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No semiconjugacy to a map of constant slope

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Abstract. We study countably piecewise continuous, piecewise monotone interval maps. We establish a necessary and sufficient criterion for the existence of a non-decreasing semiconjugacy to a map of constant slope in terms of the existence of an eigenvector of an operator acting on a space of measures. Then we give sufficient conditions under which this criterion is not satisfied. Finally, we give examples of maps not semiconjugate to a map of constant slope via a non-decreasing map. Our examples are continuous and transitive.

1. *Introduction*

The idea that some interval maps should be conjugate or semiconjugate to maps of constant slope (we use the term 'constant slope' instead of the more accurate but clumsy 'constant absolute value of the slope') appeared first about 50 years ago. Parry [**10**] proved that continuous, transitive, piecewise monotone (with finite number of pieces) interval maps are conjugate to maps of constant slope. Later, Milnor and Thurston [**8**] proved an analogous result, removing the assumption of transitivity, but replacing conjugacy by semiconjugacy. Another proof of the Milnor–Thurston theorem appeared in [**1**]. That proof, after small modifications, can be used for maps of graphs or for piecewise continuous maps (with finitely many pieces) [**2**].

In all cases we require that the (semi)conjugacy is via monotone maps preserving orientation. This is a natural requirement; if we drop it then we get a completely different, and less interesting, problem. Also in all cases the logarithm of the slope is equal to the topological entropy of the initial map. This is because the same is true for the constant slope maps (see [**1**, **9**]).

A natural question is what can be said if the map is piecewise monotone, but with *countably* many pieces. When trying to make such generalizations, one immediately encounters some basic problems.

The first problem is a definition of a countably piecewise monotone map. For such continuous maps, what should we assume about the set of turning points (local extrema)?

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For instance, if we allow the closure of this set to be a Cantor set, there may be substantial dynamics on it, not captured by our considerations. Thus, it is reasonable to assume that the closure of the set of turning points is countable.

The second problem is: what should the slope be? For countably piecewise monotone maps of constant slope it is no longer true that the entropy is the logarithm of the slope. There are obvious counterexamples, where all points of the interval, except the endpoints, are moved to the right, so the entropy is zero but the slope is larger than 1. Thus, there is no natural choice of the slope of the map to which our map should be (semi)conjugate.

Recently, Bobok [**6**] considered the case of continuous, Markov, countably piecewise monotone interval maps. He found a necessary and sufficient condition for the existence of a non-decreasing semiconjugacy to a map of constant slope in terms of the existence of an eigenvector for a certain operator. The operator is given by a countably infinite 0-1 matrix representing the transitions in the Markov system, and the criterion asks for a non-negative eigenvector in the sequence space ℓ^1. Bobok described a rich class of examples satisfying this criterion and proved that for many of these examples the constant slope so obtained is the exponential of the topological entropy of the original interval map. However, he did not give any examples that violate the criterion.

In the present paper, we study the general case of countably piecewise continuous, piecewise monotone interval maps without any Markov assumption. We also establish a necessary and sufficient criterion—analogous to Bobok's—for the existence of a non-decreasing semiconjugacy to a map of constant slope. It is given, like Bobok's criterion, in terms of existence of an eigenvector of some operator, but the operator acts on measures rather than on sequences. Then we construct a class of examples which violate that criterion. Our examples are continuous and transitive and thus have positive topological entropy.

The paper is organized as follows. In §2 we present notation and define objects used in the next sections. In §3 we establish the criterion for the semiconjugacy to a map of constant slope. In §4 we produce sufficient conditions for this criterion being not satisfied. Theorem 4.5 can be considered the main technical result of the paper. Section 5 contains the main theorem of the paper, Theorem 5.1, which gives us a large class of continuous transitive interval maps for which Theorem 4.5 applies. Finally, in §6 we give a concrete example of a one-parameter family of continuous transitive countably piecewise monotone interval maps that are not semiconjugate to a map of constant slope via a non-decreasing map. We show that in this family there are uncountably many Markov and uncountably many non-Markov maps.

2. *Notation and definitions*

Let us introduce the various notations and definitions required to address the problem more clearly.

If P is a closed, countable subset of $[0, 1]$, then a component of the complement of P will be called a *P-basic interval*, and the set of all P-basic intervals will be denoted $\mathcal{B}(P)$.

We want to consider countably piecewise monotone interval maps, but not only continuous, but also piecewise continuous. That would mean that there exists a closed countable set $P \subset [0, 1]$ such that our map is continuous and monotone on each P-basic

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interval. However, this creates a question: what should be the values of our map at the points of P? If the map is continuous, this is not a problem. However, in general there is no good answer. Even if we allow two values at those points (one-sided limits from both sides), P may have accumulation points, and there is no natural way of extending our map to those points. Therefore we choose the simplest solution—we do not define the map at all at the points of P. This is not a new idea; a similar solution is normally used for instance in the holomorphic dynamics on the complex projective spaces of dimension larger than 1.

Thus, we define a class $\mathcal{C}$ of maps f for which there exists a closed, countable set $P \subset [0, 1]$, $f : [0, 1] \setminus P \to [0, 1]$, and f is continuous and strictly monotone on each P-basic interval. Note that we assume strict monotonicity; while it is possible to do everything that we do assuming only monotonicity, the technical details would be much more involved and they would obscure the ideas.

Similarly as in measure theory where two functions are considered equal if they differ only on a set of measure zero, we will consider two elements of $\mathcal{C}$ equal if they are equal on the complement of a closed countable set. This gives us a possibility of using different sets P for a given map $f \in \mathcal{C}$. Each such set for which f is continuous and strictly monotone on P-basic intervals will be called f-admissible.

LEMMA 2.1. *A composition of two maps from $\mathcal{C}$ belongs to $\mathcal{C}$.*

Proof. We will show that if $f, g \in \mathcal{C}$, the set P is f-admissible, and Q is g-admissible, then the set $P \cup f^{-1}(Q)$ is $g \circ f$-admissible. First observe that the set $(f|_I)^{-1}(Q)$ is countable for every P-basic interval I. Moreover, there are only countably many P-basic intervals. Thus, the set $P \cup f^{-1}(Q)$ is countable.

