tests are not efficient [see Tanaka (1996, Chap. 10)].

(c) It is of some interest to note that the power envelope in the conditional case coincides with that of the panel AR unit root test for a model with heterogeneous intercepts, whereas the power envelope in the stationary case coincides with that of the test for a model with heterogeneous trends [Moon et al. (2007) and Tanaka (2017)].

(d) The limiting power envelopes for Models B, C, and D remain to be derived, but it is our conjecture that the LBI and LBIU tests applied to those models are also asymptotically efficient.

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3.4. Effect of Temporal or Cross-Sectional Dependence

Here we briefly consider the situation where there exists temporal or cross-sectional dependence of the error term $\{\varepsilon_{it}\}$ in (5) and examine the effect of such dependence on the test statistics obtained in previous subsections.

Let us first consider temporal dependence. For this purpose we assume

$$\varepsilon_{it} = \sum_{k=0}^{\infty} \phi_{ik}\, \zeta_{i,t-k} = \phi_i(L)\, \zeta_{it}, \quad \sum_{k=1}^{\infty} k\, |\phi_{ik}| < \infty, \quad \{\zeta_{it}\} \sim \text{NID}(0,\, \sigma^2), \qquad \textbf{(29)}$$

where $\phi_i(L) = 1 + \phi_{i1} L + \phi_{i2} L^2 + \cdots$ with L being the lag-operator. The distributional properties of the statistics $Q_{NT}^{(M)}$ in (15) and $R_{NT}^{(M)}$ in (19) are affected by this relaxation. In fact, it can be shown that, as $T \to \infty$ for each N,

$$Q_{NT}^{(M)} \;\Rightarrow\; a^{(M)} \sigma^2 \sum_{i=1}^{N} \phi_i^2(1) Z_i^2 \Bigg/ \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 + O_p(1),$$

$$R_{NT}^{(M)} \;\Rightarrow\; \sigma^2 \sum_{i=1}^{N} \phi_i^2(1) U_i^{(M)} \Bigg/ \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 + O_p(1),$$

where $\sigma^2 \phi_i^2(1)$ is the long-run variance of $\{\varepsilon_{it}\}$, whereas σ_i^2 is the short-run variance given by

$$\sigma_i^2 = \text{Var}(\varepsilon_{it}) = \sigma^2 \sum_{k=0}^{\infty} \phi_{ik}^2.$$

The above statistics depend on the short-run and long-run variances of the error term that characterize temporal dependence.

We next consider cross-sectional dependence, for which we assume that

$$\{\varepsilon_{it}\} \sim \text{NID}(0,\, \sigma_{ii}) \text{ for each } i, \quad \text{Cov}(\varepsilon_{it},\, \varepsilon_{ks}) = \delta_{ts}\, \sigma_{ik}.$$

It then follows that, defining $\tilde{Z}_i \sim \text{N}(0,\, 1)$,

$$Q_{NT}^{(M)} \;\Rightarrow\; a^{(M)} \sum_{i=1}^{N} \sigma_{ii} \tilde{Z}_i^2 \Bigg/ \frac{1}{N} \sum_{i=1}^{N} \sigma_{ii} + O_p(1), \quad \text{Cov}(\tilde{Z}_i,\, \tilde{Z}_k) = \frac{\sigma_{ik}}{\sqrt{\sigma_{ii}\, \sigma_{kk}}}.$$

We also have

$$R_{NT}^{(M)} \;\Rightarrow\; \sum_{i=1}^{N} \sigma_{ii} \int_0^1 \int_0^1 \left[K^{(M)}(r,s) + c_N^2 K_{(2)}^{(M)}(r,s) \right] d\tilde{W}_i(r)\, d\tilde{W}_i(s) \Bigg/ \frac{1}{N} \sum_{i=1}^{N} \sigma_{ii} + O_p(1),$$

where $\{\tilde{W}_i(r)\}$ is the standard Brownian motion with

$$\text{Cov}(\tilde{W}_i(r),\, \tilde{W}_k(s)) = \frac{\sigma_{ik}}{\sqrt{\sigma_{ii}\, \sigma_{kk}}} \min(r,s).$$

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The test statistics depend on the covariances σ_{ik} of the error term that characterize cross-sectional dependence.

It is recognized from the above observations that, when there exists temporal or cross-sectional dependence, the test statistics obtained in previous subsections contain nuisance parameters in a complicated way, and the number of such parameters increases with N. We need to find statistics that are independent of nuisance parameters. This remains to be done.

4. PANEL ERROR COMPONENTS MODELS

The panel MA unit root tests discussed so far can be used, by taking first differences of data, $y_{it} = z_{it} - z_{i,t-1}$, to check if differencing is superfluous or not. It is one method of testing the null of stationarity. This section considers another method by using raw data z_{it}. For that purpose we introduce panel error components models in Section 4.1, for which LBI tests are conducted and limiting powers are computed in Section 4.2. The power envelopes are discussed in Section 4.3. It is seen that there exists a close relationship between the panel MA and stationarity tests.

4.1. Description of Models and Testing Procedures

In this section we consider the following four models:

$$\text{Model E: } z_{it} = \beta_0 + \beta_1 t + \varepsilon_{it} + \mu_{it}, \tag{30}$$

$$\text{Model F: } z_{it} = \beta_{0i} + \beta_1 t + \varepsilon_{it} + \mu_{it}, \tag{31}$$

$$\text{Model G: } z_{it} = \beta_0 + \beta_{1i} t + \varepsilon_{it} + \mu_{it}, \tag{32}$$

$$\text{Model H: } z_{it} = \beta_{0i} + \beta_{1i} t + \varepsilon_{it} + \mu_{it}, \tag{33}$$

where

$$\mu_{it} = \mu_{i,t-1} + \xi_{it}, \quad \mu_{i0} = 0. \tag{34}$$

We assume that $\{\varepsilon_{it}\} \sim \text{NID}(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon^2 > 0$ and $\{\xi_{it}\} \sim \text{NID}(0, \sigma_\xi^2)$ with $\sigma_\xi^2 \geq 0$, and these are independent of each other. The models in (30) through (33) contain two unobserved error terms and are referred to as the panel error components models.

