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Abstract

A Laurent polynomial in two variables is tempered if its edge polynomials are cyclotomic. Variation of coefficients leads to a family of smooth complete genus-g curves carrying a canonical algebraic K_2-class over a g-dimensional base S, hence to an extension of admissible variations of mixed Hodge structure (or normal function) on S. We prove that the $\mathbb{R}$-split locus of this extension is finite. Consequently, the torsion locus of the normal function and the A-factor locus for the family of curves are also finite.

1. Introduction

Hodge theory provides the basic interface between the algebraic and transcendental worlds in modern complex geometry. To a family $\pi\colon \mathcal{X} \to \mathcal{S}$ of smooth projective varieties it associates a polarized variation of Hodge structure (PVHS), or equivalently a period mapping [Gri68]. To an algebraic cycle on such a family, homologous to zero on fibers, it associates an admissible normal function [Sai96]. This can be viewed either as a holomorphic section of a bundle of generalized Jacobians, or as an extension of the original variation of Hodge structure (VHS) by a Tate object: the simplest kind of variation of mixed Hodge structure (VMHS). We can distinguish two kinds of normal functions, 'classical' and 'higher', with the former arising from cycles in the usual Chow group of $\mathcal{X}$, and the latter from higher Chow cycles in the sense of Bloch [Blo86].

A *special subvariety* of $\mathcal{S}$ is a component of the (Hodge) locus where a given mixed variation acquires extra rational Hodge tensors, such as when a normal function takes torsion values. Although it is known that the special locus is always a countable union of algebraic varieties [CDK95, BP13, BPS10], questions remain about the finiteness and field of definition of these components, especially in the *nonclassical* case where Griffiths transversality enters. In particular, according to the *Zilber–Pink conjecture* for VMHS [BU24], the maximal *atypical* special subvarieties (see § 1.3) should be finite in number, with tame geometry providing strong evidence in the pure case [BKU24]. What those techniques have so far been unable to produce are finiteness results for *zero-dimensional* components of the Hodge locus.

In this article we address a compelling (and nonclassical) case of the Zilber–Pink conjecture for *higher* normal functions, where the VHS of a family of curves is extended by a Tate object

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in weight 4. This sort of VMHS arises in many contexts, including K_2 of curve families [DK11], Mahler measure of two-variable polynomials [Rod99], local mirror symmetry and Hori–Vafa models [BKV17], and A-polynomials of knot complements [GM21]. For a generically nontorsion normal function of this type, it turns out that all components of the torsion locus are atypical, and, under additional assumptions, of dimension 0. One may then argue (cf. § 1.3) that Zilber–Pink predicts the torsion locus is a finite set of points. Our main results below confirm this prediction; in fact, we prove something stronger, as we now describe in detail.

1.1 Statement of results

Let S be a quasi-projective complex d-manifold,[1] carrying a nonzero integral polarized variation of Hodge structure $\mathbf{V} = (\mathcal{V}, \mathcal{F}^\bullet, \mathbb{V}, \nabla, Q)$, pure of weight and level 1. Here $\mathbb{V}$ is the underlying $\mathbb{Z}$-local system, $\mathcal{V}$ ($\cong \mathbb{V} \otimes \mathcal{O}_S$) the underlying holomorphic vector bundle (or its sheaf of sections), $\mathcal{F}^\bullet$ the Hodge filtration on $\mathcal{V}$, Q the polarization, and $\nabla\colon \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S$ the flat connection with kernel $\mathbb{V}_\mathbb{C} := \mathbb{V} \otimes \mathbb{C}$. Denote by g (≥ 1) the rank of $\mathcal{F}^1\mathcal{V}$, so that the Hodge numbers of $\mathcal{V}$ are (g, g).

Consider a nontorsion admissible normal function $R \in \mathrm{ANF}_S(\mathbf{V}(2))$, or equivalently an extension

$$0 \to \mathbf{V} \to \mathbf{E} \to \mathbb{Z}(-2)_S \to 0 \tag{1.1}$$

of admissible $\mathbb{Z}$-VMHS [SZ85] on S which is not split as an extension of $\mathbb{Q}$-VMHS. We may represent R as a global holomorphic section of the generalized Jacobian bundle of $\mathbf{V}(2)$, that is, of the sheaf $\mathcal{V}/\mathbb{V}(2) = \mathcal{V}/(2\pi\mathbf{i})^2\mathbb{V}$, with the horizontality property $\nabla R \in H^0(S, \mathcal{F}^1\mathcal{V} \otimes \Omega^1_S)$. (We describe the consequences of admissibility later.) Writing $\mathcal{A}$ for the sheaf of real analytic functions on S and $\mathcal{V}_\mathbb{R} := \mathbb{V} \otimes \mathcal{A}$, we can also consider the projection $R \mapsto \mathfrak{R}^* := \mathrm{Im}(R)$ to $\mathbf{i}\mathcal{V}_\mathbb{R}$, called the *real regulator* of the extension.[2]

The normal function R defines special loci in S:

- $Z_\mathbb{Z}(R) :=$ $\mathbb{Z}$-split locus of $\mathbf{E}$ = 0-locus of R;
- $Z_\mathbb{Q}(R) :=$ $\mathbb{Q}$-split locus of $\mathbf{E}$ = torsion locus of R; and
- $Z_\mathbb{R}(R) :=$ $\mathbb{R}$-split locus of $\mathbf{E}$ = 0-locus of $\mathfrak{R}^*$.

Clearly we have $Z_\mathbb{R}(R) \supseteq Z_\mathbb{Q}(R) \supseteq Z_\mathbb{Z}(R)$. A fundamental result of Brosnan and Pearlstein [BP13] states that $Z_\mathbb{Z}(R)$ is an algebraic subvariety of S; an immediate corollary is that $Z_\mathbb{Q}(R) = \bigcup_{n \in \mathbb{Z}_{>0}} Z_\mathbb{Z}(nR)$ is a countable union of algebraic subvarieties. But we claim that in our present scenario much more is true.

Theorem 1.1. *Suppose that $d = 1$, that is, that S is a curve. Then $Z_\mathbb{Q}(R)$ is finite; and if $\mathbf{V}$ has trivial fixed part, then $Z_\mathbb{R}(R)$ is also finite.*

At any $s \in S$, we can view $\nabla R|_s$ as a map from T_s to $\mathcal{F}^1\mathcal{V}|_s$; its rank is bounded by $\min(d, g)$.

Theorem 1.2. *Suppose that $d \leq g$ and ∇R has maximal rank at every point of S. Then $Z_\mathbb{R}(R)$ (and thus $Z_\mathbb{Q}(R)$) is finite.*

Remark 1.3. As pointed out by one of the referees, these theorems are a little surprising because in general the real split locus is a real analytic subvariety, not a complex analytic one. For instance, a Kummer extension $\nu_f \in \mathrm{ANF}_S(\mathbb{Z}(1)_S)$ corresponds to a unit $f \in \mathcal{O}^\times_{\mathrm{alg}}(S)$,

[1] That is, S denotes the complex points of a smooth complex quasi-projective variety.

[2] Writing $\mathbb{V}_\mathbb{R} := \mathbb{V} \otimes_\mathbb{Z} \mathbb{R}$ and $\mathcal{V}_\mathbb{C} := \mathcal{V}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$, and noting $\mathbb{Z}(2) \subset \mathbb{R}$, this 'projection' is the composition $\mathcal{V}/\mathbb{V}_{\mathbb{Z}(2)} \twoheadrightarrow \mathcal{V}/\mathbb{V}_\mathbb{R} \twoheadrightarrow \mathcal{V}/\mathcal{V}_\mathbb{R} \hookrightarrow \mathcal{V}_\mathbb{C}/\mathcal{V}_\mathbb{R} \cong \mathbf{i}\mathcal{V}_\mathbb{R}$.

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and ν_f is $\mathbb{R}$-split where $|f| = 1$. But our hypotheses ensure that the $\mathbb{R}$-split locus in our case has dimension 0.

A geometric source for normal functions of the above type is that of *tempered Laurent polynomials*, a notion introduced by Rodriguez Villegas [Rod99]; it has been applied or generalized in various contexts in [DK11, Ker22, DKS23]. Let $\Delta \subset \mathbb{R}^2$ be a polygon with integer vertices. Label the boundary points $\partial\Delta \cap \mathbb{Z}^2$ by ℓ^i and the g interior points $\mathrm{int}(\Delta) \cap \mathbb{Z}^2$ by m^j. Let

$$F_a = \sum_i c_i x_1^{\ell_1^i} x_2^{\ell_2^i} + \sum_{j=1}^{g} a_j x_1^{m_1^j} x_2^{m_2^j} \tag{1.2}$$

be a family of Laurent polynomials in two variables, with a_j free and c_i constant, Newton polygon Δ, and cyclotomic edge polynomials (given by the c_i). Write $C_a^* := \{F_a(x_1, x_2) = 0\} \subset \mathbb{G}_m^2$ and $C_a := \overline{C_a^*} \subset \mathbb{P}_\Delta$, $S := \mathbb{A}^g \setminus \Sigma$ for the locus where C_a is smooth, and $\mathcal{V}$ for the VHS over S with fiber $H^1(C_a)$ at $a \in S$. Then some fixed multiple of the 'coordinate symbol' $\{-x_1, -x_2\} \in \mathrm{CH}^2(C_a^*, 2)$ lifts to a class $\xi_a \in \mathrm{CH}^2(C_a, 2)$ for every $a \in S$. Taking images under the integral regulator/Abel–Jacobi maps

$$\mathrm{AJ} : \mathrm{CH}^2(C_a, 2) \to H^1(C_a, \mathbb{C}/\mathbb{Z}(2)) \tag{1.3}$$

produces $R \in \mathrm{ANF}_S(\mathcal{V}(2))$ as above. We call this the *geometric setting*.

Proposition 1.4. *In the geometric setting, ∇R has maximal rank at every point of S.*

So the hypotheses of Theorem 1.2 hold, and in particular we have the following corollary.

Corollary 1.5. *In the geometric setting, ξ_a is torsion for only finitely many $a \in S$.*

1.2 Outline of the paper

We begin in § 2 with a very explicit proof of Theorem 1.1 for $g = 1$, to familiarize the reader with the asymptotic aspects of the method in the simplest case. This is then generalized to arbitrary g in § 3. The main idea is to use transversality to constrain the differential behavior of R and admissibility (together with the notions of canonical extension and singularities of normal functions) to constrain its asymptotic behavior. For background the reader may consult [KP11, § 2].