We may assume that $0, 1 \in P$. Let $[a, b]$ be the closure of a P-basic interval. Then $(f|_{(a,b)})^{-1}(Q)$ is closed in (a, b). Since $a, b \in P$, the set $(P \cup f^{-1}(Q)) \cap [a, b]$ is closed in $[a, b]$. Since P is closed in $[0, 1]$, this proves that $P \cup f^{-1}(Q)$ is closed in $[0, 1]$.

Let I be a component of the complement of $P \cup f^{-1}(Q)$. Then I is a subset of a P-basic interval and $f(I)$ is a subset of a Q-basic interval, so $g \circ f$ is continuous and strictly monotone on I. □

Using this lemma, by induction we get that if $f \in \mathcal{C}$ then $f^n \in \mathcal{C}$ for every natural n. That is, we can iterate a map from $\mathcal{C}$ without leaving this class of maps.

We do not want to abandon continuous maps. Therefore we consider the class $\mathcal{CC}$ of continuous maps $f : [0, 1] \to [0, 1]$ for which there exists a countable closed set $P \subset [0, 1]$ such that $f|_{[0,1]\setminus P} \in \mathcal{C}$. Results for maps from this class will follow easily from the results for maps from $\mathcal{C}$. In view of Lemma 2.1, a composition of maps from $\mathcal{CC}$ belongs to $\mathcal{CC}$, so in particular, iterates of a map from $\mathcal{CC}$ belong to $\mathcal{CC}$.

We will say that a map $f \in \mathcal{C}$ (or $f \in \mathcal{CC}$) has *constant slope* λ, if for some f-admissible set P, f restricted to each P-basic interval is affine with slope of absolute value λ. Clearly, this property depends only on the map f, and not on the choice of an f-admissible set P.

We will say that a non-atomic measure defined on the Borel σ-algebra on the interval $[0, 1]$ is *strongly σ-finite* if there is a closed countable set $P \subset [0, 1]$ such that each P-basic interval has finite measure. We denote by $\mathcal{M}$ the set of all such measures. Observe that $\mathcal{M}$ is closed under addition and under multiplication by positive real scalars.

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If $g : X \to Y$ is a map and ν is a measure on X, then we can always push this measure forward and define a measure $g_*\nu$ on Y by the formula $g_*\nu(A) = \nu(g^{-1}(A))$. If g is invertible and μ is a measure on Y, we can push this measure to X by g^{-1}. This defines a pull-back of μ, that is, $g^*\mu = g_*^{-1}\mu$. In terms of measures of sets, we have $g^*\mu(A) = \mu(g(A))$.

For a map $f \in \mathcal{C}$, we put all those pull-backs together to get an operator $T_f : \mathcal{M} \to \mathcal{M}$. It acts on a measure $\mu \in \mathcal{M}$ as follows. Choose an f-admissible set P. For each P-basic interval I, consider the homeomorphism $f|_I : I \to f(I)$. Pull back the measure $\mu|_{f(I)}$ by $f|_I$ to a measure on I. This defines $T_f\mu$ on the interval I:

$$(T_f\mu)|_I = (f|_I)^*(\mu|_{f(I)}), \quad I \in \mathcal{B}(P).$$

More explicitly,

$$(T_f\mu)(A) = \sum_{I \in \mathcal{B}(P)} \mu(f(I \cap A))$$

for all Borel sets A. By a common refinement argument, the definition of T_f depends only on the map f, and not on the choice of an f-admissible set P. Moreover, as in the proof of Lemma 2.1, if for a closed countable set Q each Q-basic interval has measure μ finite, then each R-basic interval, where $R = P \cup f^{-1}(Q)$, has measure $T_f\mu$ finite. This shows that indeed T_f maps $\mathcal{M}$ to $\mathcal{M}$.

Note the linearity properties $T_f(\mu + \nu) = T_f\mu + T_f\nu$ and $T_f(\alpha\mu) = \alpha T_f\mu$ for $\mu, \nu \in \mathcal{M}$ and $\alpha \geq 0$.

Instead of maps of the interval [0, 1] into itself, we can consider maps of the circle or of the real line into itself. We will use for them the same notation as for the interval maps.

3. *Semiconjugacy*

Although $\mathcal{M}$ is not a true linear space (multiplication by negative scalars is not permitted), it is nevertheless quite fruitful to consider eigenvectors for positive eigenvalues. Let us consider the meaning of an eigenvector in this setting. Fix a map $f \in \mathcal{C}$ and an f-admissible set P. The condition $T_f\mu = \lambda\mu$, $\lambda > 0$, is equivalent to the condition that for all P-basic intervals I, $(f|_I)^*(\mu|_{f(I)}) = \lambda\mu|_I$. We will suppress subscripts and write $f^*\mu = \lambda\mu$ when the context is clear. However, if $f^*\mu = \lambda\mu$ on a P-basic interval I, then the Radon–Nikodym derivative $(df^*\mu)/d\mu$ is identically λ on I. This Radon–Nikodym derivative is the measure-theoretic version of the Jacobian of f on I as defined by Parry [**11**]. Thus, a measure $\mu \in \mathcal{M}$ satisfying $T_f\mu = \lambda\mu$ is a measure of constant Jacobian λ for f. If μ is an eigenvector for T_f, then for any subinterval J contained in a single P-basic interval I, $\mu(f(J)) = \lambda\mu(J)$. In words, if μ is an eigenvector for T_f, then within each P-basic interval, f uniformly stretches the μ measures of intervals by a factor λ. It is suggestive to note that constant slope maps have the same property, but with lengths in place of measures. We will show that this has a deeper meaning. Let us denote the Lebesgue measure by m.

LEMMA 3.1. *The map $f \in \mathcal{C}$ has constant slope λ if and only if $T_f m = \lambda m$.*

Proof. Assume that f has constant slope λ and let P be f-admissible. Then f is affine with slope $\pm\lambda$ on each P-basic interval. Consider an arbitrary P-basic interval I and the

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restricted map $f|_I : I \to f(I)$. It suffices to prove that $f^*m = \lambda m$ on this interval. For every subinterval $(a, b) \subset I$ we have

$$(f^*m)(a, b) = m(f((a, b))) = |f(b) - f(a)| = \lambda(b - a) = (\lambda m)(a, b).$$

Since the two Borel measures f^*m and λm on I agree on all open intervals, they are in fact equal.