The testing problem dealt with here is to test

$$H_0 : \rho = \frac{\sigma_\xi^2}{\sigma_\varepsilon^2} = 0 \quad \text{versus} \quad H_1 : \rho = \frac{c_N^2}{T^2}, \tag{35}$$

where $c_N^2 = c^2/N^{2\kappa}$ with $0 < \kappa < 1/2$. Note that, under H_0, μ_{it} reduces to $\mu_{it} = \mu_{i0} = 0$ for each i, whereas it follows a random walk under H_1. The reason for

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this choice of H_1 can be given in terms of the stationary MA (1) model discussed in previous sections. To see this consider Model E in (30), which is equivalent to

$$\begin{aligned} z_{i1} &= \beta_0 + \beta_1 + \varepsilon_{i1} + \xi_{i1}, \\ z_{it} - z_{i,t-1} &= \beta_1 + \varepsilon_{it} - \varepsilon_{i,t-1} + \xi_{it}, \quad (t = 2, \ldots, T). \end{aligned}$$

Here it can be checked that the process $\{x_{it}\} = \{\varepsilon_{it} - \varepsilon_{i,t-1} + \xi_{it}\}$ follows MA(1) with

$$\mathrm{Var}(x_{it}) = 2\sigma_\varepsilon^2 + \sigma_\xi^2, \quad \mathrm{Cov}(x_{it}, x_{i,t-1}) = -\sigma_\varepsilon^2, \tag{36}$$

whereas the usual MA(1) process $\{y_{it}\} = \{\zeta_{it} - \alpha\,\zeta_{i,t-1}\}$ with $\{\zeta_{it}\} \sim \mathrm{NID}(0, \sigma^2)$ has

$$\mathrm{Var}(y_{it}) = (1+\alpha^2)\sigma^2, \quad \mathrm{Cov}(y_{it}, y_{i,t-1}) = -\alpha\sigma^2. \tag{37}$$

Then it follows from (36) and (37) with $\alpha = 1 - c_N/T$ that

$$\sigma_\varepsilon^2 = \alpha\sigma^2, \quad \sigma_\xi^2 = (1-\alpha)^2\sigma^2, \quad \rho = \frac{\sigma_\xi^2}{\sigma_\varepsilon^2} = \frac{(1-\alpha)^2}{\alpha} = \frac{c_N^2}{T^2} + O(c_N^3/T^3),$$

which justifies H_1 in (35).

The present problem was initially discussed in the time series context by Nyblom and Mäkeläinen (1983), which was generalized by Nabeya and Tanaka (1988) using more general models and the method based on the Fredholm determinant. An extension to panel data was earlier discussed in Hadri (2000) and Shin and Snell (2006), where only the limiting null distributions of test statistics were derived. Note, however, that Shin and Snell (2006) discussed the joint limit theory and allowed the error term $\{\varepsilon_{it}\}$ to be temporally dependent. Here we are concerned with limiting alternative distributions as $T \to \infty$ for each intermediate N by keeping the independence assumption of the error term. The case of temporal or cross-sectional dependence can be examined in the same way as in Section 3.4, and thus is omitted here.

Using similar notations given in (13), the above models may be expressed as

$$z = X\gamma + \eta, \quad \eta = \varepsilon + (I_N \otimes C)\xi \sim \mathrm{N}\Big(0, \sigma_\varepsilon^2\Big(I_N \otimes (I_T + \rho CC')\Big)\Big), \tag{38}$$

which gives the log-likelihood

$$\begin{aligned} L(\psi) = &-\frac{NT}{2}\log 2\pi\sigma_\varepsilon^2 - \frac{1}{2}\log\Big|I_N \otimes (I_T + \rho CC')\Big| \\ &-\frac{1}{2\sigma_\varepsilon^2}(z - X\gamma)'\Big(I_N \otimes (I_T + \rho CC')^{-1}\Big)(z - X\gamma), \end{aligned} \tag{39}$$

where ψ is the parameter vector composed of γ, ρ, and σ_ε^2.

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We can now conduct the LBI test for Model M (M = E, F, G, H) which rejects H_0 for large values of

$$S_{NT}^{(M)} = \frac{2}{T^2}\left(\frac{\partial L(\psi)}{\partial \rho}\bigg|_{H_0} + \frac{NT(T+1)}{4} \right) = \frac{1}{T^2 \hat{\sigma}_\varepsilon^2} \hat{\boldsymbol{\eta}}'(I_N \otimes CC')\hat{\boldsymbol{\eta}}, \tag{40}$$

where $\hat{\sigma}_\varepsilon^2 = \hat{\boldsymbol{\eta}}'\hat{\boldsymbol{\eta}}/(NT)$ and

$$\hat{\boldsymbol{\eta}} = \boldsymbol{z} - X\hat{\gamma} = M_X \boldsymbol{\eta}, \quad \hat{\gamma} = (X'X)^{-1}X'\boldsymbol{z}, \quad M_X = I_{NT} - X(X'X)^{-1}X.$$

In the next subsection we derive the asymptotic distribution of $S_{NT}^{(M)}$ as $T \to \infty$ for each intermediate N, and then $N \to \infty$.

4.2. Computation of the Limiting Powers of the LBI Tests

To derive the asymptotic distributions of $S_{NT}^{(M)}$, noting that $\boldsymbol{\eta} \overset{\mathcal{D}}{=} A_\rho \varepsilon$ with $A_\rho = I_N \otimes (I_T + \rho CC')$, we use the following relation:

$$\begin{aligned}
\hat{\boldsymbol{\eta}}'(I_N \otimes CC')\hat{\boldsymbol{\eta}} &= \Big(\varepsilon + (I_N \otimes C)\boldsymbol{\xi}\Big)' M_X (I_N \otimes CC') M_X \Big(\varepsilon + (I_N \otimes C)\boldsymbol{\xi}\Big) \\
&\overset{\mathcal{D}}{=} \varepsilon' A_\rho^{1/2} M_X (I_N \otimes C)(I_N \otimes C') M_X A_\rho^{1/2} \varepsilon \\
&\overset{\mathcal{D}}{=} \varepsilon' (I_N \otimes C') M_X A_\rho M_X (I_N \otimes C)\varepsilon \\
&= \varepsilon' \left[(I_N \otimes C') M_X (I_N \otimes C) + \rho \Big((I_N \otimes C') M_X (I_N \otimes C) \Big)^2 \right] \varepsilon.
\end{aligned}$$

Proceeding in the same way as in Section 3, we can derive the asymptotic distribution of $S_{NT}^{(M)}$ as $T \to \infty$ for each N, which is described in the following theorem.

THEOREM 6. *It holds that, as $T \to \infty$ under $\rho = c_N^2/T^2$ for each N,*

$$S_{NT}^{(M)} \Rightarrow S_N^{(M)} = V_1^{(H)} + \sum_{i=2}^{N} V_i^{(M)}, \qquad (\mathrm{M} = \mathrm{E, F, G, H}), \tag{41}$$

where

$$\begin{aligned}
V_i^{(M)} &= \int_0^1 \int_0^1 \left[K^{(M)}(r,s) + c_N^2 K_{(2)}^{(M)}(r,s) \right] dW_i(r)\, dW_i(s), \\
K^{(E)}(r,s) &= 1 - \max(r,s), \\
K^{(F)}(r,s) &= K^{(A)}(r,s) = \min(r,s) - rs, \\
K^{(G)}(r,s) &= 1 - \max(r,s) - \frac{3}{4}(1-r^2)(1-s^2), \\
K^{(H)}(r,s) &= K^{(B)}(r,s) = \min(r,s) - rs - 3rs(1-r)(1-s).
\end{aligned}$$

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The distributions of $S_N^{(M)}$ in (41) can be computed in the same way as those of $R_N^{(M)}$ in (20), using the inversion formula in (22). For this purpose we need the FDs $D^{(M)}(\lambda)$ of $K^{(M)}(r,s)$ (M = E, F, G, H) which are computed in Tanaka (1996) as

$$D^{(E)}(\lambda) = \cos\sqrt{\lambda},$$

$$D^{(F)}(\lambda) = D^{(A)}(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}},$$

$$D^{(G)}(\lambda) = \frac{3}{\lambda^{3/2}}\left(\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda}\right),$$

$$D^{(H)}(\lambda) = D^{(B)}(\lambda) = \frac{12}{\lambda^2}\left(2 - \sqrt{\lambda}\sin\sqrt{\lambda} - 2\cos\sqrt{\lambda}\right).$$

It follows from Theorems 3 and 6 that, as N becomes large, the distributions of $S_N^{(F)}$ and $S_N^{(H)}$ tend to those of $R_N^{(A)}$ and $R_N^{(B)}$, respectively. This makes clear the point mentioned before in connection with the relationship between error components and MA models.