For the proof of Theorem 1.2 in § 4, two new ingredients are needed. The first is a classical asymptotic Hodge theory result of Cattani and Kaplan [CK82], on the constancy of the monodromy weight filtration in the interior of a monodromy cone. We use this to bound $Z_{\mathbb{R}}(R)$ away from singularities of R. The second comes from the theory of o-minimal structures [vdDM96] which has recently been highly influential in Hodge theory through works such as [BBT23, BKU24]. In order to deal with boundary points where R is nonsingular, we make use of the $\mathbb{R}_{\mathrm{an,exp}}$-definability of functions arising from the real and imaginary parts of R.

In each case, the proof begins by applying a base change to make the boundary into a normal-crossing divisor and monodromies of $\mathbb{V}$ unipotent. Since finite monodromies become trivial, this results in some boundary strata with no monodromy. (Here 'boundary' refers to the complement of the finite cover of S in its compactification.) Though the VHS and normal function[3] can be extended across such strata, we may not have ∇R of maximal rank there and cannot expect the finiteness result to hold there either. So these strata remain part of the boundary for the purposes

[3]For a normal function of the type considered here (in $\mathrm{ANF}(\mathcal{V}(2))$, for $\mathcal{V}$ of level and weight 1), one cannot have R singular along a boundary stratum unless $\mathcal{V}$ is singular there. Moreover, if R is singular at a boundary point, then it is in fact singular in codimension 1 along a divisor passing through that point. See § 4.1.

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of our analysis, and there are thus four local scenarios to consider (after the base change): points at which both R and $\mathcal{V}$ are singular; points at which only $\mathcal{V}$ is singular; boundary points at which neither is singular; and interior points. The goal is to show that about any point in the compactified cover of S, there is a neighborhood whose intersection with the cover of S contains no point (except possibly the center) at which $\mathfrak{R}^*$ is zero.

Turning to the geometric case in § 5, we briefly review some consequences of an explicit representation of the regulator class R developed in [DK11, DKS23]. This is used to calculate ∇R and prove Proposition 1.4. Finally, in Appendix A, we offer a more explicit analysis (generalizing that in § 2) to the asymptotic behavior of the real and imaginary parts of R that avoids o-minimal theory. This is an appendix because the solution seems somewhat remarkable, but not part of the main article because it only works when neither the VHS nor the normal function is singular at the boundary point in question.

1.3 Relation to other works

Under the hypotheses of Theorems 1.1 and 1.2, it is straightforward to argue from [BP13, Corollary 1.3] that $Z_{\mathbb{Q}}(R)$ is a *countable* set of points. Moreover, some of the arguments which we phrase in terms of the properties of the integral regulator class R can be rephrased as in [BP13, § 4] in terms of the local normal form for the mixed variation $\mathbf{E}$.

On the other hand, the conjectural framework around atypical Hodge loci [Kli17, BKU24, BU24, KT25] predicts that $Z_{\mathbb{Q}}(R)$ is finite. To explain this, we use the formulation of Zilber–Pink for $\mathbb{Z}$-VMHS in [BU24, Conjecture 4.10]. Associated to a mixed variation $\mathbf{E}$ on S is a *Hodge datum* $(\mathcal{G}, \mathcal{D})$, comprising its (generic) Mumford–Tate group and domain, and *period map* $\Phi\colon S \to \widetilde{\Gamma}\backslash\mathcal{D}$ (with $\widetilde{\Gamma}$ the image[4] of the monodromy representation). The special subvarieties $S' \subset S$ for $\mathbf{E}$ are the irreducible components of $\Phi^{-1}(\widetilde{\Gamma}'\backslash\mathcal{D}')$ for which the Hodge datum of $\mathbf{E}|_{S'}$ is $(\mathcal{G}', \mathcal{D}') \subsetneq (\mathcal{G}, \mathcal{D})$; such an S' is atypical if

$$\operatorname{codim}_S(S') < \operatorname{codim}_{\Gamma\backslash\mathcal{D}}(\widetilde{\Gamma}'\backslash\mathcal{D}'). \tag{1.4}$$

The Zilber–Pink conjecture states that the atypical Hodge locus is a *finite* union of *maximal* such components.

Now let $\mathbf{E}$, $\mathbf{V}$ and R be as in (1.1), where we may assume (by arguing as in § 3.1 and deleting unnecessary factors) that $\mathbf{V}$ has trivial fixed part and R generically takes nontorsion values in all irreducible factors of $\mathbf{V}$. Write (G, D) for the Hodge datum of $\mathbf{V}$, $2g$ for its rank, and Γ for its monodromy group. By [KT24, Corollary 2.9], R is then *generic* in the sense that $\mathcal{G}$ is an extension of G by $\mathbb{G}_a^{2g}$, and $\widetilde{\Gamma}$ is (or enlarges to) an extension of Γ by $\mathbb{Z}^{2g}$. Accordingly, $\mathcal{D}$ (respectively, $\widetilde{\Gamma}\backslash\mathcal{D}$) is fibered over D (respectively, $\Gamma\backslash D$) with fibers $\cong \mathbb{C}^{2g}$ (respectively, $(\mathbb{C}^*)^{2g}$). The subdomains $\mathcal{D}' \subset \mathcal{D}$ corresponding to R becoming torsion are the countably many sections of $\mathcal{D} \to D$ taking values in $\mathbb{Q}(2)^{2g}$.

For S' an irreducible component of $Z_{\mathbb{Q}}(R)$, this yields RHS(1.4) $\geq 2g$. On the other hand, since the rank of ∇R is bounded above by g, [GZ26, Theorem 1.1] implies that LHS(1.4) $\leq g$, making S' atypical. According to Zilber–Pink, the *maximal* atypical special subvarieties thus consist of finitely many components of $Z_{\mathbb{Q}}(R)$, together with finitely many maximal atypical special subvarieties for $\mathbf{V}$ which are not contained in $Z_{\mathbb{Q}}(R)$. Restricting R to these components and inducing on dimension, one deduces that $Z_{\mathbb{Q}}(R)$ has only finitely many components.

Finally, if we assume that $d \leq g$ and (if $d > 1$) that ∇R has maximal rank *everywhere* on S, then components of $Z_{\mathbb{Q}}(R)$ are (zero-dimensional) points and Zilber–Pink implies that

[4] Alternatively, one may replace S by a finite cover and $\tilde{\Gamma}$ by an arithmetic subgroup of $\mathcal{G}(\mathbb{Q})$ containing the image of monodromy.

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$|Z_{\mathbb{Q}}(R)| < \infty$. It is this prediction that our Theorems 1.1 and 1.2 confirm. This contrasts with the approach via monodromic atypicality in [BKU24, BU24], which produces powerful results for positive-dimensional components of the atypical Hodge locus (including torsion loci of normal functions [KT24, GZ26]), but cannot say anything about atypical special points.

There is also an intriguing connection to knot theory. By [CCG+94, § 4] and [GM21, Proposition 5.2] (see also the expository account in [Akm25]), the Laurent polynomial F_a in (1.2) is a factor of an A-polynomial of a 3-manifold with torus boundary if and only if ξ_a is torsion. Call F_a *exact* if $a \in Z_{\mathbb{R}}(R)$; then the exact locus contains the A-factor locus. In [GM21, § 4], it was conjectured that the exact locus (hence also the A-factor locus) is finite in S, with a proof claimed for $g = 1$. However, this proof did not account for degenerations, only showing that exact polynomials are isolated in S. (This is insufficient since S is noncompact: see the argument at the end of § 2.) In any case, Theorem 1.2 and Proposition 1.4 immediately imply the Guilloux–Marché conjecture in generality.

Corollary 1.6. *In the geometric setting, F_a is exact for only finitely many $a \in S$.*

Moreover, the second author's paper [Akm25] shows how to reinterpret Corollary 1.5 in this context without reference to a family. Namely, define the *overgenus* of an A-factor to be the minimum genus of an A-polynomial curve containing the A-factor curve as an irreducible component. Then according to Theorem 4.1 in [Akm25], there are only finitely many smooth A-factors with overgenus bounded by any fixed positive integer.

A brief word on notation: for a holomorphic function F, we write $\mathfrak{F} = \mathrm{Re}(F)$ and $\mathfrak{F}^* = \mathrm{Im}(F)$; the same goes for holomorphic sections (or vector-valued functions), when the meaning is clear.

2. Families of elliptic curves

In this section we assume that $\mathbf{V}$ is a weight-1, rank-2 $\mathbb{Z}$-PVHS on a curve S with unipotent (possibly trivial) monodromies about points of $\Sigma = \overline{S} \setminus S$, and consider an extension $\mathbf{E}$ as in (1.1). Its nonzero mixed Hodge numbers are $h^{1,0} = h^{0,1} = h^{2,2} = 1$. Let R denote the corresponding admissible normal function.[5] On any simply connected open subset of S, R has a lift to a holomorphic section $\tilde{R}$ of $\mathcal{V}$. Assuming that no such local lift is constant, we claim that any $x \in \overline{S}$ has a neighborhood U such that $U \setminus \{x\} \subset S$ and $\mathfrak{R}^*|_{U\setminus\{x\}}$ is nowhere zero. We defer other aspects of the one-dimensional base story (such as justifying the last assumption) to § 3.

Identifying U with a disk $\Delta = \{z \in \mathbb{C} \,|\, |z| < \epsilon\}$, and taking $\tilde{R}$ to denote a multivalued lift on Δ^*, we need to show, after shrinking ϵ if necessary, that $\mathfrak{R}^* = \mathrm{Im}(\tilde{R})$ is nowhere 0 on $\Delta^* := \Delta \setminus \{0\}$.