Conversely, assume that $T_f m = \lambda m$. Let I be a P-basic interval. Then for every subinterval $(a, b) \subset I$ we have

$$|f(b) - f(a)| = m(f((a, b))) = (f^*m)(a, b) = \lambda m(a, b) = \lambda(b - a).$$

Therefore $f|_I$ is affine with slope $\pm\lambda$. But I was arbitrary. Therefore f has constant slope λ. □

Remark 3.2. With only slight modifications in wording, the proof of Lemma 3.1 goes through for maps of the real line into itself.

THEOREM 3.3. *Let $f \in \mathcal{C}$ and let $\lambda > 0$. Then f is semiconjugate via a non-decreasing map φ to some map $g \in \mathcal{C}$ of constant slope λ if and only if there exists a probability measure $\mu \in \mathcal{M}$ such that $T_f \mu = \lambda\mu$.*

Proof. Assume that f is semiconjugate to a map g of constant slope λ by a non-decreasing map φ. Let P be an f-admissible set; it is clear that then $\varphi(P)$ is a g-admissible set. By Lemma 3.1, $T_g m = \lambda m$. Since m is non-atomic, it can be pulled back by the non-decreasing map φ to define a measure $\mu = \varphi^*m$; explicitly, $\mu(A) = m(\varphi(A))$ for all Borel sets A. Then μ is a non-atomic Borel probability measure. Now let I be any P-basic interval. Take restrictions of f, g, φ, m, and μ to the appropriate domains in the following commutative diagram:

$$\begin{array}{ccc} I & \xrightarrow{f} & f(I) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \varphi(I) & \xrightarrow[g]{} & g(\varphi(I)) \end{array} \tag{1}$$

Using these restricted maps and measures, $f^*\mu$ may be computed on I as

$$f^*\mu = f^*(\varphi^*m) = \varphi^*(g^*m) = \varphi^*(\lambda m) = \lambda\varphi^*m = \lambda\mu.$$

But I was an arbitrary P-basic interval. Therefore $T_f\mu = \lambda\mu$.

Conversely, assume that there exists a probability measure $\mu \in \mathcal{M}$ such that $T_f\mu = \lambda\mu$. Define a map φ by $\varphi(x) = \mu([0, x])$. This map is continuous, non-decreasing, and maps $[0, 1]$ onto $[0, 1]$. To see that φ induces a well-defined factor map g, suppose that $x_1 < x_2$ and $\varphi(x_1) = \varphi(x_2)$. If the interval $[x_1, x_2]$ contains a point of P, then $\varphi(x_i)$ belongs to $\varphi(P)$, which will be a g-admissible set, so there is no need to define g at $\varphi(x_i)$. Otherwise, the interval $[x_1, x_2]$ is contained in a P-basic set I, $\mu([x_1, x_2]) = 0$, and thus

$$\mu(f([x_1, x_2])) = (T_f\mu)([x_1, x_2]) = \lambda\mu([x_1, x_2]) = 0.$$

Therefore $\varphi(f(x_1)) = \varphi(f(x_2))$. This shows that we can define our map g by the equation $g(\varphi(x)) = \varphi(f(x))$, and such g will be monotone and continuous on every $\varphi(P)$-basic

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interval. By construction, $\varphi_*\mu$ is the Lebesgue measure m. It remains to consider $T_g m$. Using the same restricted maps and measures as in diagram (1), we get

$$g^*m = g^*(\varphi_*\mu) = \varphi_*(f^*\mu) = \varphi_*(\lambda\mu) = \lambda\varphi_*\mu = \lambda m.$$

Since I was arbitrary, this shows that $T_g m = \lambda m$. By Lemma 3.1, g has constant slope λ. □

Remark 3.4. If $f \in \mathcal{CC}$, then the only way we could get a discontinuity of g in the above construction was when $x_1 < x_2$, $\varphi(x_1) = \varphi(x_2)$, and there is a point of P between x_1 and x_2. However, then

$$\mu(f([x_1, x_2])) = \mu\left(\bigcup_{I\in\mathcal{B}(P)} f(I \cap [x_1, x_2])\right) \le \sum_{I\in\mathcal{B}(P)} \mu(f(I \cap [x_1, x_2]))$$
$$= (T_f\mu)([x_1, x_2]) = \lambda\mu([x_1, x_2]) = 0.$$

By the continuity of f, the set $f([x_1, x_2])$ includes the interval with endpoints $f(x_1)$, $f(x_2)$. Therefore $\varphi(f(x_1)) = \varphi(f(x_2))$, so no discontinuity is created. This shows that Theorem 3.3 holds with $\mathcal{C}$ replaced by $\mathcal{CC}$, for both f and g.

Remark 3.5. With only slight modifications in wording, the proof of Theorem 3.3 and the considerations in Remark 3.4 go through for circle maps.

4. *No semiconjugacy*

Now we want to find conditions that prevent semiconjugacy to a map of constant slope. We start with a technical lemma. One of our assumptions is that $\lambda > 2$. For piecewise monotone maps with finite number of pieces this type of an assumption is usually circumvented by taking a sufficiently high iterate of the map. If $\lambda > 1$ then for some large n we get $\lambda^n > 2$. However, here we have another assumption, that the measures of P-basic intervals are bounded away from 0, and taking an iterate of a map could lead to this condition being violated.

LEMMA 4.1. *Let $f \in \mathcal{C}$. Suppose that there exist $\lambda > 2$, $\delta > 0$, $\mu \in \mathcal{M}$ and an f-admissible set P such that $T_f\mu = \lambda\mu$ and the measure of every P-basic interval I satisfies $\delta \le \mu(I) < \infty$. Then for μ almost every x in $[0, 1]$ there exist infinitely many times $n_1 < n_2 < \cdots$ such that x belongs to an interval which is mapped monotonically by f^{n_k} to an interval of μ-measure at least δ.*

Proof. Fix an arbitrarily large natural number N and choose an arbitrary P^N-basic interval J. A P^N-basic interval means a component of the complement of $\bigcup_{i=0}^{N-1} f^{-i}(P)$; thus, f^N is monotone and continuous on each P^N-basic interval. It suffices to prove that for μ almost every x in J there exists a time $n \ge N$ and a P^n-basic interval $L \subset J$ such that $x \in L$ and $\mu(f^n(L)) \ge \delta$.