The following theorem describes the asymptotic distribution of $S_N^{(M)}$ as $N \to \infty$ under $c_N^2 = c^2/N^{2\kappa}$ by choosing κ suitably.

THEOREM 7. *Suppose that we conduct the LBI tests based on $S_N^{(M)}$ in (41) at the 100γ % level. Then the limiting powers as $N \to \infty$ under $c_N^2 = c^2/\sqrt{N}$ are given as follows*:

$$P\left(\sqrt{\frac{3}{N}}\left(S_N^{(E)} - \frac{N}{2}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{3}}{6}c^2\right) = \Phi\left(z_\gamma + 0.2887\,c^2\right),$$

$$P\left(\sqrt{\frac{45}{N}}\left(S_N^{(F)} - \frac{N}{6}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{5}}{30}c^2\right) = \Phi\left(z_\gamma + 0.0745\,c^2\right),$$

$$P\left(\sqrt{\frac{175}{N}}\left(S_N^{(G)} - \frac{N}{10}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{7}}{70}c^2\right) = \Phi\left(z_\gamma + 0.0378\,c^2\right),$$

$$P\left(\sqrt{\frac{6300}{11N}}\left(S_N^{(H)} - \frac{N}{15}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{77}}{420}c^2\right) = \Phi\left(z_\gamma + 0.0209\,c^2\right).$$

Note that the limiting powers are nontrivial against $\rho = c_N^2/T^2 = c^2/(N^{1/2}T^2)$, and decrease in the order of E, F, G, and H. It follows from Theorems 4 and 7 that the limiting powers for Models F and H coincide with those for Models A and B, respectively, as was mentioned before.

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Figure 7 presents the probability densities of $S_N^{(E)}/N$ under H_0 for $N = 1, 10, 100$. It follows from Theorem 7 that the density converges to $1/2$ as $N \to \infty$. Figure 8 shows powers of the LBI $S_N^{(E)}$-test when $N = 1, 10, 100, \infty$. It is seen that the powers for $N = 100$ are reasonably well approximated by the powers for $N = \infty$. Figure 9 compares the powers for Models E, F, G, and H when $N = \infty$. The powers decrease as model complexity increases, as Theorem 7 indicates. It may be concluded that the general features of the LBI $S_N^{(M)}$-tests are quite similar to those of the LBIU $R_N^{(M)}$-tests discussed in Section 3.2.

Table 1 reports the simulated powers for Models F and H at the 5% level obtained in Hadri (2000) and the approximated powers in parentheses based on Theorem 7 for some values of N and T. As an example, for Model F with $\rho = 0.001$, $T = 25$, $N = 100$, the approximated power was computed from $\Phi(-1.6449 + \sqrt{5} \times \rho \times \sqrt{N} \times T^2/30) = \Phi(-1.179) = 0.1192$. It is seen from Table 1 that the simulated powers are reasonably close to the approximated powers. In particular the approximation performs well for Model F even when $T = 25$.

It is noted that we can conduct the LBI tests for some other error components models, using the LBI statistics $S_{NT}^{(\cdot)}$. For example, we have

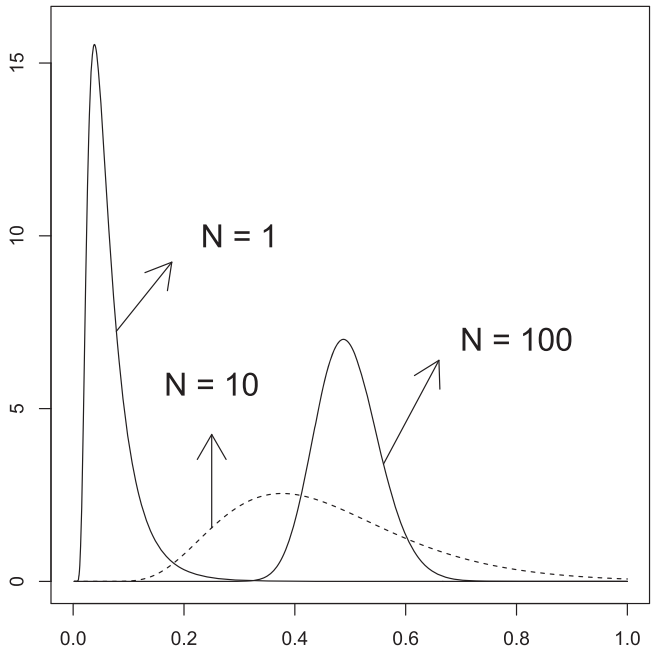


FIGURE 7. Densities of $S_N^{(E)}/N$ under H_0 for $N = 1, 10, 100$.

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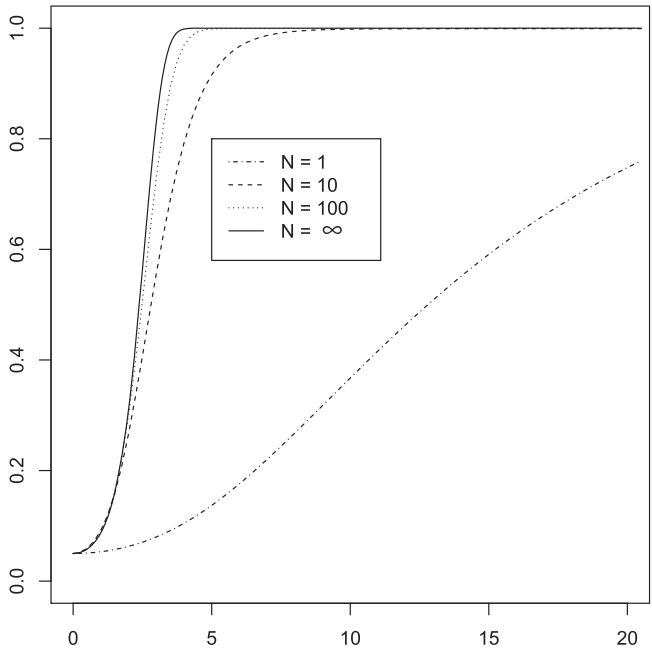


FIGURE 8. Local powers of the LBI $S_N^{(E)}$-test ($N = 1, 10, 100, \infty$).