The variation $\mathbf{V}$ is very easy to describe on Δ^*. Let α, β denote a basis of a 'nearby fiber' $V := \mathbb{V}|_{z_0}$ of the local system; we may assume that the monodromy operator $T = e^N$ sends $\alpha \mapsto \alpha - n\beta$ and $\beta \mapsto \beta$ for some $n \in \mathbb{Z}_{\geq 0}$. (Dually, we think of $V^\vee = \mathbb{Z}\langle \alpha^\vee, \beta^\vee \rangle$ as H_1 of an elliptic curve, with the Dehn twist sending $\beta^\vee \mapsto \beta^\vee + n\alpha^\vee$.) Write $\ell(z) := (1/2\pi\mathbf{i}) \log(z)$ and $\widetilde{\alpha} := \alpha + n\ell(z)\beta$, which is a *single valued* section of $\mathcal{V}$. The canonical extension $\mathcal{V}_e$ of $\mathcal{V}$ to Δ is $\mathcal{O}\langle \widetilde{\alpha}, \beta \rangle$; this comes with a logarithmic connection $\nabla \colon \mathcal{V}_e \to \mathcal{V}_e \otimes \Omega^1_\Delta \langle \log\{0\} \rangle$. A holomorphic generator of $\mathcal{F}^1_e$ takes the form

$$\omega = \alpha + \tau(z)\beta = \widetilde{\alpha} + \eta(z)\beta = \alpha + (\eta(z) + n\ell(z))\beta, \tag{2.1}$$

where the multivalued function τ should be thought of as the period ratio and $\eta(0)$ as indicating the limit mixed Hodge structure (LMHS) $V_{\lim} := \psi_z \mathbf{V}$ at the origin.

[5] Such normal functions are discussed, for example, in [DK11, § 4] and [GKS24, § 4].

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Now by Griffiths transversality and admissibility for $\mathbf{E}$, $(1/2\pi\mathbf{i})\nabla_{z\partial_z}\tilde{R}$ extends to a holomorphic section of $\mathcal{F}_e^1$ on Δ. Let a be the value at 0 of its $\widetilde{\alpha}$-coefficient; this is an integer, the *singularity class* of R. It is worth explaining its meaning briefly: the LMHS $E_{\lim} = \psi_z\mathbf{E}$ has mixed Hodge numbers $h^{0,0} = h^{1,1} = h^{2,2} = 1$. The action of the extended monodromy logarithm (also denoted N) on $E_{\lim}$ is the same thing as $-2\pi\mathbf{i}\mathrm{Res}_0(\nabla)$. The class $(-1/(2\pi\mathbf{i})^2)\tilde{R}$ may be seen as the difference of Hodge and integral lifts of a generator of $\mathbb{Z}(-2)$, and so the $(-1,-1)$-morphism N sends this generator in $E_{\lim}^{2,2}$ to $a\omega(0)$ in $E_{\lim}^{1,1}$ (equivalently, $a\widetilde{\alpha}(0)$ in $\mathrm{Gr}_2^W E_{\lim}$). If a is nonzero, then we can write on Δ^* (really on its universal cover)

$$\tilde{R} = (2\pi\mathbf{i}a\log(z) + \text{holomorphic})\alpha + (\cdots)\beta. \tag{2.2}$$

Taking the imaginary part gives $\mathfrak{R}^* = (2\pi a\log|z| + \text{harmonic})\alpha + (\cdots)\beta$, whence $\mathfrak{R}^*$ is nonzero on Δ^* (shrinking the disk as needed).

Henceforth we assume that $a = 0$, that is, that R is *nonsingular* at the origin. In this case, N is zero on $E_{\lim}^{2,2}$ and $\nabla_{z\partial_z}\tilde{R}$ vanishes at 0. That is, we can write $\nabla_{\partial_z}\tilde{R} =: f\omega$ on Δ^*, where f extends to a holomorphic function on Δ. (By the nonconstancy assumption, f is not identically zero.) Indeed, if R is nonsingular,[6] then R has a single-valued lift $\tilde{R}$ to $\mathcal{V}_e$, with *limit* $\tilde{R}(0) \in \ker(N) \subseteq V_{\lim}$. Thus, if $n > 0$ (i.e., $\mathcal{V}$ is singular at 0), then $\tilde{R}(0) = \Psi\beta$ for some $\Psi \in \mathbb{C}$, since $N\widetilde{\alpha} = -n\beta \notin \ker(N)$. If $n = 0$, then $\tilde{R}(0) = \Psi_\alpha\alpha + \Psi_\beta\beta$ is arbitrary.

Set $k = \mathrm{ord}_0(f)$, $F := \int_0 f\,dz = z^{k+1}U$, $\tilde{F} := \int_0 F(dz/z) = z^{k+1}\tilde{U}$, and $G := \int_0 \eta f\,dz = FH_0$, where $U, \tilde{U}, H_0$ are holomorphic and $U, \tilde{U}$ are units (on Δ). From this point on, the argument is different for $n = 0$ and $n > 0$.

If $n > 0$, integrating $f\omega$ gives

$$\begin{aligned}\tilde{R} &= \int_0 \{f\alpha + (\eta f + n\ell(z)f)\beta\}\,dz \\ &= z^{k+1}U\alpha + \left\{\Psi + z^{k+1}\left(UH_0 + nU\ell(z) - \frac{n}{2\pi\mathbf{i}}\tilde{U}\right)\right\}\beta \\ &= z^{k+1}U\alpha + \{\Psi + z^{k+1}U(H + n\ell(z))\}\beta,\end{aligned}$$

where $H := H_0 - (n/2\pi\mathbf{i})(\tilde{U}/U)$ is holomorphic. Taking imaginary parts and writing $\psi := \mathrm{Im}(\Psi)$ gives

$$\mathfrak{R}^* := \mathrm{Im}(\tilde{R}) = \mathrm{Im}(z^{k+1}U)\alpha + \{\psi + \mathrm{Im}[z^{k+1}U(H + n\ell(z))]\}\beta.$$

If $\psi \neq 0$, then $\mathfrak{R}^*$ is nonzero[7] on Δ and we are done. If $\psi = 0$, then

$$\begin{aligned}\mathfrak{R}^* &= \mathrm{Im}(z^{k+1}U)\widetilde{\alpha} + \{\mathrm{Im}[z^{k+1}U(H + n\ell(z))] - n\ell(z)\mathrm{Im}(z^{k+1}U)\}\beta \\ &= \mathrm{Im}(z^{k+1}U)\widetilde{\alpha} + \left\{\mathrm{Im}(z^{k+1}UH) - \frac{n}{2\pi}\log|z|\bar{z}^{k+1}\bar{U}\right\}\beta \\ &= \mathrm{Im}(z^{k+1}U)\widetilde{\alpha} - \frac{n}{2\pi}\bar{z}^{k+1}\log|z|U\left\{1 - \frac{2\pi}{n}\frac{\mathrm{Im}(z^{k+1}UH)}{\bar{z}^{k+1}\log|z|\bar{U}}\right\}\beta.\end{aligned}$$

Now $|H| \leq B$ for some $B > 0$ on Δ. Shrinking Δ so that $|z| < e^{-4\pi B/n}$, we have

$$\frac{2\pi}{n}\left|\frac{\mathrm{Im}(z^{k+1}UH)}{\bar{z}^{k+1}\log|z|\bar{U}}\right| \leq \frac{2\pi}{n}\left|\frac{z^{k+1}UH}{\bar{z}^{k+1}\bar{U}}\right|\frac{1}{-\log|z|} \leq \frac{2\pi B/n}{-\log|z|} < \frac{1}{2},$$

and thus the coefficient of β is nonzero on Δ^*.

[6] See [KP11, § 2.12] for a general discussion of asymptotic invariants of normal functions; the main point here is that if the singularity class vanishes, then the limit is defined in the Jacobian of the kernel of N.

[7] Always after possibly shrinking Δ in this argument.

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If $n=0$, then $\eta=\tau$ is nonzero, and integration by parts shows that[8] $H:=H_0/\tau=1+\mathcal{O}(z)$ is a holomorphic unit. We thus have $G=FH\tau$ and $\tilde{R}=(\Psi_\alpha+F)\alpha+(\Psi_\beta+G)\beta$. If the imaginary part of either constant Ψ_α or Ψ_β is nonzero, we are done; if both are zero, then[9] $\mathfrak{R}^*=\mathfrak{F}^*\alpha+\mathfrak{G}^*\beta$. Suppose we have a set $\mathscr{S}\subset\Delta^*$ of common zeros of $\mathfrak{F}^*$ and $\mathfrak{G}^*$ with 0 as limit point. We may shrink Δ so that $\mathrm{Im}(\tau H)$ is nonzero there. Then on $\mathscr{S}$ we have $\mathfrak{G}^*=\mathrm{Im}(\tau H)\mathfrak{F}+\mathrm{Re}(\tau H)\mathfrak{F}^*=\mathrm{Im}(\tau H)\mathfrak{F}$ so that F vanishes on $\mathscr{S}$. Since F is holomorphic and $G=FH\tau$, F and G vanish identically on Δ, contradicting the local nonconstancy of $\tilde{R}$.

Thus, we have shown that every $x\in\overline{S}$ has an analytic open neighborhood U_x such that $U_x\backslash\{x\}\subset S$ and

$$Z_{\mathbb{R}}(R)\cap(U_x\backslash\{x\})=\emptyset. \tag{2.3}$$

The $\{U_x\}_{x\in\overline{S}}$ constitute an open cover of $\overline{S}$; so by compactness of $\overline{S}$, it has a finite subcover of the form $\{U_x\}_{x\in\Sigma\cup\Sigma'}$, with $\Sigma'\subset S$ finite. By (2.3), we have $Z_{\mathbb{R}}(R)\subseteq\Sigma'$, which gives $|Z_{\mathbb{R}}(R)|<\infty$ as desired.

Though it is nice to see a very explicit analytic argument once, it would clearly become unwieldy in the most general setting. Much of the analysis is subsumed in results of Bierstone and Milman [BM88] and van den Dries and Miller [vdDM96] in what follows.

3. One-parameter families

In this section, we give a complete proof of Theorem 1.1.

3.1 Preliminaries

Let $\mathbf{V}$ be a $\mathbb{Z}$-PVHS with Hodge numbers (g,g) on a curve S, and $R\in\mathrm{ANF}_S(\mathbf{V}(2))$ a nontorsion normal function.