If $\mu(f^N(J)) \ge \delta$, then we are done. Otherwise, J is a 'bad' interval, and we subdivide it at all the points of intersection $J \cap f^{-N}(P)$ into P^{N+1}-basic intervals, which we classify as either 'good' or 'bad' according as $\mu(f^{N+1}(L))$ is either at least δ or smaller than δ, respectively. For points x in the good P^{N+1}-basic intervals, the claim holds. But if

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any of these intervals L is bad, we subdivide it further at all the points of intersection $L \cap f^{-(N+1)}(P)$ into P^{N+2}-basic intervals, which we then classify as good or bad, and so on.

To be more precise, we define $\mathcal{B}_0 = \{J\}$ and we recursively define

$$\mathcal{B}_{i+1} = \{M \in \mathcal{B}(P^{N+i+1}) : \mu(f^{N+i+1}(M)) < \delta \text{ and } \exists L \in \mathcal{B}_i,\ M \subset L\}.$$

Now observe that each bad interval L at stage i subdivides into at most two bad intervals at stage $i+1$, because $\#(L \cap f^{-(N+i)}(P)) = \#(f^{N+i}(L) \cap P)$, and by hypothesis, an interval of measure less than δ never contains more than one point of P. It follows that $\#\mathcal{B}_i \le 2^i$. The hypothesis $T_f\mu = \lambda\mu$ means that wherever f is monotone and continuous, it stretches μ measures uniformly by the factor λ. This provides an upper bound on the measures of bad intervals. If $L \in \mathcal{B}_i$, then

$$\mu(L) = \lambda^{-(N+i)}\mu(f^{N+i}(L)) \le \lambda^{-(N+i)}\delta.$$

It follows that

$$\sum_{L \in \mathcal{B}_i} \mu(L) \le \left(\frac{2}{\lambda}\right)^i \lambda^{-N}\delta,$$

and this quantity tends to zero as $i \to \infty$. Therefore almost every point of J falls at some stage of the process into a good basic interval, and this proves our claim. □

The following lemma is an analog of Lebesgue's density theorem. While it is known, it is difficult to find in the literature the statement we want. Usually statements with one-sided neighborhoods are only about the Lebesgue measure, while statements about more general measures use balls around the density point. Therefore we show how to deduce what we need from the statement about the Lebesgue measure.

LEMMA 4.2. *Let $\mu \in \mathcal{M}$ and let A be a Borel set. Then for μ almost every $x \in A$ the measures of all one-sided neighborhoods of x are positive, and*

$$\lim_{\delta \searrow 0} \frac{\mu(A \cap [x, x+\delta))}{\mu([x, x+\delta))} = \lim_{\delta \searrow 0} \frac{\mu(A \cap (x-\delta, x])}{\mu((x-\delta, x])} = 1. \tag{2}$$

Proof. Let P be a countable closed subset of $[0, 1]$ such that the measure of every P-basic interval is finite. Let L be a P-basic interval, and let a denote the left endpoint of L. It suffices to prove the claim for μ almost every x in $A \cap L$; therefore we may restrict everything to L.

There may exist closed subintervals of L of measure zero. There are countably many of such maximal intervals, so their union has measure zero. Hence, we are free to remove this union, as well as the endpoints of L, from A. Then, for each $x \in A$ and each positive δ, the measures $\mu([x, x+\delta))$ and $\mu((x-\delta, x])$ are non-zero. Since we restrict everything to L, and $\mu(L)$ is finite, we see that the ratios under consideration have finite numerators and denominators, and so are well-defined.

Introduce a map $\varphi : L \to Y$, where $Y = [0, \mu(L)]$, by $\varphi(x) = \mu((a, x))$. By construction, φ is non-decreasing. Moreover, φ is continuous because μ is non-atomic.

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By the definition of φ, every interval $I \subset Y$ enjoys the property $m(I) = \mu(\varphi^{-1}(I))$. It follows that this property holds for all measurable sets $I \subset Y$. Therefore

$$\lim_{\delta \searrow 0} \frac{\mu(A \cap [x, x+\delta))}{\mu([x, x+\delta))} = \lim_{\eta \searrow 0} \frac{m(\varphi(A) \cap [\varphi(x), \varphi(x)+\eta))}{m([\varphi(x), \varphi(x)+\eta))} \tag{3}$$

and

$$\lim_{\delta \searrow 0} \frac{\mu(A \cap (x-\delta, x])}{\mu((x-\delta, x])} = \lim_{\eta \searrow 0} \frac{m(\varphi(A) \cap (\varphi(x)-\eta, \varphi(x)])}{m((\varphi(x)-\eta, \varphi(x)])}. \tag{4}$$

The preimage under φ of a set of full Lebesgue measure in Y has full μ measure in L. By the Lebesgue density theorem (see, e.g. [**7**]), the limits of the right-hand sides of (3) and (4) are 1 for Lebesgue almost all $x \in \varphi(A)$. Therefore (2) holds for μ almost all $x \in A$. □

In the next theorem we need an assumption stronger than transitivity. For a continuous map f on a topological space X, the usual definition of (topological) transitivity is that for every pair of non-empty open sets U and V in X, there is a positive integer k such that $f^k(U) \cap V \neq \emptyset$. The term 'strong transitivity' is sometimes used for the stronger property that for every non-empty open set U, the union $\bigcup_{n=0}^{\infty} f^n(U)$ is the whole space X. Let us make an appropriate modification of this notion for the class $\mathcal{C}$ in which countable closed sets are negligible. We will say that a map $f \in \mathcal{C}$ is *substantially transitive* if for every non-empty open set $U \in [0, 1]$ the set $[0, 1] \setminus \bigcup_{n=0}^{\infty} f^n(U)$ has countable closure.

If $f \in \mathcal{CC}$ is transitive, we get substantial transitivity automatically. We will need this later also for continuous circle maps, so we will state a lemma for graph maps. Again, this lemma is known, but it is easier to prove it than to look for it in the literature.