$$\text{Model E1: } z_{it} = \varepsilon_{it} + \mu_{it}, \quad S_{NT}^{(E1)} \Rightarrow \sum_{i=1}^{N} V_i^{(E)},$$

$$\text{Model E2: } z_{it} = \beta_0 + \varepsilon_{it} + \mu_{it}, \quad S_{NT}^{(E2)} \Rightarrow V_1^{(F)} + \sum_{i=2}^{N} V_i^{(E)},$$

$$\text{Model E3: } z_{it} = \beta_1 t + \varepsilon_{it} + \mu_{it}, \quad S_{NT}^{(E3)} \Rightarrow V_1^{(G)} + \sum_{i=2}^{N} V_i^{(E)},$$

$$\text{Model F1: } z_{it} = \beta_{0i} + \varepsilon_{it} + \mu_{it}, \quad S_{NT}^{(F1)} \Rightarrow \sum_{i=1}^{N} V_i^{(F)},$$

$$\text{Model G1: } z_{it} = \beta_{1i} t + \varepsilon_{it} + \mu_{it}, \quad S_{NT}^{(G1)} \Rightarrow \sum_{i=1}^{N} V_i^{(G)}.$$

Then it holds that the models in the group composed of E, E1, E2, and E3 have the same asymptotic property of tests. The same is true of the group of Models F and F1, and the group of G and G1. Thus we conclude that the existence of common regressors does not affect the asymptotic behavior of the LBI tests as $N \to \infty$, although that of heterogeneous regressors does.

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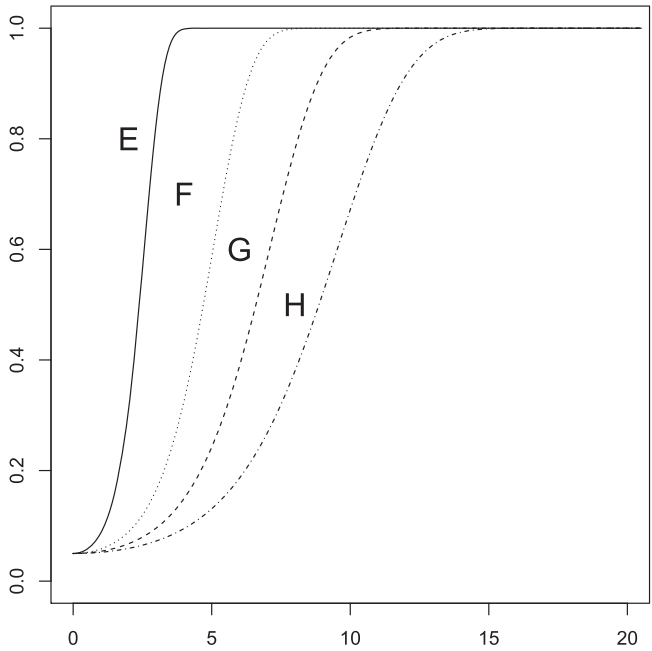


FIGURE 9. Limiting powers of the LBI $S_N^{(M)}$-tests ($N = \infty$; $M = E, F, G, H$).

4.3. Derivation of the Power Envelope

As the last topic in this section we consider the power envelopes of the LBI tests based on $S_N^{(M)}$. Since Model F has the same asymptotic properties as Model A, and the LBIU $R_N^{(A)}$-test is asymptotically efficient, we can deduce that the LBI $S_N^{(F)}$-test is also asymptotically efficient. To prove the asymptotic efficiency of the LBI $S_N^{(E)}$-test, let us consider

$$\text{Model E1: } z_{it} = \varepsilon_{it} + \mu_{it}, \quad \mu_{it} = \mu_{i,t-1} + \xi_{it}, \quad \mu_{i,0} = 0. \tag{42}$$

Note that the asymptotic properties of the tests for Model E1 are the same as those for Model E. Thus the results obtained for Model E1 also apply to Model E.

Let us consider the testing problem

$$H_0 : \rho = 0 \text{ versus } H_1 : \rho(\theta) = \theta_N^2 / T^2,$$

where $\theta_N = \theta / N^\kappa$ with θ being a known constant. We assume that the true value of ρ is given by $\rho(c) = c_N^2 / T^2$ with $c_N = c / N^\kappa$. Then the MPI test rejects H_0 for large values of

$$S_{NT}(\theta) = T \frac{\sum_{i=1}^N \boldsymbol{z}_i' \left[I_T - (I_T + \rho(\theta) CC')^{-1} \right] \boldsymbol{z}_i}{\sum_{i=1}^N \boldsymbol{z}_i' \boldsymbol{z}_i}. \tag{43}$$

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TABLE 1. Simulated powers in Hadri (2000) and approximated powers

	Values of (T, N)					
ρ	(25, 25)	(25, 50)	(25, 100)	(50, 25)	(50, 50)	(50, 100)
			Model F			
0.0001	0.0557	0.0591	0.0517	0.0622	0.0807	0.0802
	(0.0524)	(0.0535)	(0.0550)	(0.0604)	(0.0651)	(0.0723)
0.001	0.0845	0.1003	0.1154	0.2294	0.3803	0.5298
	(0.0790)	(0.0942)	(0.1192)	(0.2378)	(0.3714)	(0.5865)
0.01	0.5763	0.7999	0.9638	0.9965	1.0	1.0
	(0.7531)	(0.9504)	(0.9987)	(1.0)	(1.0)	(1.0)
			Model H			
0.0001	0.0427	0.0389	0.0285	0.0439	0.0566	0.0466
	(0.507)	(0.0510)	(0.0514)	(0.0528)	(0.0539)	(0.0556)
0.001	0.0477	0.0464	0.0365	0.0729	0.1084	0.1184
	(0.0571)	(0.0603)	(0.0650)	(0.0832)	(0.1011)	(0.1308)
0.01	0.1310	0.1814	0.2317	0.6362	0.8655	0.9871
	(0.1607)	(0.2353)	(0.3673)	(0.8332)	(0.9797)	(0.9998)

The following theorem gives the power envelope of the MPI test for Model E1.

THEOREM 8. *For Model E1 in (42), suppose that* $\rho(c) = c^2/(\sqrt{N}T^2)$ *and* $\rho(\theta) = \theta^2/(\sqrt{N}T^2)$. *Then the limiting power envelope of the MPI test at the* 100γ *% level is given by*

$$P\left(\sqrt{3N}\left(\frac{S_{NT}(\theta)}{\theta_N^2} - \frac{1}{2} + \frac{\theta_N^2}{6}\right) > z_{1-\gamma}\right) \to \Phi\left(z_\gamma + \frac{\sqrt{3}}{6}c^2\right) = \Phi\left(z_\gamma + 0.2887\,c^2\right).$$

It follows from Theorems 7 and 8 that the limiting power of the LBI test for Model E1 and thus Model E coincides with the power envelope, which implies that the LBI tests for these models are asymptotically efficient. This contrasts with the time series case ($N = 1$), where the LBI test is asymptotically inefficient [Tanaka (1996, Chap. 10)]. It is our conjecture that the LBI tests for Models G and H are also asymptotically efficient.