Lemma 3.1. *If $\mathbf{V}$ has trivial fixed part, then every local lift $\tilde{R}$ of R is nonconstant.*

Proof. Suppose otherwise: R is nontorsion, but on some disk $D\subset S$, $\tilde{R}$ is the flat continuation of a vector $(2\pi\mathbf{i})^2v\in V_{\mathbb{C}}$ in the fiber of the local system $\mathbb{V}_{\mathbb{C}}$ over a base point of D. Now R is a holomorphic section of $\mathcal{V}/\mathbb{V}_{\mathbb{Z}(2)}$, so by analytic continuation its local lifts are all flat, which forces R to be a section of $\mathbb{V}_{\mathbb{C}}/\mathbb{V}_{\mathbb{Z}(2)}$. Equivalently, we have $(T-I)v\in V_{\mathbb{Z}}$ for every $T\in\Gamma\leq\mathrm{GL}(V_{\mathbb{Z}})$, where Γ is the monodromy group of $\mathbb{V}$. Now since R is nontorsion, $v\notin V_{\mathbb{Q}}$; accordingly, let $\sigma\in\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ be such that $v-v^\sigma\neq0$. By triviality of the fixed part, $V_{\mathbb{C}}^\Gamma=\{0\}$; so for *some* $T\in\Gamma$ we have $T(v-v^\sigma)\neq v-v^\sigma$, hence $((T-I)v)^\sigma\neq(T-I)v$. This is a contradiction. □

Now assume that we have shown in the trivial fixed part case that $Z_{\mathbb{R}}(R)$ is finite. Dropping the fixed-part hypothesis, we claim that $Z_{\mathbb{Q}}(R)$ is finite. By the theorem of the fixed part, we can decompose $\mathbf{V}_{\mathbb{Q}}=\mathbf{V}_{\mathrm{fix}}\oplus\mathbf{V}_{\mathrm{var}}$ as a direct sum in the category of $\mathbb{Q}$-PVHS, where $\mathbf{V}_{\mathrm{fix}}$ is constant and $\mathbf{V}_{\mathrm{var}}$ has trivial fixed part; decompose $R=R_{\mathrm{fix}}+R_{\mathrm{var}}$ accordingly. If R_{var} is nontorsion, then by assumption $Z_{\mathbb{R}}(R_{\mathrm{var}})\supseteq Z_{\mathbb{Q}}(R)$ is finite. So suppose that R_{var} is torsion, and observe that R_{fix} lives in $\mathrm{ANF}_S(\mathbf{V}_{\mathrm{fix}}(2))$, which is an extension of $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0),H^1(S,\mathbb{V}_{\mathrm{fix}}(2)))$ by (constant sections)

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0),H^0(S,\mathbb{V}_{\mathrm{fix}}(2)))\cong V_{\mathrm{fix},\mathbb{C}}/V_{\mathrm{fix},\mathbb{Q}(2)}.$$

[8] Note that this is different from the H in the last paragraph.
[9] Recall that we write $\mathfrak{F}=\mathrm{Re}(F)$, $\mathfrak{F}^*=\mathrm{Im}(F)$, etc. throughout this paper.

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But since $H^1(S, \mathbb{V}_{\mathrm{fix}}(2)) \cong H^1(S) \otimes V_{\mathrm{fix}}(2)$ has weights -2 and -1, it has no Hodge $(0,0)$-classes and the Hom-space is trivial. Hence, R_{fix} is constant and nontorsion (since R is nontorsion), whence $Z_{\mathbb{Q}}(R)$ is empty.

Henceforth, we assume $\mathbf{V}$ has trivial fixed part. We need a general result which allows us to pass to an appropriate finite cover, both in this and the next section. We do not need to worry about whether this introduces a nontrivial fixed part, since nonconstant local lifts of R cannot become constant on the cover.

Lemma 3.2. *Let $\mathbf{H}$ be a $\mathbb{Z}$-PVHS on a smooth complex quasi-projective variety Y. Then there exist a finite étale cover $\widehat{Y} \to Y$ and smooth projective compactification $\overline{Y} \supseteq \widehat{Y}$ such that $\Sigma = \overline{Y} \setminus \widehat{Y}$ is a normal crossing divisor about which the base change $\widehat{\mathbf{H}}$ has unipotent monodromies.*

Proof. Write $H_{\mathbb{Z}}$ for a fiber of $\mathbb{H}$, and let $\Gamma \leq \mathrm{Aut}(H_{\mathbb{Z}}, Q)$ be the image of the monodromy representation. First, $\mathrm{Aut}(H_{\mathbb{Z}}, Q)$ has a finite-index subgroup which is *neat* [Bor69, 17.4], that is, no element has a torsion eigenvalue (root of unity different from 1 itself). Let $\widehat{\Gamma}$ denote its intersection with Γ. Second, the étale covering $\widehat{Y^{\mathrm{an}}} \to Y^{\mathrm{an}}$ of complex manifolds induced by $\widehat{\Gamma}$ (or rather, its pullback to $\pi_1(Y^{\mathrm{an}})$) is in fact algebraic by the 'Riemann existence theorem' [SGA1, XII]. Let $\widehat{Y}$ denote the underlying variety. Third, by resolution of singularities [Hir64, p. 146], $\widehat{Y}$ admits a smooth compactification $\overline{Y}$ with normal-crossing complement. The monodromies of the base change $\widehat{\mathbf{H}}$ are then unipotent by combining the monodromy theorem [Gri70, 3.1] with neatness of $\widehat{\Gamma}$. □

Let $\widehat{S} \subset \overline{S}$ be the cover of S guaranteed by Lemma 3.2, and write $\widehat{R}$ for the base change of R. So by covering $\overline{S}$ with disks $U \cong \Delta$ centered at every point (with $\Delta^* \subset S$), we are reduced as in § 2 to the following scenario:

- a $\mathbb{Z}$-PVHS $\mathbf{V}$ on Δ^* with Hodge numbers (g, g) and unipotent monodromy operator T; and
- a nontorsion normal function $\widehat{R}|_{\Delta^*} \in \mathrm{ANF}_{\Delta^*}(\mathbf{V}(2))$ with nonconstant local lifts.[10]

Now assume we have shown the following claim.

Claim 3.3. After shrinking Δ, $\mathrm{Im}(\widehat{R}|_{\Delta^*})$ is nowhere zero on Δ^*.

Then compactness of $\overline{S}$ produces a finite subcover; and since the centers of this finite set of disks must contain $Z_{\mathbb{R}}(\widehat{R}) \cup \Sigma$, $Z_{\mathbb{R}}(\widehat{R})$ (and thus $Z_{\mathbb{R}}(R)$) is finite, proving Theorem 1.1.

3.2 The asymptotic part

In this subsection we prove Claim 3.3. We begin with two lemmas on real analytic functions, which are also used in the next section.

Lemma 3.4. *If $A(t)$ and $B(t)$ are real analytic functions on $(-\epsilon, \epsilon)$, and $A(t) - B(t)\log|t|$ has zeros converging to $t = 0$, then A and B are identically zero.*

Proof. If one is identically 0, the other must be too by the local connectedness of analytic varieties [BM88, Corollary 2.7]. If neither is identically 0, divide into two cases, which amount to $u(t) = t^c \log|t|$ $(c > 0)$ and $t^c u(t) = \log|t|$ $(c \geq 0)$ with u an analytic unit. Neither has solutions on $(-\epsilon, 0) \cup (0, \epsilon)$ for sufficiently small $\epsilon > 0$. Contradiction. □

[10] This is simply denoted by R in the paragraphs that follow.

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Lemma 3.5. *Let $A_1,\dots,A_p$ be real analytic functions on a neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^n$, and set $X := \{x \in \mathcal{U} \mid A_i(x) = 0,\ i = 1,\dots,p\}$. Let $W \subset \mathcal{U}$ be another closed analytic subset containing 0, and set $X^- := X \setminus X \cap W$. Then either (i) it is possible to shrink $\mathcal{U}$ so that X^- becomes empty, or (ii) X^- contains a connected real analytic manifold Y of positive dimension.*

Proof. By [BM88, Proposition 2.10],[11] after shrinking $\mathcal{U}$, X has a *finite* stratification into connected semianalytic sets X_i which are analytic submanifolds of $\mathcal{U}$ of pure dimensions δ_i. So after further shrinking $\mathcal{U}$, the only possible zero-dimensional stratum is the origin. If some X_i is not contained in W, then $\delta_i > 0$ and $X_i \cap W$ is a closed analytic subset, and we take Y to be a connected component of $X_i \setminus X_i \cap W$. □

At this point it suffices to work with rational coefficients. Let V be a fiber of the local system $\mathbb{V}_{\mathbb{Q}}$ on Δ^*, and $\{\alpha_1,\dots,\alpha_g;\beta_1,\dots,\beta_g\}$ denote a symplectic basis. We may assume that the monodromy logarithm about $z=0$ takes the form $N = \left(\begin{smallmatrix} 0 & 0 \\ -\mathcal{N} & 0 \end{smallmatrix}\right)$ (with $\mathcal{N} = \mathcal{N}_{ij}$ a $g \times g$ block). Define $\widetilde{\alpha}_i := \alpha_i + \sum_j \ell(z)\mathcal{N}_{ij}\beta_j$ to get (together with $\beta_1,\dots,\beta_g$) a basis for the canonical extension $\mathcal{V}_e$. A basis of $\mathcal{F}_e^1\mathcal{V}_e$ is written $\omega_i = \alpha_i + \sum_j \tau_{ij}\beta_j = \widetilde{\alpha}_i + \sum_j \eta_{ij}\beta_j$, with η holomorphic at 0 and $\tau = \mathfrak{t} + \mathbf{i}\mathfrak{t}^*$ symmetric and $\mathfrak{t}^*$ positive definite (on Δ^*).

For a local (multivalued) lift of R to $\mathcal{V}$, we can write $\tilde{R} = \sum_{j=1}^g F_j\widetilde{\alpha_j} + G_j\beta_j$. To compute $\mathfrak{R}^*$, we need to rewrite $\tilde{R}$ with respect to the local system basis $\{\alpha_j;\beta_j\}$, take imaginary parts of the coefficients, and finally rewrite again in the canonical extension basis, which yields single-valued coefficients. However, the coefficient of $\widetilde{\alpha_j}$ in $\mathfrak{R}^*$ is simply $\mathfrak{F}_j^* = \mathrm{Im}(F_j)$. Deferring a detailed discussion of singularities to § 4.1, we keep things simple here: the image of a generator of $E_{\lim}^{2,2}$ under N is given by $\sum_{j=1}^g a_j\widetilde{\alpha_j}(0)$ for some $a_j \in \mathbb{Q}$; and $F_j = 2\pi\mathbf{i}a_j\log(z) + \{\text{holomorphic on } \Delta\}$, whence $(1/2\pi)\mathfrak{F}_j^* = a_j\log|z| + \{\text{harmonic on } \Delta\}$. If any $a_j \neq 0$, after shrinking the disk the harmonic part is bounded by some C which $|a_j\log|z||$ everywhere exceeds. Hence $\mathfrak{F}_j^*$, *a fortiori* $\mathfrak{R}^*$, has no zero in Δ^*. This reduces the problem to the case where R is nonsingular at 0.