LEMMA 4.3. *Let X be a graph and let f be a topologically transitive continuous map of X to X. Let U be a non-empty open subset of X. Then the set $X \setminus \bigcup_{n=0}^{\infty} f^n(U)$ is finite.*

Proof. By replacing U by one of its connected components, we may assume that U is connected. Let V be the connected component of the set $W = \bigcup_{n=0}^{\infty} f^n(U)$ that contains U. By transitivity, there is N such that $f^N(V) \cap V \neq \emptyset$. However, V is a component of a forward invariant set, so $f^N(V) \subset V$. It follows that $W = \bigcup_{n=0}^{N-1} f^n(V)$, and thus W has only finitely many connected components. On the other hand, by transitivity W is dense in X. Therefore W excludes only finitely many points of X. □

Remark 4.4. If $X = \mathbb{R}$ and f is a lifting of a degree one circle map, the same proof shows that $\bigcup_{n=0}^{\infty} f^n(U) = \mathbb{R}$. For such f there is a constant $M > 0$ such that $|f(x) - x| < M$ for every $x \in \mathbb{R}$, so W is contained in the set of points whose distance from V is smaller than NM. Since V is connected and W is dense, we must have $V = \mathbb{R}$.

Now we can prove the main technical result of the paper.

THEOREM 4.5. *Let $f \in \mathcal{C}$ be a substantially transitive map and let $\lambda > 2$. Assume that there exist $\delta > 0$, an infinite measure $\mu \in \mathcal{M}$, and an f-admissible set P, such that $T_f\mu = \lambda\mu$ and the measure of every P-basic interval I satisfies $\delta \leq \mu(I) < \infty$. Then there is no probability measure $\nu \in \mathcal{M}$ such that $T_f\nu = \lambda\nu$.*

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Proof. Suppose that such measure ν exists. The measure $\mu + \nu$ is an infinite measure in $\mathcal{M}$ such that $T_f(\mu + \nu) = \lambda(\mu + \nu)$ and for every P-basic interval I we have $\delta \le (\mu + \nu)(I) < \infty$. Replace μ by $\mu + \nu$ if necessary to obtain absolute continuity of ν with respect to μ. Now take the Radon–Nikodym derivative, $\xi = d\nu/d\mu$ and write $d\nu = \xi d\mu$. Integrate the function $\xi \circ f$ over any Borel set A contained in any P-basic interval I:

$$\int_A \xi \circ f \, d\mu|_I = \int_{f(A)} \xi \, df_*(\mu|_I) = \frac{1}{\lambda} \int_{f(A)} \xi \, d\mu|_{f(I)} = \frac{1}{\lambda} \int_{f(A)} d\nu|_{f(I)}$$
$$= \int_{f(A)} df_*(\nu|_I) = \int_A 1 \circ f \, d\nu|_I = \int_A d\nu|_I = \int_A \xi \, d\mu|_I.$$

This shows that the equality $\xi \circ f = \xi$ holds μ almost everywhere; that is, that up to a set of μ measure zero, the function ξ is constant along the orbits of f.

By the definition of the Radon–Nikodym derivative, $\int_0^1 \xi \, d\mu = \nu([0, 1]) = 1$. Therefore there exists a positive real number ε such that the measurable set $E = \xi^{-1}([\varepsilon, \infty))$ has positive μ measure. This measure cannot be infinite, because then $\int_0^1 \xi \, d\mu$ would be infinite. Thus, $0 < \mu(E) < \infty$. Because ξ is constant along orbits, this set E is fully invariant; that is, $f^{-1}(E) = E$. While this is μ almost everywhere, we can modify E by adding/subtracting a set of μ measure zero so that it holds everywhere.

The plan of the rest of the proof is as follows. We use Lemma 4.2 to get high density of E in a small interval, then Lemma 4.1 to transport it to a long interval, and then substantial transitivity to transport it to the whole space. By the invariance of E this construction gives us infinite measure of E, which is impossible.

There is a point $x \in E$ which satisfies conclusions of both Lemmas 4.1 and 4.2 (with $A = E$). In particular, there exists a sequence of times n_k and a sequence of intervals $L_k = [a_k, b_k]$ containing x such that f^{n_k} is monotone on L_k and such that $\mu(f^{n_k}(L_k)) \ge \delta$. Trimming the intervals L_k (but keeping $x \in L_k$), we can achieve equality $\mu(f^{n_k}(L_k)) = \delta$. Therefore $\mu(L_k) = \lambda^{-n_k}\delta$, and this decreases to zero. But every neighborhood (both two-sided and one-sided) of x has positive μ measure. Therefore, $a_k \to x$ and $b_k \to x$ as $k \to \infty$. Now we may use the fact that x is a density point of E to conclude that $\mu(E \cap L_k)/\mu(L_k) \to 1$.

Next, we show the density of E in an interval at the large scale. By compactness, after passing to subsequences we may assume that $f^{n_k}(a_k)$ converges to some point $a \in [0, 1]$ and $f^{n_k}(b_k)$ converges to some point $b \in [0, 1]$. Let L denote the interval $[a, b]$. We claim that the points a and b are distinct. Indeed, if $a = b$, then for each j take the interval $(a - 1/j, a + 1/j)$. It contains infinitely many intervals $f^{n_k}(L_k)$ and $\mu(E \cap f^{n_k}(L_k))$ is approaching δ, so the measure of $E \cap (a - 1/j, a + 1/j)$ is at least δ. Since the measure of E is finite, we may send j to infinity and find by the continuity of measure that there is an atom at a, which is a contradiction. Thus $a \ne b$. As k grows, the endpoints $f^{n_k}(a_k)$, $f^{n_k}(b_k)$ of $f^{n_k}(L_k)$ eventually draw nearer to the respective endpoints a, b of L than half the distance between a and b. Therefore, for sufficiently large k, the symmetric difference $f^{n_k}(L_k) \triangle L$ consists of two intervals; one with endpoints a, $f^{n_k}(a_k)$, and the other with endpoints b, $f^{n_k}(b_k)$. Again by the continuity of measure, each of these intervals has μ measure converging to zero. Therefore $\mu(L \triangle f^{n_k}(L_k)) \to 0$ as $k \to \infty$.