5. CONCLUDING REMARKS

Under a simple setting, we have computed the limiting local powers of panel MA unit root tests and stationarity tests. It was found for panel MA models that the initial value plays a very important role and does affect the asymptotic behavior of the tests. For the conditional case we suggested the LBI test whose power is nontrivial in a $N^{-1/2}T^{-1}$ neighborhood of unity, whereas, for the stationary case, we considered the LBIU test whose power is nontrivial in a $N^{-1/4}T^{-1}$ neighborhood

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of unity. We also conducted panel stationarity tests using error components models that have a close relationship with stationary MA models, and computed the limiting local powers of the LBI test in a $N^{-1/2}T^{-2}$ neighborhood of the origin.

For the above models we examined the cross-sectional effect of N as N becomes large, given $T \to \infty$, and found that the existence of common regressors does not affect the asymptotic property of the tests. Similar results can be shown to hold for panel AR models. For example, consider a simple panel AR model

$$y_{it} = \beta_0 + \eta_{it}, \quad \eta_{it} = \rho\, \eta_{i,t-1} + \varepsilon_{it}, \quad \eta_{i,0} = 0, \quad \{\varepsilon_{it}\} \sim \text{ i.i.d.}(0, \sigma^2),$$

where $\rho = 1 - c_N/T$, and define the OLSE $\hat{\rho}$ by

$$\hat{\rho} = \frac{\sum_{i=1}^{N}\sum_{t=2}^{T} \hat{\eta}_{i,t-1}\,\hat{\eta}_{it}}{\sum_{i=1}^{N}\sum_{t=2}^{T} \hat{\eta}_{i,t-1}^2}, \quad \hat{\eta}_{it} = y_{it} - \hat{\beta}_0 = y_{it} - \frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T} y_{it}.$$

Then it holds that, as $T \to \infty$ under $\rho = 1 - c_N/T$ for each N,

$$T(\hat{\rho} - 1) \;\Rightarrow\; R_N = \frac{U_1^{(B)} + \sum_{i=2}^{N} U_i^{(A)}}{V_1^{(B)} + \sum_{i=2}^{N} V_i^{(A)}}, \tag{44}$$

where

$$U_1^{(B)} = \int_0^1 \left(Y_1(r) - \int_0^1 Y_1(s)\, ds \right) dY_1(r), \quad U_i^{(A)} = \int_0^1 Y_i(r)\, dY_i(r),$$

$$V_1^{(B)} = \int_0^1 \left(Y_1(r) - \int_0^1 Y_1(s)\, ds \right)^2 dr, \quad V_i^{(A)} = \int_0^1 Y_i^2(r)\, dr,$$

and $\{Y_i(r)\}$ is the Ornstein-Uhlenbeck process defined by

$$dY_i(r) = -c_N\, Y_i(r)\, dr + dW_i(r).$$

It is seen that the numerator and denominator of R_N in (44) have a distributional property similar to the statistics dealt with in this article, and it holds that, as N becomes large, the distribution of R_N does not depend on $U_1^{(B)}$ or $V_1^{(B)}$. This means that the panel AR models also have the property that the effect of common regressors fades away asymptotically. More details will be reported elsewhere.

We also obtained the limiting power envelopes for some simple MA and error components models, and found that the LBI and LBIU tests are asymptotically efficient. This contrasts with the corresponding tests in the time series case.

A caveat in our analysis is that it is based on an i.i.d. assumption of the error term, which needs to be extended to cover the case of temporal or cross-sectional dependence in future work.

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APPENDIX

Proof of Theorem 1. Let us consider Model A, for which we have

$$Q_{NT}^{(A)} = \frac{1}{T\tilde{\sigma}^2}\boldsymbol{\varepsilon}' A_{NT}' A_{NT}\boldsymbol{\varepsilon} \overset{\mathcal{D}}{=} \frac{\sigma^2}{T\tilde{\sigma}^2}\boldsymbol{Y}' A_{NT} A_{NT}'\boldsymbol{Y},$$

where $\boldsymbol{Y} \sim \mathrm{N}(\boldsymbol{0}, I_N)$ and

$$A_{NT} = (I_N \otimes \boldsymbol{i}_T' C)\Big[I_{NT} - X(X'(I_N \otimes C'C)X)^{-1}X'(I_N \otimes C'C)\Big](I_N \otimes C^{-1}(\alpha))$$

with $X = (\boldsymbol{i}_N \otimes \boldsymbol{i}_T \quad I_N \otimes \boldsymbol{d}_T)$. We have, after some manipulations,

$$\boldsymbol{Y}' A_{NT} A_{NT}'\boldsymbol{Y} = \boldsymbol{a}_T'\boldsymbol{a}_T\boldsymbol{Y}'\boldsymbol{Y} - \frac{2}{N}\boldsymbol{a}_T'\boldsymbol{b}_T\boldsymbol{Y}'\boldsymbol{i}_N\boldsymbol{i}_N'\boldsymbol{Y} + \frac{1}{N}\boldsymbol{b}_T'\boldsymbol{b}_T\boldsymbol{Y}'\boldsymbol{i}_N\boldsymbol{i}_N'\boldsymbol{Y} + O_p(1).$$

Following Nabeya (2000), define $\boldsymbol{Z} = H\boldsymbol{Y}$, where H is the $N \times N$ orthogonal matrix with the first row being $\boldsymbol{i}_N'/\sqrt{N}$ so that $\boldsymbol{Y}'\boldsymbol{i}_N = \boldsymbol{Y}'H'H\boldsymbol{i}_N = \boldsymbol{Z}'\sqrt{N}(1, 0, \ldots, 0)' = \sqrt{N}Z_1$. Then it holds that

$$\begin{aligned}\boldsymbol{Y}' A_{NT} A_{NT}'\boldsymbol{Y} &\overset{\mathcal{D}}{=} \boldsymbol{a}_T'\boldsymbol{a}_T\boldsymbol{Z}'\boldsymbol{Z} - 2\boldsymbol{a}_T'\boldsymbol{b}_T Z_1^2 + \boldsymbol{b}_T'\boldsymbol{b}_T Z_1^2 + O_p(1)\\ &= \| \boldsymbol{a}_T - \boldsymbol{b}_T \|^2 Z_1^2 + \| \boldsymbol{a}_T \|^2 \sum_{i=2}^{N} Z_i^2 + O_p(1),\end{aligned}$$

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and it can be shown that

$$\lim_{T\to\infty}\frac{1}{T}\parallel \boldsymbol{a}_T-\boldsymbol{b}_T\parallel^2=a^{(D)},\quad \lim_{T\to\infty}\frac{1}{T}\parallel \boldsymbol{a}_T\parallel^2=a^{(A)},$$

where $a^{(D)}$ and $a^{(A)}$ are defined in Theorem 1. We can also show that $\tilde{\sigma}^2$ converges to σ^2 in probability as $T\to\infty$ with N fixed. Thus (16) has been proved for Model A. The proofs for the other models can be accomplished similarly, which establishes Theorem 1. ■

Proof of Theorem 2. For Model A it follows from Theorem 1 that

$$Q_N^{(A)}=a^{(D)}Z_1^2+a^{(A)}\sum_{i=2}^{N}Z_i^2,$$

where $\{Z_i\}\sim \mathrm{NID}(0,1)$, which yields, as N becomes large under $c_N=c/\sqrt{N}$,

$$\mathrm{E}(Q_N^{(A)})=N+c\sqrt{N}+O(1),\quad \mathrm{Var}(Q_N^{(A)})=2N+O(1).$$