In this case[12] R admits a single-valued lift

$$\begin{aligned}\tilde{R} &= \sum_i (F_i\alpha_i + \hat{G}_i\beta_i) = \sum_i F_i\widetilde{\alpha}_i + \sum_i \left(\hat{G}_i - \ell(z)\sum_j \mathcal{N}_{ij}F_j\right)\beta_i \\ &=: \sum_i (F_i\widetilde{\alpha}_i + G_i\beta_i)\end{aligned}$$

in which $F_i, G_i \in \mathcal{O}(\Delta)$ and $N\tilde{R}(0) = 0$ (which means that for each i we have $\sum_j \mathcal{N}_{ij}F_j(0) = 0$). Differentiating gives

$$\nabla_{\partial_z}\tilde{R} = \sum_{i=1}^g (f_i\alpha_i + \hat{g}_i\beta_i), \quad \hat{g}_i = \sum_j \tau_{ij}f_j = g_i + \ell(z)\sum_j \mathcal{N}_{ij}f_j,$$

for some f_i and $g_i (= \sum_j \eta_{ij}f_j)$ which are holomorphic on Δ.[13] Taking imaginary part of $\tilde{R}$ gives

$$\mathfrak{R}^* = \sum_i (\mathfrak{F}_i^*\alpha_i + \hat{\mathfrak{G}}_i^*\beta_i) = \sum_i \mathfrak{F}_i^*\widetilde{\alpha}_i + \sum_i \left(\hat{\mathfrak{G}}_i^* - \ell(z)\sum_j \mathcal{N}_{ij}\mathfrak{F}_j^*\right)\beta_i,$$

where $\hat{\mathfrak{G}}_i^* = \mathrm{Im}(\hat{G}_i) = \mathrm{Im}(G_i + \ell(z)\sum_j \mathcal{N}_{ij}F_j) = \mathfrak{G}_i^* - (\log|z|/2\pi)\sum \mathcal{N}_{ij}\mathfrak{F}_j + (\arg(z)/2\pi)\sum_j \mathcal{N}_{ij}\mathfrak{F}_j^*$.

[11] See also Theorem 3.14 and the beginning of the proof of Theorem 5.1 in [BM88].
[12] Note that here we do not break the argument into subcases $N = 0$ and $N \neq 0$ as we did in § 2.
[13] Note that $F_i' = f_i$, $\hat{G}_i' = \hat{g}_i$, and $G_i' = g_i - (1/2\pi\mathbf{i}z)\sum_j \mathcal{N}_{ij}F_j$, which is also holomorphic on Δ.

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Suppose there exists a sequence of points, limiting to $z=0$, on which all $\mathfrak{F}_i^*$ and $\hat{\mathfrak{G}}_i^*$ vanish (or equivalently, that it is not possible to shrink Δ so that $Z_{\mathbb{R}}(\hat{R}|_{\Delta^*})$ is finite). By Lemma 3.5 applied to $A_i=\mathfrak{F}_i^*$ ($p=g$, $n=2$, $W=\{0\}$), we find that their vanishing locus is either Δ or a real analytic variety of dimension 1. In the first case, a holomorphic function with zero imaginary part is constant, which contradicts local nonconstancy of $\tilde{R}$. In the second case, by [BM88, Corollary 4.9] there is a sequence of (real) blowups at 0 that makes the preimage normal-crossing. This guarantees the existence of an injective analytic map $\gamma\colon I:=(-\epsilon,\epsilon)\to\Delta$ (sending $t\mapsto\gamma(t)=(x(t),y(t))$, and $0\mapsto(0,0)$) with image in the zero locus of the $\{\mathfrak{F}_i^*\}$. Write γ for both the image curve and the map, and $\gamma'(t)=\begin{pmatrix}x'(t)\\y'(t)\end{pmatrix}$.

The derivative of the restriction of the imaginary part to this curve is now

$$\frac{d}{dt}\gamma^*\mathfrak{R}^*=\nabla_{\gamma'(t)}\mathfrak{R}^*(\gamma(t))=\begin{pmatrix}\mathfrak{F}_x^* & \mathfrak{F}_y^*\\ \hat{\mathfrak{G}}_x^* & \hat{\mathfrak{G}}_y^*\end{pmatrix}\gamma'(t)=\begin{pmatrix}\mathfrak{f}^* & \mathfrak{f}\\ \hat{\mathfrak{g}}^* & \hat{\mathfrak{g}}\end{pmatrix}\gamma'(t),$$

where the matrix entries are all evaluated at $\gamma(t)$, and the product is '$2g\times 2$ matrix times 2×1 vector'. By construction, the first g entries of the resulting $g\times 1$ vector (as analytic functions on I) are identically zero. The last g entries are derivatives of functions $\hat{\mathfrak{G}}_i^*(\gamma(t))$ (analytic on $I\setminus\{0\}$, continuous on I) with infinitely many zeros approaching the origin. So these derivatives must themselves have infinitely many zeros, reflecting stationary points of $\hat{\mathfrak{G}}_i^*(\gamma(t))$. So each entry of

$$(\hat{\mathfrak{g}}^*\ \hat{\mathfrak{g}})\gamma'(t)=\mathfrak{t}(\mathfrak{f}^*\ \mathfrak{f})\gamma'(t)+\mathfrak{t}^*(\mathfrak{f}\ -\mathfrak{f}^*)\gamma'(t)=\mathfrak{t}^*(\mathfrak{f}\ -\mathfrak{f}^*)\gamma'(t) \tag{3.1}$$

has zeros limiting to the origin.

At this point we cannot simply invert $\mathfrak{t}^*$, since the zeros could occur in different entries at different sequences of values of t. However,[14] observe that $\mathfrak{t}^*(\gamma(t))=\mathfrak{n}^*(\gamma(t))-(1/2\pi)\log|z(t)|\mathcal{N}=\widetilde{\mathfrak{n}^*}(t)-c(\log|t|/2\pi)\mathcal{N}$, where $c=(1/2)\mathrm{ord}_0(x(t)^2+y(t)^2)$. Then (3.1) is

$$\widetilde{\mathfrak{n}^*}(\mathfrak{f}\ -\mathfrak{f}^*)\gamma'(t)-\log|t|\frac{c}{2\pi}\mathcal{N}(\mathfrak{f}\ -\mathfrak{f}^*)\gamma'(t)=:\mathbf{A}(t)-\log|t|\mathbf{B}(t),$$

where $\mathbf{A}(t)$ and $\mathbf{B}(t)$ have analytic entries on I. Applying Lemma 3.4 entry by entry, both are identically zero; hence, so is (3.1) and thus (*now* inverting $\mathfrak{t}^*$) $(\mathfrak{f}\ -\mathfrak{f}^*)\gamma'(t)$. Conclude that

$$(F_x\ F_y)\gamma'(t)=(\mathfrak{f}\ -\mathfrak{f}^*)\gamma'(t)+\mathbf{i}(\mathfrak{f}^*\ \mathfrak{f})\gamma'(t)\equiv 0,$$

making all $F_i(\gamma(t))$ constant. But since the F_i are holomorphic, they are then constant on the complex analytic closure of γ, which is Δ. This makes all f_i zero, again contradicting the local nonconstancy of $\tilde{R}$. So there was no sequence of points as described above, and we can shrink Δ to make $\mathfrak{R}^*$ nowhere vanishing on Δ^*. This completes the proof of Claim 3.3.

4. The multiparameter setting

In this section, we prove Theorem 1.2. Let $\mathbf{V}$ be a $\mathbb{Z}$-PVHS with Hodge numbers (g,g) on a smooth complex quasi-projective variety S of dimension $d\le g$, and $R\in\mathrm{ANF}_S(\mathbf{V}(2))$ a nontorsion normal function for which ∇R has maximal rank at every point of S.

Lemma 4.1. *Let $s\in S$ be a point and $\mathscr{D}\subset S$ be a neighborhood of s. Then, after possibly shrinking $\mathscr{D}$, $\mathfrak{R}^*|_{\mathscr{D}}$ is zero either nowhere or only at s. That is, zeros of $\mathfrak{R}^*$ in S are isolated.*

Proof. Regard $\mathfrak{R}^*=\mathrm{Im}(R)$ as a real analytic function from $\mathscr{D}\subset\mathbb{C}^d\cong\mathbb{R}^{2d}$ to $\mathbb{R}^{2g}$. By Lemma 3.5 (with $W=\{0\}$), we can shrink $\mathscr{D}$ so that $\{\mathfrak{R}^*=0\}\cap(\mathscr{D}\setminus\{0\})$ either is empty or contains a real

[14] Here $\mathfrak{n}^*=\mathrm{Im}(\eta)$.

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analytic manifold of positive dimension. Assume the latter and place a coordinate neighborhood Δ^d at a point in this manifold. (Make this point the new origin.)

By the maximal rank hypothesis, the $\mu_j(z) := (\nabla_{\partial/\partial z_j} R)(z)$ $(j = 1, \dots, d)$ span a subspace of rank d in $(\mathcal{F}^1\mathcal{V})_z$ for every $z \in \Delta^d$. Writing $z_j = x_j + \mathbf{i}y_j$, we have $\nabla_{\partial/\partial x_j}\mathfrak{R}^* = \mathrm{Im}(\nabla_{\partial/\partial x_j} R) = \mathrm{Im}(\nabla_{\partial/\partial z_j} R) = \mathrm{Im}(\mu_j)$ and $\nabla_{\partial/\partial y_j}\mathfrak{R}^* = \mathrm{Re}(\mu_j)$.

Since $\mathfrak{R}^*$ is zero on some manifold through 0, for some nonzero tangent vector[15] $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \mathbb{R}^{2d} \cong T_0\Delta^d$ we have $(\nabla_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}\mathfrak{R}^*)|_0 = 0$. Then

$$\begin{aligned} 0 &= \sum_{j=1}^{d} (a_j \mathrm{Im}(\mu_j) + b_j \mathrm{Re}(\mu_j))(0) \\ &= \sum_{j=1}^{d} \tfrac{1}{2}(b_j - \mathbf{i}a_j)\mu_j(0) + \sum_{j=1}^{d} \tfrac{1}{2}(b_j + \mathbf{i}a_j)\overline{\mu_j(0)} \end{aligned}$$

yields a contradiction as $\{\mu_j(0), \overline{\mu_j(0)}\}_{j=1}^d$ are independent over $\mathbb{C}$. □

Now apply the covering Lemma 3.2 to replace S by $\widehat{S} \subseteq \overline{S}$ with $\Sigma = \overline{S} \setminus \widehat{S}$ a normal crossing divisor, about which the base change $\hat{\mathbf{V}}$ has unipotent monodromies. Clearly, since the cover is étale, $\nabla\hat{R}$ is everywhere of maximal rank and zeros of $\hat{\mathfrak{R}}^*$ in $\widehat{S}$ are still isolated by Lemma 4.1.