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Together with the invariance of E and the monotonicity of f^{n_k} on L_k, this shows that

$$\frac{\mu(E\cap L)}{\mu(L)} \ge \lim_{k\to\infty} \frac{\mu(E\cap f^{n_k}(L_k)) - \mu(f^{n_k}(L_k)\triangle L)}{\mu(f^{n_k}(L_k)) + \mu(f^{n_k}(L_k)\triangle L)} = \lim_{k\to\infty} \frac{\mu(E\cap f^{n_k}(L_k))}{\mu(f^{n_k}(L_k))}$$
$$= \lim_{k\to\infty} \frac{\mu(f^{n_k}(E\cap L_k))}{\mu(f^{n_k}(L_k))} = \lim_{k\to\infty} \frac{\lambda^{n_k}\cdot\mu(E\cap L_k)}{\lambda^{n_k}\cdot\mu(L_k)} = 1.$$

Therefore $E\cap L$ has full measure in the interval L; that is, $\mu(L\setminus E)=0$.

If the invariant set E fills L, then it must also fill all the images $f^n(L)$, $n\in\mathbb{N}$. Indeed, $\mu(f(L)\setminus E)=\mu(f(L\setminus E))$ by the f-invariance of E. However,

$$\mu(f(L\setminus E)) \le \sum_{I\in\mathcal{B}(P)} \mu(f(I\cap(L\setminus E))) = (T_f\mu)(L\setminus E) = \lambda\mu(L\setminus E) = 0.$$

This shows that $\mu(f(L)\setminus E)=0$, and it follows inductively that $\mu(f^n(L)\setminus E)=0$ for all $n\in\mathbb{N}$.

The interval L has non-empty interior, so by substantial transitivity of f the set $\bigcup_{n=0}^{\infty} f^n(L)$ excludes at most countably many points of $[0,1]$, and hence has full μ measure in $[0,1]$. But E has full measure in $\bigcup_{n=0}^{\infty} f^n(L)$, and therefore E has full measure in $[0,1]$. This is a contradiction because E has finite μ measure, while by the assumption, $\mu([0,1])=\infty$. Therefore it is impossible for a probability measure $\nu\in\mathcal{M}$ to satisfy $T_f\nu=\lambda\nu$. □

5. *Liftings of circle maps*

While Theorem 4.5, together with Theorem 3.3, provides sufficient conditions for a map $f\in\mathcal{C}$ to have no semiconjugacy to a map of constant slope, it is not immediately obvious how to construct concrete examples. In particular, even if we use those theorems to exclude semiconjugacy to a map with a given slope λ, how can we exclude other λs? In this section we provide tools to do it for a large class of maps. These maps additionally will be continuous (formally, they will be restrictions of continuous maps to $[0,1]\setminus P$). We denote the circle $\mathbb{R}/\mathbb{Z}$ by $\mathbb{S}^1$.

THEOREM 5.1. *Assume that $f:\mathbb{S}^1\to\mathbb{S}^1$ is a continuous degree one map that is piecewise monotone with finitely many pieces and has constant slope $\lambda>1$. Assume also that f has a lifting $F:\mathbb{R}\to\mathbb{R}$ that is topologically transitive. Take any continuous interval map $g:[0,1]\to[0,1]$ such that $g|_{(0,1)}$ is topologically conjugate to F. Then there does not exist any non-decreasing semiconjugacy of g to an interval map of constant slope.*

Proof. Fix n sufficiently large so that $\lambda^n>2$ and consider the iterates g^n, f^n, and F^n. The map $g^n|_{(0,1)}$ is conjugate to F^n and F^n is a lifting of the degree one circle map f^n of constant slope λ^n. *A priori*, a transitive map need not have transitive iterates. But F is a transitive map on the real line, and therefore either all iterates of F are transitive, or else there exists a point $y\in\mathbb{R}$ such that $F((-\infty,y])=[y,\infty)$ and $F([y,\infty))=(-\infty,y]$ (see [**3**, pp. 156–157]—the statements are for interval maps but the proofs also hold in $\mathbb{R}$). This latter alternative is impossible for a lifting of a degree one circle map. Therefore F^n is transitive. If there exists any non-decreasing map that semiconjugates g with a constant slope interval map, then the same map also conjugates g^n with a constant slope interval

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map. This shows that after replacing g, f, and F by some suitably high iterates, we may assume that $\lambda > 2$.

Let $h : (0, 1) \to \mathbb{R}$ denote the homeomorphism that conjugates $g|_{(0,1)}$ with F. Let $\pi : \mathbb{R} \to \mathbb{S}^1$, denote the natural projection (that semiconjugates F with f). Moreover, let $s : \mathbb{R} \to \mathbb{R}$, given by $s(x) = x + 1$, denote the deck transformation; then $s \circ F = F \circ s$. Thus, the following diagrams commute.

$$\begin{array}{ccc} (0,1) & \xrightarrow{g} & (0,1) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{R} & \xrightarrow[F]{} & \mathbb{R} \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{S}^1 & \xrightarrow[f]{} & \mathbb{S}^1 \end{array} \qquad \begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{R} & \xrightarrow[F]{} & \mathbb{R} \end{array}$$

The hypothesis of piecewise monotonicity means that there exists a finite set $P \subset \mathbb{S}^1$ such that f is monotone on each P-basic arc, and because of the constant slope, that monotonicity is strict. In the circle, strict monotonicity does not guarantee injectivity, but after adjoining the finite set $f^{-1}(x)$ for some $x \in \mathbb{S}^1$ to P we have also injectivity of f on each P-basic arc, so that the restriction of f to any P-basic arc is then a homeomorphism onto its image. Just as for interval maps we define the operator T_f acting on the space of non-atomic, strongly σ-finite, Borel measures on the circle. Let $P_F = \pi^{-1}(P)$. Then P_F is a closed, countable set, invariant under the integer translation map s, and F is strictly monotone on each P_F-basic interval. Let $P_g = h^{-1}(P_F) \cup \{0, 1\}$. Then $g \in \mathcal{CC}$ and P_g is a g-admissible set.

Suppose that there is a non-decreasing semiconjugacy of g to a constant slope interval map, say, with slope λ'. Then by Theorem 3.3 there exists a probability measure $\nu_g \in \mathcal{M}$ such that $T_g \nu_g = \lambda' \nu_g$. Push this measure down to a measure $\nu_F = h_*(\nu_g)$ on the real line. Then h gives not only a topological conjugacy, but also a measure-theoretic isomorphism of $((0, 1), g, \nu_g)$ with $(\mathbb{R}, F, \nu_F)$. It follows that $T_F \nu_F = \lambda' \nu_F$.