Thus it follows from the WLLN and CLT that, as $N\to\infty$ under $c_N=c/\sqrt{N}$,

$$\frac{1}{\sqrt{N}}\left(Q_N^{(A)}-N\right)\Rightarrow \mathrm{N}\left(c,2\right),$$

which proves the first relation in Theorem 2. Similarly we can prove the other relations, which establishes Theorem 2. ■

Proof of Theorem 3. The statistic $R_{NT}^{(D)}$ was dealt with in Tanaka (2017), which yields

$$\tilde{\boldsymbol{\eta}}'\left(I_N\otimes\Omega^{-2}\right)\tilde{\boldsymbol{\eta}}=\sum_{i=1}^{N}\tilde{\boldsymbol{\eta}}_i'\Omega^{-2}\tilde{\boldsymbol{\eta}}_i\overset{\mathcal{D}}{=}\sum_{i=1}^{N}\left[\left(1-\frac{c_N}{T}\right)\boldsymbol{\varepsilon}_i'A_T\boldsymbol{\varepsilon}_i+\frac{c_N^2}{T^2}\boldsymbol{\varepsilon}_i'A_T^2\boldsymbol{\varepsilon}_i\right],$$

where

$$A_T=\Omega^{-1}-\Omega^{-1}X_T(X_T'\Omega^{-1}X_T)^{-1}X_T'\Omega^{-1},\quad X_T=(\boldsymbol{i}_T\quad \boldsymbol{d}_T).$$

Then it can be shown that the (j,k)th element $A_T(j,k)$ of A_T satisfies

$$\lim_{T\to\infty}\max_{1\le j,k\le T}\left|\frac{1}{T}A_T(j,k)-K^{(D)}\left(\frac{j}{T},\frac{k}{T}\right)\right|=0.$$

Thus it follows from Tanaka (1996, Chap. 5) and the consistency of $\tilde{\sigma}^2$ that the relation (20) holds for $R_{NT}^{(D)}$.

We next consider $R_{NT}^{(B)}$, for which we have

$$\tilde{\boldsymbol{\eta}}'\left(I_N\otimes\Omega^{-2}\right)\tilde{\boldsymbol{\eta}}\overset{\mathcal{D}}{=}\boldsymbol{\varepsilon}'\left(\alpha B_{NT}+\frac{c_N^2}{T^2}B_{NT}^2\right)\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}'\left(G_{NT}+\frac{c_N^2}{T^2}G_{NT}^2\right)\boldsymbol{\varepsilon}+o_p(T^2),$$

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where

$$
\begin{aligned}
B_{NT} &= \Phi_{NT} - \Phi_{NT} X (X' \Phi_{NT} X)^{-1} X' \Phi_{NT}, \quad \Phi_{NT} = I_N \otimes \Omega^{-1}, \\
X &= (I_N \otimes \boldsymbol{i}_T \quad \boldsymbol{i}_N \otimes \boldsymbol{d}_T), \quad G_{NT} = I_N \otimes P_T - \frac{1}{N} \boldsymbol{i}_N \boldsymbol{i}_N' \otimes Q_T, \\
P_T &= \Omega^{-1} - \frac{12}{T^3} \Omega^{-1} \boldsymbol{i}_T \boldsymbol{i}_T' \Omega^{-1} = \Big[\min(j,k) - \frac{jk}{T} - \frac{3}{T^3} jk(T-j)(T-k) + O(1) \Big], \\
Q_T &= \frac{180}{T^3} \Omega^{-1} \boldsymbol{i}_T \boldsymbol{i}_T' \Omega^{-1} - \frac{360}{T^4} \Omega^{-1} \boldsymbol{i}_T \boldsymbol{d}_T' \Omega^{-1} - \frac{360}{T^4} \Omega^{-1} \boldsymbol{d}_T \boldsymbol{i}_T' \Omega^{-1} + \frac{720}{T^5} \Omega^{-1} \boldsymbol{d}_T \boldsymbol{d}_T' \Omega^{-1} \\
&= \Big[\frac{45}{T^3} jk(T-j)(T-k) - \frac{30}{T^4} jk(T-j)(T^2-k^2) - \frac{30}{T^4} jk(T^2-j^2)(T-k) \\
&\qquad + \frac{20}{T^5} jk(T^2-j^2)(T^2-k^2) + O(1) \Big].
\end{aligned}
$$

Since it holds that $\tilde{\sigma}^2 \to \sigma^2$ in probability, and

$$
\lim_{T\to\infty} \max_{1 \le j,k \le T} \left| \frac{1}{T} P_T(j,k) - K_1\Big(\frac{j}{T}, \frac{k}{T}\Big) \right| = 0, \quad \lim_{T\to\infty} \max_{1 \le j,k \le T} \left| \frac{1}{T} Q_T(j,k) - K_2\Big(\frac{j}{T}, \frac{k}{T}\Big) \right| = 0,
$$

where $K_1 = K^{(B)}$ and $K_2 = K^{(B)} - K^{(D)}$, we have that

$$
\begin{aligned}
\frac{1}{T^2 \tilde{\sigma}^2} \varepsilon' G_{NT} \varepsilon &= \frac{1}{T^2 \tilde{\sigma}^2} \left[\sum_{i=1}^N \varepsilon_i' P_T \varepsilon_i - \frac{1}{N} \sum_{i=1}^N \sum_{k=1}^N \varepsilon_i' Q_T \varepsilon_k \right] \\
&\Rightarrow \sum_{i=1}^N \int_0^1 \int_0^1 K^{(B)}(r,s)\, dW_i(r)\, dW_i(s) \\
&\qquad - \frac{1}{N} \sum_{i=1}^N \sum_{k=1}^N \int_0^1 \int_0^1 \Big(K^{(B)}(r,s) - K^{(D)}(r,s) \Big) dW_i(r)\, dW_k(s) \\
&= \int_0^1 \int_0^1 K^{(B)}(r,s)\, d\boldsymbol{W}'(r)\, d\boldsymbol{W}(s) \\
&\qquad - \frac{1}{N} \int_0^1 \int_0^1 \Big(K^{(B)}(r,s) - K^{(D)}(r,s) \Big) d\boldsymbol{W}'(r) \boldsymbol{i}_N \boldsymbol{i}_N' d\boldsymbol{W}(s).
\end{aligned}
$$