Let $x \in \overline{S}$ be an arbitrary point, and $\Delta^d \cong U \subset \overline{S}$ a coordinate neighborhood of x with $U \cap \widehat{S} \cong (\Delta^*)^c \times \Delta^{d-c}$ and coordinates $z_1, \dots, z_d$. We show (taking U sufficiently small) that $\hat{\mathfrak{R}}^*|_{U \cap \hat{S}}$ is (i) empty if $U \not\subset \widehat{S}$ and (ii) either empty or $\{x\}$ if $U \subset \widehat{S}$. (Indeed, part (ii) is already clear by the lemma.) The compactness of $\overline{S}$ then yields a finite cover by polydisks, in which (excluding the intersection of each polydisk with Σ, where $\hat{\mathfrak{R}}^*$ is not defined)[16] the only possible zeros of $\hat{\mathfrak{R}}^*$ are the centers of those polydisks which are contained in $\widehat{S}$. This gives the desired finiteness of $Z_{\mathbb{R}}(\widehat{R})$ hence that of $Z_{\mathbb{R}}(R)$.

Before proceeding we spell out what is meant by part (i) and simplify notation (dropping the hats).[17] Suppose we have, for $1 \le c \le d \le g$:

- a $\mathbb{Q}$-PVHS $\mathbf{V}$ on $U := (\Delta^*)^c \times \Delta^{d-c}$ with Hodge numbers (g, g) and (possibly trivial) unipotent monodromy operators $T_1, \dots, T_c$; and
- a nontorsion normal function $R \in \mathrm{ANF}_U(\mathbf{V}(2))$ with the properties that zeros of $\mathfrak{R}^*$ are isolated in U and ∇R has rank d everywhere.

In the subsections that follow we show the following claim.

Claim 4.2. After shrinking $\overline{U} = \Delta^d$, $\mathfrak{R}^*$ is nowhere zero on U.

As in § 3.2, we let V denote a fiber of $\mathbb{V}_{\mathbb{Q}}$ on U, and $\{\alpha_j, \beta_j\}_{j=1}^g$ denote a symplectic basis. The latter may be chosen so that $N_1, \dots, N_c$ take the form $N_k = \left(\begin{smallmatrix} 0 & 0 \\ -\mathcal{N}^k & 0 \end{smallmatrix}\right)$ with $\mathcal{N}^k = (\mathcal{N}^k_{ij})$ symmetric; and then the β_i and $\widetilde{\alpha}_i := \alpha_i + \sum_{j=1}^g \sum_{k=1}^c \ell(z_k)\mathcal{N}^k_{ij}\beta_j$ yield a basis for the unipotent extension.

[15] Here a and b are both $d \times 1$ column vectors with real entries.

[16] Of course, we can extend $\hat{R}$ to those components of Σ along which it is nonsingular. It is entirely possible that zeros of $\mathfrak{R}^*$ in this extension may not be isolated. We are making no claim about this.

[17] We can also pass from $\mathbb{Z}$- to $\mathbb{Q}$-VMHS, since we have already used the fact that $\mathbf{V}$ is a $\mathbb{Z}$-PVHS to invoke the monodromy theorem.

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4.1 Nontrivial singularities

In this part, where $0 \neq \mathrm{sing}_0(R) \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^1(\imath_0^* R\jmath_* \mathbf{V}(2)))$,[18] we make no use of the maximal rank assumption or its consequence (that zeros of $\mathfrak{R}^*$ are isolated). So the bounding of zeros of $\mathfrak{R}^*$ away from a singularity of R is a more general phenomenon.

Let $V_{\lim}$ denote the LMHS of $\mathbf{V}$ at 0, with monodromy weight filtration $M_\bullet := W(N)[-1]_\bullet$, where $N = N_1 + \cdots + N_c$. We choose our symplectic basis so that it has graded pieces $\mathrm{Gr}_2^M V_{\lim} = \langle \widetilde{\alpha_1}(0), \ldots, \widetilde{\alpha_r}(0)\rangle$, $\mathrm{Gr}_0^M V_{\lim} = \langle \beta_1, \ldots, \beta_r\rangle$, and $\mathrm{Gr}_1^M V_{\lim} = \langle \alpha_{r+1}, \ldots, \alpha_g; \beta_{r+1}, \ldots, \beta_g\rangle$. The monodromy logarithms $N_1, \ldots, N_c$ kill $M_1 V_{\lim}$ but (unlike N) may not individually induce isomorphisms from Gr_2^M to Gr_0^M. A result from [CK82] (which will be crucial for us) is that *any* $N' \in \mathbb{R}_{>0}\langle N_1, \ldots, N_c\rangle$ *induces the same filtration* $M_\bullet$. Write $M_\bullet^i$ for the filtrations induced by the N_i.

Now consider the extension $0 \to \mathbf{V}(2) \to \mathbf{E} \to \mathbb{Q}(0) \to 0$ of admissible $\mathbb{Q}$-VMHS on U produced by R. There are the monodromy logarithms $\tilde{N}_1, \ldots, \tilde{N}_c$ on $E_{\lim}$ lifting the ones on $V_{\lim}$, with sum inducing a weight filtration $\tilde{M}_\bullet$ on $E_{\lim}$ (restricting to $M[4]_\bullet$ on $V_{\lim}$). Denoting by $\mathbf{1} \in E_{\lim,\mathbb{Q}}$ a lift of $1 \in \mathbb{Q}(0) \cong \mathrm{Gr}_0^{\tilde{M}} E_{\lim,\mathbb{Q}}$, we set $\mathfrak{s}_i := \tilde{N}_i \mathbf{1} \in \mathrm{Gr}_{-2}^{\tilde{M}} E_{\lim,\mathbb{Q}} \cong \mathrm{Gr}_2^M V_{\lim,\mathbb{Q}}$. We can represent $\mathfrak{s}_i =: \sum_{j=1}^r a_j^i \widetilde{\alpha_j}(0)$ by a vector $a^i = (a_j^i) \in \mathbb{Q}^r$.

Each of these $\mathfrak{s}_i$ is the limit of a singularity class along the ith coordinate hyperplane. Hence, if $\mathfrak{s}_i \neq 0$, then $\mathfrak{s}_i$ is nonzero in $\mathrm{Gr}_2^{M^i} V_{\lim}$, which N_i sends isomorphically to $\mathrm{Gr}_0^{M^i} V_{\lim}$, and

$$\mathfrak{s}_i \neq 0 \implies N_i \mathfrak{s}_i \neq 0. \tag{4.1}$$

Moreover, as $(\mathfrak{s}_1, \ldots, \mathfrak{s}_c)$ gives a class in H^1 of the standard Koszul-type complex $V_{\lim} \overset{\oplus N_i}{\to} \bigoplus_{i=1}^c V_{\lim}(-1) \to \bigoplus_{i<j} V_{\lim}(-2)$, we see that

$$N_i \mathfrak{s}_j = N_j \mathfrak{s}_i \ (\forall i, j). \tag{4.2}$$

Finally, since the singularity class of R at the origin is nonzero by assumption, some $\mathfrak{s}_i \neq 0$, without loss of generality $\mathfrak{s}_1$.

For a local (multivalued) lift of R to $\mathcal{V}$, we can write $\tilde{R} = \sum_{j=1}^g F_j \widetilde{\alpha_j} + G_j \beta_j$, so that $\mathfrak{F}_j^* = \mathrm{im}(F_j)$ is the coefficient of $\widetilde{\alpha_j}$ in $\mathfrak{R}^*$. The singularity class implies that $F_j = 2\pi\mathbf{i} \sum_{i=1}^c a_j^i \log(z_i) + \{\text{holomorphic on } \Delta^d\}$, whence $\mathfrak{R}_j^* := (1/2\pi)\mathfrak{F}_j^* = \sum_{i=1}^c a_j^i \log|z_i| + \{\text{harmonic on } \Delta^d\}$. If $c = 1$, then at least one period blows up like $\log|z_1|$, which beats the (bounded) harmonic part on a possibly smaller polydisk.

So assume $2 \leq c \leq d \leq g$, and suppose there is a sequence of points $z^\gamma = (z_i^\gamma) \to 0$ in U with $\mathfrak{R}^*(z^\gamma) = 0$ (*a fortiori* all $\mathfrak{R}_j^*(z^\gamma) = 0$). Since the harmonic parts are bounded below $1/\sqrt{g}$ on a polydisk, for each $K \in \mathbb{N}$ there must exist $L_i(K) \in \mathbb{R}_{>K}$ $(i = 1, \ldots, c)$ such that $\|\sum_i L_i(K) a^i\| < 1$. (This is simply by taking γ sufficiently large that $-\log|z_i^\gamma| > K$.) Since $\mathbb{P}^{c-1}(\mathbb{R}_{\geq 0})$ is compact, there is a subsequence K_p such that $[L_i(K_p)]$ has a limit $[\mathcal{L}_i] \in \mathbb{P}^{c-1}(\mathbb{R}_{\geq 0})$. We may assume that $\mathcal{L}_1 \neq 0$, so that $\|a^1 + \sum_{i>1}(L_i(K_p)/L_1(K_p))a^i\| < 1/L_1(K_p) < 1/K_p \to 0 \implies$

$$\left\| a^1 + \sum_{i>1} \frac{\mathcal{L}_i}{\mathcal{L}_1} a^i \right\| \leq \left\| \sum_{i>1} \left(\frac{\mathcal{L}_i}{\mathcal{L}_1} - \frac{L_i(K_p)}{L_1(K_p)} \right) a^i \right\| + \left\| a^1 + \sum_{i>1} \frac{L_i(K_p)}{L_1(K_p)} a^i \right\| \to 0$$

$\implies \|a^1 + \sum_{i>1}(\mathcal{L}_i/\mathcal{L}_1)a^i\| = 0$. So $a^1 = -\sum_{i=2}^c \mathcal{L}_i a^i = 0$, or equivalently,

$$\mathfrak{s}_1 = -\sum_{i \geq 2} \frac{\mathcal{L}_i}{\mathcal{L}_1} \mathfrak{s}_i. \tag{4.3}$$

[18] Here $\jmath\colon U \hookrightarrow \Delta^d$ and $\imath_0\colon \{0\} \hookrightarrow \Delta^d$ are the inclusions, and $R\jmath_* \mathbf{V}(2) \in D^b\mathrm{MHM}(\Delta^d)$. We spell out the singularity class explicitly (up to twist) as a tuple $(\mathfrak{s}_1, \ldots, \mathfrak{s}_c)$ in the following.