Now push this measure down to the circle, defining $\nu_f = \pi_* \nu_F$. If A is a Borel subset of a P-basic arc in $\mathbb{S}^1$, then its preimage in the covering space $\mathbb{R}$ can be expressed as a disjoint union $\pi^{-1}(A) = \bigcup_{n=-\infty}^{\infty} s^n(B)$ in such a way that B is a subset of a P_F-basic interval. Then for each $n \in \mathbb{N}$, $s^n(B)$ is also a subset of a P_F-basic interval, because P_F is s-invariant. By the injectivity of f on each P-basic arc, it follows that the sets $F(s^n(B))$, $n \in \mathbb{N}$ are pairwise disjoint. Therefore $\pi^{-1}(f(A)) = \bigcup_{n=-\infty}^{\infty} F(s^n(B))$ is also a disjoint union. Now we can calculate

$$\begin{aligned} (T_f \nu_f)(A) &= \nu_f(f(A)) = \nu_F(\pi^{-1}(f(A))) = \nu_F\Bigg(\bigcup_{n=-\infty}^{\infty} F(s^n(B))\Bigg) \\ &= \sum_{n=-\infty}^{\infty} \nu_F(F(s^n(B))) = \sum_{n=-\infty}^{\infty} \lambda' \nu_F(s^n(B)) = \lambda' \nu_F\Bigg(\bigcup_{n=-\infty}^{\infty} s^n(B)\Bigg) = \lambda' \nu_f(A). \end{aligned}$$

It follows that $T_f \nu_f = \lambda' \nu_f$.

By Theorem 3.3 and Remarks 3.4 and 3.5, there is a non-decreasing semiconjugacy of f to a circle map of constant slope λ'. But f, being a factor of a transitive map, is itself transitive. Therefore a non-decreasing semiconjugacy is automatically a conjugacy

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(see [**1**]). Thus, f is conjugate to a circle map of constant slope λ'. The topological entropy of a constant slope circle map is the logarithm of the slope (see [**1**, **9**]), and topological entropy is a conjugacy invariant. In such a way we have shown that if there exists a non-decreasing semiconjugacy of g to an interval map of constant slope λ', then in fact $\lambda' = \lambda$, the constant slope of f.

Let us push the Lebesgue measure m on $\mathbb{R}$ via the homeomorphism h^{-1}, and denote $\mu = (h^{-1})_*(m)$. The μ measures of P_g-basic intervals are the same as m measures of corresponding P_F-basic intervals. Since F is a lifting of a piecewise monotone map with finite number of pieces, those measures take only finitely many values, all of them finite. Moreover F, as the lifting of a map of constant slope, has also constant slope. By Lemma 3.1 and Remark 3.2, the Lebesgue measure m satisfies $T_F m = \lambda m$. Since h is a measure-theoretic isomorphism of $((0, 1), g, \mu)$ with $(\mathbb{R}, F, m)$ it follows that $T_g \mu = \lambda \mu$. Substantial transitivity of g follows from transitivity of F and Lemma 4.3. All this shows that μ belongs to $\mathcal{M}$ and that g and μ satisfy the assumptions of Theorem 4.5. Therefore there is no probability measure $\nu \in \mathcal{M}$ with $T_g \nu = \lambda \nu$, and consequently, there is no non-decreasing semiconjugacy of g to a constant slope interval map. □

Remark 5.2. In Theorem 5.1 there is no difficulty in finding a continuous interval map $g : [0, 1] \to [0, 1]$ such that $g|_{(0,1)}$ is topologically conjugate to F. Let h denote any homeomorphism of $(0, 1)$ with $\mathbb{R}$ and define $g = h^{-1} \circ F \circ h$ with additional fixed points at 0 and 1. We obtain continuity of g at the points 0, 1, because F was assumed to be the lifting of a degree one circle map.

Remark 5.3. There are trivial examples of maps with zero topological entropy for which there is no semiconjugacy to a map of constant slope, for instance, the map $f : [0, 1] \to [0, 1]$ given by $f(x) = x^2$. Theorem 5.1 is non-trivial in that it applies to continuous and transitive interval maps, and by [**4**, **5**], such maps always have topological entropy of at least $\log \sqrt{2}$.

If we wish to construct explicit examples that satisfy the hypotheses of Theorem 5.1, the only possible difficulty is in verifying the transitivity of the lifting F. Fortunately, there is a simple condition, broadly applicable and easy to verify, that guarantees transitivity.

THEOREM 5.4. *Assume that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous degree one map that is piecewise monotone with finitely many pieces and has constant slope. Assume also that $F : \mathbb{R} \to \mathbb{R}$ is a lifting of f. Let P denote the set of turning points of F. If for each P-basic interval I there are points x_L, x_R in the closure of I such that $F(x_L) = x_L - 1$ and $F(x_R) = x_R + 1$, then F is topologically transitive.*

Proof. The sets $R := \{x \in \mathbb{R} : F(x) - x = 1\}$ and $L := \{x \in \mathbb{R} : F(x) - x = -1\}$ are both invariant under integer translations and are both non-empty by hypothesis. Choose a point $x_L \in L$ and let x_R be the smallest element of R that is larger than x_L. Then $x_R - x_L < 1$ and $F(x_R) - F(x_L) > 2$. Since F has constant slope λ, this shows that $\lambda > 2$.

Let $U \subset \mathbb{R}$ be any open interval. As n grows, the successive images $F^n(U)$ grow in length by a factor of at least $\lambda/2 > 1$ until some image $F^N(U)$ contains an entire P-basic interval. Within the closure of this P-basic interval there are points $x_L \in L$,

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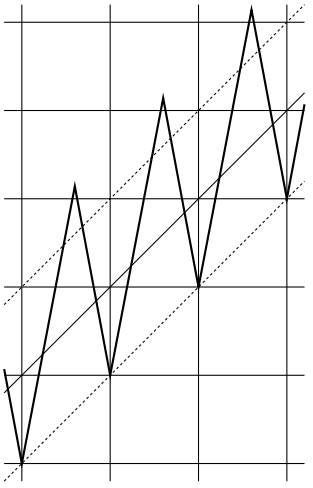


FIGURE 1. The map F_λ with $\lambda = 5.28$.