Define $\boldsymbol{Z}(r) = H \boldsymbol{W}(r)$, where H is the $N \times N$ orthogonal matrix with the first row being $\boldsymbol{i}_N'/\sqrt{N}$ so that $\boldsymbol{W}'(r) \boldsymbol{i}_N = \boldsymbol{W}'(r) H' H \boldsymbol{i}_N = \boldsymbol{Z}'(r) \sqrt{N} (1, 0, \ldots, 0)' = \sqrt{N} Z_1(r)$. Then it holds that

$$
\begin{aligned}
\frac{1}{T^2 \tilde{\sigma}^2} \varepsilon' G_{NT} \varepsilon &\Rightarrow \int_0^1 \int_0^1 K^{(B)}(r,s)\, d\boldsymbol{Z}'(r)\, d\boldsymbol{Z}(s) \\
&\qquad - \int_0^1 \int_0^1 \Big(K^{(B)}(r,s) - K^{(D)}(r,s) \Big) dZ_1(r)\, dZ_1(s) \\
&= \int_0^1 \int_0^1 K^{(D)}(r,s)\, dZ_1(r)\, dZ_1(s) + \sum_{i=2}^N \int_0^1 \int_0^1 K^{(B)}(r,s)\, dZ_i(r)\, dZ_i(s).
\end{aligned}
$$

Since this last quantity can be expressed as

$$
S_N = \sum_{i=1}^N \int_0^1 \int_0^1 L_i(r,s)\, dZ_i(r)\, dZ_i(s), \quad L_1 = K^{(D)}, \quad L_i = K^{(B)} \ (i \ge 2),
$$

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it follows from Tanaka (1996, Chap. 5) that the relation (20) holds for M = B. The other cases can be proved similarly, which establishes Theorem 3. ■

Proof of Theorem 4. Let us consider Model B, for which we have, by putting $c_N = c/N^{1/4}$,

$$\mathrm{E}(R_N^{(B)}) = N\left(\frac{1}{15} + \frac{11c^2}{12600\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(R_N^{(B)}) = \frac{11N}{6300} + O(\sqrt{N}),$$

where we used computerized algebra to compute these moments. Then it follows from the CLT that

$$\frac{1}{\sqrt{N}}\left(R_N^{(B)} - \frac{N}{15}\right) \Rightarrow \mathrm{N}\left(\frac{11c^2}{12600}, \frac{11}{6300}\right),$$

which yields

$$\sqrt{\frac{6300}{11N}}\left(R_N^{(B)} - \frac{N}{15}\right) \Rightarrow \mathrm{N}\left(\frac{\sqrt{77}c^2}{420}, 1\right).$$

This result gives the expression for limiting powers for Model B. For the other models we have

$$\mathrm{E}(R_N^{(A)}) = N\left(\frac{1}{6} + \frac{c^2}{90\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(R_N^{(A)}) = \frac{N}{45} + O(\sqrt{N}),$$

$$\mathrm{E}(R_N^{(C)}) = N\left(\frac{1}{14} + \frac{c^2}{882\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(R_N^{(C)}) = \frac{N}{441} + O(\sqrt{N}),$$

$$\mathrm{E}(R_N^{(D)}) = N\left(\frac{3}{70} + \frac{11c^2}{44100\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(R_N^{(D)}) = \frac{11N}{22050} + O(\sqrt{N}).$$

Then the limiting powers for the other models can be obtained from these results, which establishes Theorem 4. ■

Proof of Theorem 5. Let us first consider the conditional case. It follows from Tanaka (1996, p. 376) that, as $T \to \infty$ with N fixed under $\alpha(c) = 1 - c_N/T$ and $\alpha(\theta) = 1 - \theta_N/T$,

$$Q_{NT}(\theta) \Rightarrow \frac{1}{N}\sum_{i=1}^{N} U_i(c,\theta),$$

where

$$U_i(c,\theta) = \int_0^1 \int_0^1 K^C(r,s)\, dW_i(r)\, dW_i(s) - \theta_N,$$

$$K^C(r,s) = c_N + c_N^2\left(1 - \max(r,s)\right) + (\theta_N - c_N)e^{-\theta_N|r-s|} - \frac{(\theta_N - c_N)^2}{2\theta_N}\left(e^{-\theta_N|r-s|} - e^{-\theta_N(2-r-s)}\right).$$

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Then we have

$$\mathrm{E}(U_i(c,\theta)) = \theta_N(c_N + O(c_N^2)) - \frac{\theta_N^2}{2} + O(c_N\theta_N^2),$$

$$\mathrm{Var}(U_i(c,\theta)) = 2\theta_N^2(1 + O(c_N)) + O(\theta_N^3),$$

which proves (27).

We next consider the stationary case. It follows from Tanaka (1996, p. 382) that, as $T \to \infty$ with N fixed under $\alpha(c)$ and $\alpha(\theta)$,

$$R_{NT}(\theta) \Rightarrow \frac{1}{N}\sum_{i=1}^{N} V_i(c,\theta),$$

where

$$V_i(c,\theta) = \theta_N^2 \sum_{n=1}^{\infty} \frac{n^2\pi^2 + c_N^2}{n^2\pi^2(n^2\pi^2 + \theta_N^2)} Z_{in}^2 - \theta_N, \quad \{Z_{in}\} \sim \mathrm{NID}(0,1).$$

Then it can be shown that

$$\begin{aligned} X_i(c,\theta) &= \frac{V_i(c,\theta) + \theta_N}{\theta_N^2} = \sum_{n=1}^{\infty} \frac{n^2\pi^2 + c_N^2}{n^2\pi^2(n^2\pi^2 + \theta_N^2)} Z_{in}^2 \\ &\overset{\mathcal{D}}{=} \int_0^1 \int_0^1 K^S(r,s)\, dW_i(r)\, dW_i(s), \\ K^S(r,s) &= \frac{1}{\theta_N^2}\Big[-1 - c_N + c_N(r+s) + c_N^2\Big(\min(r,s) - rs\Big) \\ &\quad + (\theta_N - c_N)e^{-\theta_N|r-s|} - \frac{(\theta_N - c_N)^2}{2\theta_N}\Big(e^{-\theta_N|r-s|} - e^{-\theta_N(2-r-s)}\Big) \\ &\quad + \frac{\theta_N e^{-\theta_N}}{\sinh\,\theta_N}\left(\cosh\,\theta_N(1-r) + \frac{c_N}{\theta_N}\sinh\,\theta_N(1-r)\right) \\ &\qquad \times \left(\cosh\,\theta_N(1-s) + \frac{c_N}{\theta_N}\sinh\,\theta_N(1-s)\right)\Big]. \end{aligned}$$

Now we obtain

$$\mathrm{E}(X_i(c,\theta)) = \frac{1}{6} + \frac{c_N^2}{90} - \frac{\theta_N^2}{90} + O(c_N^2\theta_N^2) + O(\theta_N^4),$$

$$\mathrm{Var}(X_i(c,\theta)) = \frac{1}{45} + O(c_N^2) + O(\theta_N^2),$$

which proves (28) by the CLT. ■

Proof of Theorem 6. The statistic $S_{NT}^{(H)}$ was dealt with in Tanaka (2017), which yields

$$\hat{\boldsymbol{\eta}}'\Big(I_N \otimes CC'\Big)\hat{\boldsymbol{\eta}} = \sum_{i=1}^{N} \hat{\boldsymbol{\eta}}_i' CC' \hat{\boldsymbol{\eta}}_i \overset{\mathcal{D}}{=} \sum_{i=1}^{N} \boldsymbol{\varepsilon}_i'\Big(C'MC + \frac{c_N^2}{T^2}(C'MC)^2\Big)\boldsymbol{\varepsilon}_i,$$

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where $M = I_T - X_T(X_T'X_T)^{-1}X_T'$ and $X_T = (\boldsymbol{i}_T \quad \boldsymbol{d}_T)$. It can be shown that, by denoting the (j,k)th element of $C'MC$ as $B_T(j,k)$,

$$\lim_{T\to\infty} \max_{1\le j,k\le T} \left| \frac{1}{T} B_T(j,k) - K^{(H)}\Big(\frac{j}{T}, \frac{k}{T}\Big) \right| = 0.$$

Thus it follows from Tanaka (1996, Chap. 5) and the consistency of $\tilde{\sigma}^2$ that the relation (41) holds for $S_{NT}^{(H)}$.