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Now by (4.3) and (4.2) we have

$$\left(\sum_{i=1}^{c}\mathcal{L}_iN_i\right)\mathfrak{s}_1=\mathcal{L}_1N_1\mathfrak{s}_1+\sum_{i\geq 2}\mathcal{L}_iN_i\mathfrak{s}_1$$
$$=-\sum_{i\geq 2}\mathcal{L}_iN_1\mathfrak{s}_i+\sum_{i\geq 2}\mathcal{L}_iN_i\mathfrak{s}_1=0.$$

But combining this with (4.1) gives $(N_1+\sum_{i=1}^{c}\mathcal{L}_iN_i)\mathfrak{s}_1=N_1\mathfrak{s}_1\neq 0$. Hence, $N'=\sum_{i=1}^{c}\mathcal{L}_iN_i$ and $N''=N_1+\sum_{i=1}^{c}\mathcal{L}_iN_i$, which are both in $\mathbb{R}_{>0}\langle N_1,\ldots,N_c\rangle$, induce different weight filtrations: for the first, $\mathfrak{s}_1\in M_1'$ (as N' kills it); for the second, $\mathfrak{s}_1\notin M_1''$ (as N'' does not). This contradicts [CK82, Theorem 3.3], proving Claim 4.2 in this case.

4.2 Trivial singularities

To make the idea clear, we explain first the case where the T_k are trivial ($N_k=0$), so that *both* $\mathbf{V}$ and R are nonsingular at 0. Write $\tilde{R}=\sum_{i=1}^{g}\{F_i\alpha_i+G_i\beta_i\}$, with $F_i=\mathfrak{F}_i+\mathbf{i}\mathfrak{F}_i^*$ and $G_i=\mathfrak{G}_i+\mathbf{i}\mathfrak{G}_i^*$ for the decomposition into real and imaginary parts. Since R is nonsingular these extend to holomorphic functions on Δ^d, with harmonic (hence real analytic) real and imaginary parts. We apply Lemma 3.5 with $n=2d$, $W=\Delta^d\setminus U$, and X the vanishing locus of $\mathfrak{R}^*$, that is, of the harmonic functions $\{\mathfrak{F}_1^*,\ldots,\mathfrak{F}_g^*;\mathfrak{G}_1^*,\ldots,\mathfrak{G}_g^*\}$ of $\{x_1,\ldots,x_d;y_1,\ldots,y_d\}$ (real and imaginary parts of $z_1,\ldots,z_d$).

Suppose that we are in alternative (ii) of the lemma applied to this setting. In particular, these $2g$ functions are constant along a positive-dimensional real-analytic manifold $Y\subset U$. Write:

- $\mathfrak{F}_x^*:=(\partial\mathfrak{F}_i^*/\partial x_j)$ ($g\times d$ matrix) and so forth (for x and y; $F,\mathfrak{F}^*,\mathfrak{F},G,\mathfrak{G}^*$, and $\mathfrak{G}$), so that the Cauchy–Riemann equations give, for example, $\mathfrak{F}_x^*=-\mathfrak{F}_y$ and $\mathfrak{F}_y^*=\mathfrak{F}_x$;
- $\omega_i=\alpha_i+\sum_j\tau_{ij}\beta_j$ $(i=1,\ldots,g)$ for the standard basis of $\mathcal{F}^1\mathcal{V}$, with $\tau=\mathfrak{t}+\mathbf{i}\mathfrak{t}^*$ the real/imaginary decomposition, and τ symmetric with $\mathfrak{t}^*$ positive definite on Δ^d; and
- $\nabla_{\partial_{z_k}}\tilde{R}=\sum_i f_{ik}\omega_i$ $(k=1,\ldots,d;\ f=\mathfrak{f}+\mathbf{i}\mathfrak{f}^*)$ so that in matrix notation $DF:=(\partial F_i/\partial z_k)=f$ and $DG=\tau f$ (also note that $DF=F_x=-\mathbf{i}F_y$, so $\mathfrak{f}=\mathfrak{F}_x$, etc.).

Then for any tangent vector $\binom{a}{b}\in\mathbb{R}^{2d}$ at a point $z\in Y$, we have

$$\begin{pmatrix}\mathfrak{f}^* & \mathfrak{f}\\ \mathrm{Im}(\tau f) & \mathrm{Re}(\tau f)\end{pmatrix}\binom{a}{b}=\begin{pmatrix}\mathfrak{F}_x^* & \mathfrak{F}_x\\ \mathfrak{G}_x^* & \mathfrak{G}_x\end{pmatrix}\binom{a}{b}=\begin{pmatrix}\mathfrak{F}_x^* & \mathfrak{F}_y^*\\ \mathfrak{G}_x^* & \mathfrak{G}_y^*\end{pmatrix}\binom{a}{b}=0,$$

hence $(\mathfrak{f}^*\,\mathfrak{f})\binom{a}{b}=0$ and $0=\mathfrak{t}(\mathfrak{f}^*\,\mathfrak{f})\binom{a}{b}+\mathfrak{t}^*(\mathfrak{f}\,-\mathfrak{f}^*)\binom{a}{b}=\mathfrak{t}^*(\mathfrak{f}\,-\mathfrak{f}^*)\binom{a}{b}$. As $\mathfrak{t}^*$ is invertible on Δ^d, this gives $(\mathfrak{f}\,-\mathfrak{f}^*)\binom{a}{b}=0$. Since we have $(\mathfrak{f}\,-\mathfrak{f}^*)+\mathbf{i}(\mathfrak{f}^*\,\mathfrak{f})=(F_x\;F_y)$, we get that $D_{\binom{a}{b}}F=0$, and conclude that F is constant along Y. But then since F is complex analytic, it is constant along the complex analytic closure $\mathcal{Y}$ of Y in U. Since $DG=\tau DF$, $\tilde{R}$ is constant on $\mathcal{Y}$ and so $\mathrm{rk}(\nabla\tilde{R})<d$ along $\mathcal{Y}$, a contradiction. So we are in alternative (i) of the Lemma 3.5, which yields Claim 4.2 in this subcase.

However, we would still be a long way from being able to deal with nontrivial $\{T_k\}$ without one additional tool. This is the place in the proof where an input from o-minimal theory becomes critical. By a *log-real-analytic* function on Δ^d, we shall mean a polynomial in the $\log|z_i|=\frac{1}{2}\log(x_i^2+y_i^2)$ with coefficients in real analytic functions on Δ^d.

Lemma 4.3. *Let $A_1,\ldots,A_p$ be log-real-analytic functions on Δ^d, with common vanishing locus X. Then either (i) it is possible to shrink Δ^d so that $X^-:=X\cap U$ becomes empty, or (ii) X^- contains a connected real analytic curve.*

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Proof. Let $\mathscr{C}$ be the analytic-geometric category corresponding to the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$ (see [vdDM96, § 2.5(6)]). Then X^- belongs to $\mathscr{C}(\Delta^d)$ (not just $\mathscr{C}(U)$). If case (i) does not hold, there is a sequence of points in X^- limiting to $0 \in \Delta^d$, and so 0 belongs to the *frontier* of X^-. By § 1.17 in [vdDM96], there exists a continuous injective map $\gamma\colon [0,1) \to \Delta^d$ such that $\gamma(0,1) \subseteq X^-$, $\gamma|_{(0,1)}$ is real analytic, and $\gamma(0) = 0$. □

To eliminate case (ii) and verify Claim 4.2, we now modify the above proof as follows. First, we may write

$$\begin{aligned}\tilde{R} &= \sum_{i=1}^{g}(F_i\alpha_i + \hat{G}_i\beta_i)\\ &= \sum_{i=1}^{g} F_i\widetilde{\alpha}_i + \sum_{i=1}^{g}\left(\hat{G}_i - \sum_{j=1}^{g}\sum_{k=1}^{c}\ell(z_k)\mathcal{N}_{ij}^k F_j\right)\beta_i\\ &=: \sum_{i=1}^{g}(F_i\widetilde{\alpha}_i + G_i\beta_i)\end{aligned}$$

with $F_i, G_i \in \mathcal{O}(\Delta^d)$. Taking imaginary parts gives

$$\begin{aligned}\mathfrak{R}^* &= \sum_{i=1}^{g}(\mathfrak{F}_i^*\alpha_i + \hat{\mathfrak{G}}_i^*\beta_i) = \sum_i \mathfrak{F}_i^*\widetilde{\alpha}_i + \sum_i\left(\hat{\mathfrak{G}}_i^* - \sum_{j,k}\ell(z_k)\mathcal{N}_{ij}^k\mathfrak{F}_j^*\right)\beta_i\\ &= \sum_i \mathfrak{F}_i^*\widetilde{\alpha}_i + \sum_i\left(\mathfrak{G}_i^* - \frac{1}{2\pi}\sum_{j,k}\log|z_k|\mathcal{N}_{ij}^k(\mathfrak{F}_j - \mathbf{i}\mathfrak{F}_j^*)\right)\beta_i.\end{aligned}$$

If there is a sequence of points in U on which $\mathfrak{R}^*$ vanishes, then they belong to the common 0-locus of the (respectively real-analytic and log-real-analytic) functions $\mathfrak{F}_i^*$ and $\widetilde{\mathfrak{G}_i^*} := \mathfrak{G}_i^* - (1/2\pi)\sum_{j,k}\log|z_k|\mathcal{N}_{ij}^k\mathfrak{F}_j$ $(i = 1,\dots,g)$. By Lemma 4.3, these functions must vanish along a real-analytic curve γ in U. Clearly $\mathfrak{R}^*$ does as well.