$x_R \in R$. Then, in the next steps, $F^{N+1}(U)$ contains $(x_L - 1, x_R + 1)$, $F^{N+2}(U)$ contains $(x_L - 2, x_r + 2)$, and so on. Therefore the union of all images of U is all of $\mathbb{R}$, and this proves transitivity of F. □

Remark 5.5. We can immediately verify the hypothesis of Theorem 5.4 by superimposing the diagonal lines $y = x + 1$, $y = x - 1$ on the graph $y = F(x)$. Each piece of monotonicity of the graph of F should intersect both diagonal lines.

6. *Examples*

In this section we provide a concrete example of a one-parameter family of circle maps of degree one with transitive liftings and constant slope. In such a way we will have examples of where our theorems apply and there is no non-decreasing semiconjugacy to a map of a constant slope.

Let us describe a lifting F_λ in our family. Choose a real parameter $\lambda \geq 2 + \sqrt{5}$. Let F_λ be the 'connect the dots' map (the graph of F_λ consists of straight line segments connecting the dots) with the dots $(k, k - 1)$ and $(k + b, k + c)$, where $k \in \mathbb{Z}$, $b = (\lambda + 1)/(2\lambda)$, and $c = (\lambda - 1)/2$ (see Figure 1). On the interval $[k, k + b]$ the slope is $(c + 1)/b = \lambda$, and on $[k + b, k + 1]$ it is $-c/(1 - b) = -\lambda$, so the map has constant slope λ. We have

$$F_\lambda(k + b) - (k + b) - 1 = \frac{\lambda^2 - 4\lambda - 1}{2\lambda} = \frac{\big(\lambda - \big(2 + \sqrt{5}\big)\big)\big(\lambda - \big(2 - \sqrt{5}\big)\big)}{2\lambda} \geq 0,$$

and therefore $F_\lambda(k + b) - (k + b) \geq 1$. Moreover, $F_\lambda(k) - k = -1$, so by the Intermediate Value Theorem the assumptions of Theorem 5.4 are satisfied. Thus F_λ is topologically transitive. Now if we choose any homeomorphism $h : (0, 1) \to \mathbb{R}$, we will get a map $g_\lambda = h^{-1} \circ F_\lambda \circ h$ (with additional fixed points at 0 and 1), which belongs to $\mathcal{CC}$, but is not semiconjugate by a non-decreasing map to a map of constant slope. If we want really concrete examples, we can even specify h, for instance $h(x) = \ln(y/(1 - y))$ (then $h^{-1}(x) = e^x/(e^x + 1)$).

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We would like to have in our family both maps that are and are not Markov. Remember that 'Markov' means countably Markov, so F_λ being Markov means that for the corresponding circle map the trajectory of the local maximum has countable closure (the local minimum is always a fixed point). Of course F_λ is Markov if and only if the map g_λ is Markov.

We start with a lemma on the one-sided full 2-shift $\sigma : \Sigma \to \Sigma$, where $\Sigma = \{0, 1\}^{\mathbb{N}}$.

LEMMA 6.1. *Let D be the set of those points $s \in \Sigma$ for which the closure of the trajectory $\{\sigma^n(s)\}_{n=0}^{\infty}$ is countable. Then both sets D and $\Sigma \setminus D$ are uncountable.*

Proof. Each element of Σ is a 0-1 sequence. Let E be the set of those sequences that are built of alternating blocks of 0's and 1's, and the length of the n-th block is n or $n+1$. Since we have to choose between the lengths n and $n+1$ for each n, the set E is uncountable. We claim that $E \subset D$. Fix an element $s \in E$. The trajectory of s is of course countable. It remains to count the accumulation points of this trajectory (that is, of the ω-set of s). If we fix the size of a window and slide it sufficiently far to the right along the sequence s, we see in this window only one or two blocks. This means that every element of the ω-limit set of s will consist of one or two blocks. However, there are only countably many such sequences. This proves our claim, and hence D is uncountable.

The 2-shift is transitive, and therefore the set of points with dense trajectories contains a dense G_δ set G. If G is countable, then for every $s \in G$ the set $\Sigma \setminus \{s\}$ is open and dense, so by the Baire Category Theorem the intersection of all those sets, $\Sigma \setminus G$, is a dense G_δ set. Therefore $(\Sigma \setminus G) \cap G$ is also a dense G_δ set, but it is empty. This contradiction shows that G is uncountable. Since for every $s \in G$ the closure of the trajectory of S is Σ, we have $G \subset \Sigma \setminus D$, and thus $\Sigma \setminus D$ is uncountable. □

THEOREM 6.2. *Fix an integer $n \geq 2$. Then there are uncountable sets $\Lambda_M \subset [2n+1, 2n+3]$ and $\Lambda_{nM} \subset [2n+1, 2n+3]$ such that for every $\lambda \in \Lambda_M$ the lifting F_λ is Markov and for every $\lambda \in \Lambda_{nM}$ the lifting F_λ is not Markov.*

Proof. Let A_λ be the set of those points x such that $F_\lambda^i(x) \in [0, 1]$ for $i = 0, 1, 2, \ldots$. Perform the standard coding procedure, using the left and right subintervals of $[0, 1] \cap F_\lambda^{-1}([0, 1])$. This shows that A_λ is a Cantor set, and F_λ restricted to this set is conjugate to the one-sided full 2-shift. By the standard argument, for any given itinerary the corresponding point of A_λ depends continuously on λ. By Lemma 6.1, uncountably many itineraries correspond to points whose trajectories have countable closures, and uncountably many itineraries correspond to points whose trajectories have uncountable closures. As λ varies from $2n+1$ to $2n+3$ then the image $-n + (\lambda - 1)/2$ under F_λ of the local maximum $-n + (\lambda + 1)/(2\lambda)$ sweeps the interval $[0, 1]$. When it meets a point with a countable closure of the trajectory, the corresponding map F_λ is Markov; when it meets a point with an uncountable closure of the trajectory, it is not Markov. This completes the proof. □

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