Let us next consider $S_{NT}^{(F)}$, which yields

$$\hat{\boldsymbol{\eta}}'\Big(I_N \otimes CC'\Big)\hat{\boldsymbol{\eta}} \overset{\mathcal{D}}{=} \varepsilon' \left[G_{NT} + \frac{c_N^2}{T^2} G_{NT}^2 \right] \varepsilon + o_p(T^2),$$

where

$$G_{NT} = I_N \otimes C'\Big(I_T - \frac{1}{T}\boldsymbol{i}_T\boldsymbol{i}_T'\Big)C - \frac{1}{N}\boldsymbol{i}_N\boldsymbol{i}_N' \otimes C'Q_TC,$$

$$Q_T = \frac{3}{T}\boldsymbol{i}_T\boldsymbol{i}_T' - \frac{6}{T^2}\boldsymbol{i}_T\boldsymbol{d}_T' - \frac{6}{T^2}\boldsymbol{d}_T\boldsymbol{i}_T' + \frac{12}{T^3}\boldsymbol{d}_T\boldsymbol{d}_T'.$$

Since we have

$$\varepsilon' G_{NT}\varepsilon = \sum_{i=1}^N \varepsilon_i' \left(C'\Big(I_T - \frac{1}{T}\boldsymbol{i}_T\boldsymbol{i}_T'\Big)C \right)\varepsilon_i - \frac{1}{N}\sum_{i=1}^N\sum_{k=1}^N \varepsilon_i' C'Q_TC\varepsilon_k$$

$$= \sum_{i=1}^N \varepsilon_i' E_T \varepsilon_i - \frac{1}{N}\sum_{i=1}^N\sum_{k=1}^N \varepsilon_i' F_T \varepsilon_k,$$

$$\lim_{T\to\infty} \max_{1\le j,k\le T} \left| \frac{1}{T} E_T(j,k) - K_3\Big(\frac{j}{T}, \frac{k}{T}\Big) \right| = 0, \quad \lim_{T\to\infty} \max_{1\le j,k\le T} \left| \frac{1}{T} F_T(j,k) - K_4\Big(\frac{j}{T}, \frac{k}{T}\Big) \right| = 0,$$

where $K_3 = K^{(F)}$ and $K_4 = K^{(F)} - K^{(H)}$, it follows that

$$\frac{1}{T^2\tilde{\sigma}^2}\varepsilon' G_{NT} \Rightarrow \sum_{i=1}^N \int_0^1 \int_0^1 K^{(F)}(r,s)\, dW_i(r)\, dW_i(s)$$

$$- \frac{1}{N}\sum_{i=1}^N\sum_{k=1}^N \int_0^1 \int_0^1 \Big(K^{(F)}(r,s) - K^{(H)}(r,s)\Big)\, dW_i(r)\, dW_k(s),$$

which proves the relation (41) for M=F by the same approach as used in the proof of Theorem 3. The other cases can be proved similarly, which establishes Theorem 6. ■

Proof of Theorem 7. Putting $c_N = c/N^{1/4}$, we have

$$\mathrm{E}(S_N^{(E)}) = N\left(\frac{1}{2} + \frac{c^2}{6\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(S_N^{(E)}) = \frac{N}{3} + O(\sqrt{N}),$$

$$\mathrm{E}(S_N^{(F)}) = N\left(\frac{1}{6} + \frac{c^2}{90\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(S_N^{(F)}) = \frac{N}{45} + O(\sqrt{N}),$$

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$$\mathrm{E}(S_N^{(G)}) = N\left(\frac{1}{10} + \frac{c^2}{350\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(S_N^{(G)}) = \frac{N}{175} + O(\sqrt{N}),$$

$$\mathrm{E}(S_N^{(H)}) = N\left(\frac{1}{15} + \frac{11c^2}{12600\sqrt{N}}\right) + O(1), \quad \mathrm{Var}(S_N^{(H)}) = \frac{11N}{6300} + O(\sqrt{N}).$$

Theorem 7 can be established from these moments and the CLT. ■

Proof of Theorem 8. It follows from Tanaka (1996, p. 413) that, as $T \to \infty$ with N fixed under $\rho(\theta)$ and the true value $\rho(c) = c_N^2/T^2$ with $c_N^2 = c^2/N^{2\kappa}$,

$$S_{NT}(\theta) \Rightarrow \frac{1}{N}\sum_{i=1}^{N} V_i(c, \theta),$$

where

$$V_i(c, \theta) = \theta_N^2 \sum_{n=1}^{\infty} \frac{(n-1/2)^2\pi^2 + c_N^2}{(n-1/2)^2\pi^2\left((n-1/2)^2\pi^2 + \theta_N^2\right)} Z_{in}^2, \quad \{Z_{in}\} \sim \mathrm{NID}(0, 1).$$

The first two moments of $V_i(c, \theta)$ can be computed from its m.g.f. given in Tanaka (1996, Chap. 10) by

$$m(x) = \mathrm{E}(e^{xV_i(c,\theta)}) = \left[\cosh\sqrt{a(x)+b(x)}\,\cosh\sqrt{a(x)-b(x)}\,\Big/\,\cosh\theta_N^2\right]^{-1/2},$$

where

$$a(x) = \frac{(1-2x)\theta_N^2}{2}, \quad b(x) = \frac{1}{2}\sqrt{(1-2x)^2\theta_N^4 + 8c_N^2\theta_N^2 x}.$$

Then we have, by differentiating $m(x)$,

$$\mathrm{E}(V_i(c, \theta)) = \frac{\theta_N^2}{2} + \frac{c_N^2\theta_N^2}{6} - \frac{\theta_N^4}{6} + O(c_N^2\theta_N^4) + O(\theta_N^6),$$

$$\mathrm{Var}(V_i(c, \theta)) = \frac{\theta_N^4}{3} + O(c_N^2\theta_N^4) + O(\theta_N^6),$$

which yields, as $N \to \infty$ under $\theta_N^2 = \theta^2/\sqrt{N}$ and $c_N^2 = c^2/\sqrt{N}$,

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{V_i(c, \theta)}{\theta_N^2} - \frac{1}{2} + \frac{\theta_N^2}{6}\right) \Rightarrow \mathrm{N}\left(\frac{c^2}{6}, \frac{1}{3}\right).$$

This leads us to the asymptotic distribution of $S_{NT}(\theta)$ and the limiting power, which establishes Theorem 8. ■

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