Writing $DF = f$ and $D\hat{G} = \hat{g} = \tau f$, we then have

$$0 = \frac{d}{dt}\gamma^*\mathfrak{R}^* = \begin{pmatrix}\mathfrak{F}_x^* & \mathfrak{F}_y^*\\ \hat{\mathfrak{G}}_x^* & \hat{\mathfrak{G}}_y^*\end{pmatrix}\gamma'(t) = \begin{pmatrix}\mathfrak{f}^* & \mathfrak{f}\\ \hat{\mathfrak{g}}^* & \hat{\mathfrak{g}}\end{pmatrix}\gamma'(t) = \begin{pmatrix}\mathfrak{f}^*\\ \mathrm{Im}(\tau f) & \mathrm{Re}(\tau f)\end{pmatrix}\gamma'(t).$$

At this point the proof is as before, using invertibility of $\mathfrak{t}^*$ along γ to show that F is constant on the complex analytic closure of γ in U, contradicting the maximal rank assumption there. So there cannot have been such a sequence of points, and Claim 4.2 holds in this case too.

5. Application to Laurent polynomials

It remains to provide a geometric scenario which produces normal functions as in Theorem 1.2. As in the Introduction, let $\Delta \subset \mathbb{R}^2$ be a polygon with integer vertices and g interior integer points; and let $F = F_a$ be the family of tempered Laurent polynomials in (1.2), depending on $a \in S := \mathbb{A}^g \setminus \Sigma$. This produces a family of smooth complete curves $\mathbb{P}_\Delta \times S \supset \mathcal{C} \xrightarrow{\pi} S$ and a higher Chow cycle $\xi \in \mathrm{CH}^2(\mathcal{C}, 2)$ whose restriction to $\mathcal{C}^* := \mathcal{C} \cap (\mathbb{G}_m^2 \times S)$ is the coordinate symbol $\{-x_1, -x_2\}$ (or a nonzero integer multiple thereof).

The images of $\xi_a \in \mathrm{CH}^2(\mathcal{C}_a, 2)$ under (1.3) give rise to an admissible normal function $R \in \mathrm{ANF}_S(\mathbf{V}(2))$, where $\mathbf{V} = R^1\pi_*\mathbb{Z}$. We only need the following (see [DK11, §§ 3–4] and [DKS23, Appendix A] for more details). The restriction of $R_a \in H^1(C_a, \mathbb{C}/\mathbb{Z}(2))$ to $R_a^* \in H^1(C_a^*, \mathbb{C}/\mathbb{Z}(2))$ is represented by the restriction to X_a^* of a global current $R\{-x_1, -x_2\}$. The exterior derivative

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of this current is $dx_1/x_1 \wedge dx_2/x_2$, up to a $\mathbb{Z}(2)$-valued current which we can bound locally over the base. So we have $\nabla R^* = dx_1/x_1 \wedge dx_2/x_2$ on $\mathcal{C}^*$.

Now the residue of $dx_1/x_1 \wedge dx_2/x_2$ on a component of $\mathcal{C} \setminus \mathcal{C}^*$ is dlog of the corresponding root of an edge polynomial of F; as this is a root of unity, hence constant, the residue is zero. Thus, $dx_1/x_1 \wedge dx_2/x_2$ extends to a holomorphic form $\Omega \in \Omega^2(\mathcal{C})$. Moreover, ∇R is represented by a holomorphic 2-form on $\mathcal{C}$ by Griffiths transversality. Since $\Omega^2(\mathcal{C}) \hookrightarrow \Omega^2(\mathcal{C}^*)$, we must therefore have $\nabla R = \Omega$. Since $\mathcal{C}^*$ is cut out of $\mathbb{G}_m^2 \times S$ by F, we can also write Ω as $\mathrm{Res}_{\mathcal{C}}(dF/F \wedge dx_1/x_1 \wedge dx_2/x_2)$ on $\mathcal{C}^*$ hence on $\mathcal{C}$. For the 'directional' derivatives we therefore have

$$\begin{aligned}(\nabla_{\partial_{a_j}} R)(a) = \langle \nabla R_a, \partial_{a_j} \rangle|_{C_a} &= \left\langle \mathrm{Res}_{\mathcal{C}}\left(\frac{dF}{F} \wedge \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right), \widetilde{\partial_{a_j}} \right\rangle\Bigg|_{C_a} \\ &= \mathrm{Res}_{C_a}\left(\left\langle \frac{dF}{F} \wedge \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}, \partial_{a_j} \right\rangle\right) \\ &= \mathrm{Res}_{C_a}\left(\frac{\partial_{a_j} F}{F} \wedge \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}\right) \\ &= \mathrm{Res}_{C_a}\left(\frac{x_1^{m_1^j} x_2^{m_2^j} (dx_1/x_1) \wedge (dx_2/x_2)}{F_a(x_1, x_2)}\right) =: \mu_j(a).\end{aligned}$$

By [Bat93, p. 386], for *every* $a \in S$ these $\mu_1, \ldots, \mu_g$ yield a basis of $\Omega^1(C_a)$. In particular, they are independent, and ∇R has maximal rank. This proves Proposition 1.4.

Appendix A. Some matrix differential equations

In this appendix we present an alternate proof of Claim 4.2 (without using [BM88] or [vdDM96]) in the case where both R and $\mathbf{V}$ have trivial singularities. The key here is Lemma A.2, which solves an interesting problem in partial differential equations for which we could not find a reference. We thank J. McCarthy for discussions on this point.

Theorem A.1. *Let $B \subset \mathbb{C}^d$ be an open ball centered at the origin, $F, G\colon B \to \mathbb{C}^g$ vectors of holomorphic functions vanishing at the origin, and $\tau\colon B \to M_{g\times g}(\mathbb{C})$ a matrix of holomorphic functions satisfying $dG = \tau dF$,*[19] *and $\det(\mathfrak{t}^*(0)) \neq 0$. If $V(\mathfrak{F}^*, \mathfrak{G}^*)$ contains a sequence of points $\{x_i \neq 0\}$ converging to the origin, then a subsequence lies on a positive-dimensional component of $V(F, G)$ passing through the origin.*

Proof. We apply complex analytic embedded resolution of singularities to obtain a proper surjective holomorphic map $\pi\colon (\tilde{B}, \tilde{D}) \to (B, D)$, where D is the zero divisor of the product $F_1 F_2 \cdots F_g$. The properties of π yield a sequence $\{\tilde{x}_i\}$ in the preimage of $\{x_i\}$ converging to a point $p \in \pi^{-1}(0)$. There is a coordinate ball U centered at p in which the components of $\tilde{F} := F \circ \pi$ take the form $\tilde{F}_j = u_j z^{a_j}$, with u_j a unit in the analytic local ring $\mathcal{O}_{U,0} \cong \mathbb{C}\{z_1, z_2, \ldots, z_g\}$ and a_j a multi-index. We will henceforth work in U and drop the tildes. We construct a matrix S as in the lemma below. Shrinking U so that $\mathfrak{S}^*$ is everywhere invertible, we observe

$$0 = \mathfrak{G}^*(x_i) = \mathfrak{S}^*(x_i)\mathfrak{F}(x_i) + \mathfrak{S}(x_i)\mathfrak{F}^*(x_i) = \mathfrak{S}^*(x_i)\mathfrak{F}(x_i),$$

which implies $\mathfrak{F}(x_i) = 0$ by invertibility of $\mathfrak{S}^*(x_i)$ and hence $G(x_i) = F(x_i) = 0$. The image of these points is a sequence downstairs converging to the origin and contained in $V(F, G)$, which is a complex analytic variety. □

[19] We think of dF and dG as vectors of 1-forms.

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Lemma A.2. *Let F, G, τ be defined on U as above, with $V(F_1 \cdots F_g)$ a union of coordinate hyperplanes. Then there is a matrix of holomorphic functions $S\colon U \to M_{g\times g}(\mathbb{C})$ satisfying $G = SF$ and $S(0) = \tau(0)$.*

Proof. Define $S \coloneqq \tau - \varepsilon$, where

$$\varepsilon_{ij}(z) \coloneqq \frac{1}{F_j(z)} \int_{\gamma_z} F_j \, d\tau_{ij}, \tag{A.1}$$

and $\gamma_z\colon [0,1] \to U$ is the line segment $t \mapsto tz$. (If F_j vanishes identically, then we set $\varepsilon_{ij} \coloneqq 0$ for all i.) A routine computation verifies that $G = SF$ where S is defined. We claim that after possibly shrinking the radius, S is defined on all of B.

The restriction of ε_{ij} to the z_1-axis is a one-variable holomorphic function vanishing at the origin. Therefore, ε_{ij} is either zero or indeterminate at the origin. Now by hypothesis, each F_j is associated to a monomial, whose zero divisor is a union of coordinate hyperplanes. This forces the integral (A.1) to vanish whenever F_j does, meaning that every ε_{ij} is holomorphic after possibly shrinking B. Consequently, $\varepsilon(0) = 0$. (We may also proceed by a power series argument. Working in the analytic local ring and ignoring units, let $F_j = z^a$ and $\partial\tau_{ij}/\partial z_k = \sum_{b\geq 0} c_b z^b$. Then

$$\frac{1}{F_j(z)} \int_{\gamma_z} F_j \frac{\partial \tau_{ij}}{\partial z_k} \, dz_k = z_k \sum_{b\geq 0} \frac{c_b}{|a| + |b| + 1} z^b,$$

showing that $\varepsilon(0) = 0$.) This completes the proof. □

Remark A.3. In the proof of Lemma A.2, we at least need the divisors of the $\{F_j\}$ to be star-shaped with respect to the origin. Otherwise indeterminacy cannot be avoided: for instance, in

$$F = \begin{bmatrix} z_1 + z_2^2 \\ z_2^2 \end{bmatrix}, \quad \tau = \begin{bmatrix} z_1 + \mathbf{i} & -z \\ -z & z_1 + \mathbf{i} \end{bmatrix}, \quad \varepsilon_{11} = \frac{z_1^2/2 + z_1 z_2^2/3}{z_1 + z_2^2},$$

ε_{11} is indeterminate at 0.

Finally, if we remove the assumption that F and G vanish at 0, then applying Theorem A.1 to $F - \mathrm{Re}(F(0))$ and $G - \mathrm{Re}(G(0))$ yields a positive-dimensional complex analytic variety Y through the origin on which F and G are constant. (If the sequence $\{x_i\}$ avoids some $\Sigma \subset B$, then Y is obviously not contained in Σ.) This contradicts the maximal rank assumption on ∇R and yields the alternative proof claimed at the start of this appendix.

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Conflicts of interest

None.

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