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Flows without minimal set

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Abstract. In this paper we prove, using explicit constructions, that every non-compact manifold (except, of course, surfaces of finite genus) can be endowed with a non-singular flow without minimal subset, that is to say: a flow such that each orbit closure contains a smaller one.

0. *Introduction*

One of the objectives of the theory of dynamical systems, since its very beginning as a branch of celestial mechanics, has been to find periodic orbits. When there are none, one hopes to find at least a *minimal subset*. In this paper, we build counter-examples. Of course, the ambient space has to be non-compact. We show that most of the non-compact manifolds carry flows without minimal subset.

Let us define what a 'minimal set' is in this non-compact frame: a minimal set is a minimal, closed, non-empty, saturated set; or equivalently, an orbit closure that contains no smaller one; for example a single orbit which is periodic or properly embedded.

This problem has been somewhat classical since the seventies: Hector proved that $\mathbb{R}^3$ carries a two-dimensional foliation without minimal subset [**1**]; it follows immediately that $\mathbb{R}^n$ carries a p-dimensional one, for every $n \geq 3$ and every $2 \leq p \leq n-1$. Thus, attention was called to the case of flows. Schweitzer asked whether $\mathbb{R}^3$ carries one, and more generally if there exists some open manifold with such a flow [**5**]. In a lecture at Oberwolfach, July 1995, Kuperberg announced an example in $\mathbb{R}^3$. The same year, Inaba proposed a construction on a surface. The lecture of his preprint was the source of our interest in the subject.

Let us first recall the list of manifolds on which it is known that every flow does contain a minimal subset. Compact manifolds, as results from Zorn's lemma. The real line, obviously. Surfaces of finite genus, because [**2**] any flow on such a space has a height of at most three. It means the following: take a leaf closure. If it happens not to be minimal, take a smaller one. If it happens not to be minimal, take a smaller one. Then this third one is necessarily minimal.

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THEOREM. *Every non-compact manifold carries a flow without minimal set, except of course the real line and surfaces of finite genus.*

We work in the smooth (C^∞) category; but on one side our demonstration would also lead to the same result in the C^0 category; and on the other side, in dimension two our construction is in fact real analytic. It can also be noted that our flows have all leaves proper, although of course none is properly embedded.

The plan of the proof is as follows. In §1, one builds a first flow without minimal subset on a certain surface. In §2, cut-and-paste operations performed on this surface, transversely to the flow, give an example on every surface of infinite genus. In §3, we prove the rest of the theorem by induction on the dimension: examples of dimension $n-1$ are used to trap orbits of arbitrary n-dimensional Smale flows—much like in Schweitzer's counter-example to the so-called Seifert conjecture—and the result is an n-dimensional flow without minimal subset.

Conventions. Throughout this paper manifolds are assumed smooth, connected, paracompact, Hausdorff; and unless otherwise specified, without boundary. A surface is a two-dimensional manifold. A flow (or foliation) is a smooth, tangentially orientable, one-dimensional foliation.

The ends of non-compact manifolds will play a crucial role. Recall that the space of ends of a locally compact space X is defined as the inverse limit of the discrete space $\pi_0(X\backslash K)$ when the compact subset $K \subset X$. For X a non-compact manifold with boundary, we keep a cautious distinction between, on one side the connected components of the boundary of X, and on the other side the ends of X. The components of ∂X are contained in X; the ends are not.

1. *First example of a foliated surface without minimal subset*

In this section and the next one, we work in the real analytic differentiability class. Let the band $B = \mathbb{R} \times [0,1]$ be equipped with the coordinates $x \in \mathbb{R}$, $y \in [0,1]$ and endowed with the flow $\partial/\partial x$. It contains the *ladder* $H \subset B$ defined by $x \in \mathbb{Z}$ or $y = 0$ or $y = 1$. One defines a strip-like surface as a 2-manifold S with boundary, endowed with a flow, that fulfills the following three conditions.

(i) There exists a proper embedding $h : H \to S$ such that $h|_{\partial B}$ is a diffeomorphism of $\partial B = \{0,1\} \times \mathbb{R}$ onto ∂S; call h a ladder and $h(\{n\} \times [0,1])$ the nth round.

(ii) The flow is tangent to ∂S and transverse to every round.

(iii) Every leaf meets infinitely many different rounds.

We can think of (i) and (ii) as meaning that S is obtained from B by adding handles and ends and modifying the flow, but only in the complement of H (Figure 1). The third condition allows some control on the global dynamics.

Call a map between strip-like surfaces, $\phi : S \to S'$, *centrifugal* if it is a proper, foliation-preserving embedding of S into the interior of S' (thus $\phi(S)$ is a saturated subset of the interior of S') such that each leaf in $\phi(S)$ accumulates on $\partial S'$ (in particular, ϕ is not proper as a map into the whole of S'). Consequently $\phi(S)$ does not intersect any minimal subset of S'.

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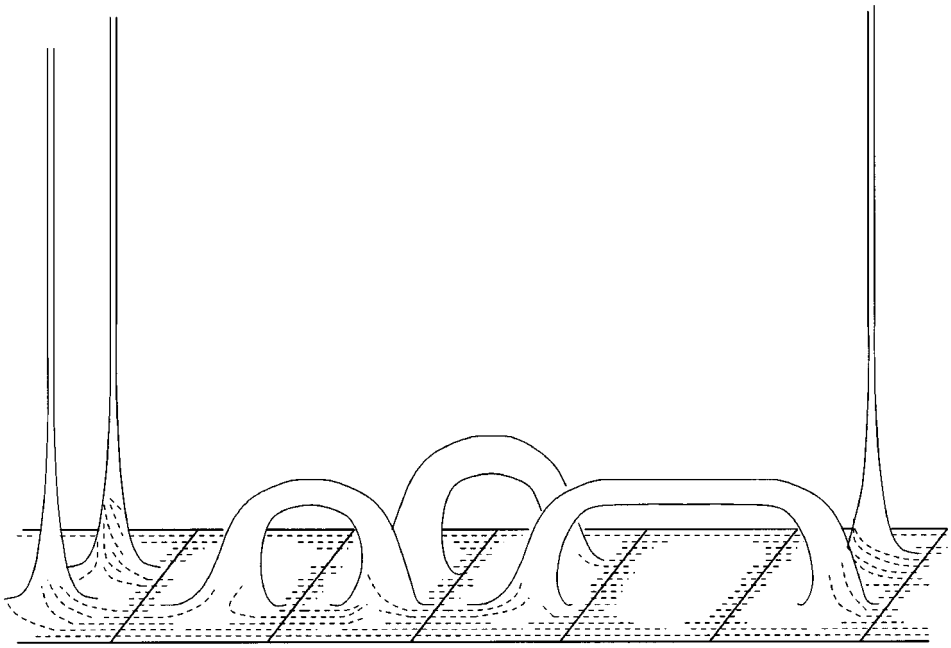


FIGURE 1.

We shall build an infinite sequence of centrifugal embeddings of strip-like surfaces:

$$M_0 \overset{\phi_0}{\to} M_1 \overset{\phi_1}{\to} \cdots .$$

It follows immediately that the direct limit M is a foliated surface without boundary and without minimal subset.

Let $M_0 = B$. To make a strip-like surface M_1 that admits a centrifugal embedding of M_0, fix a diffeomorphism f of the interval $[0, 1]$ such that $f(y) > y$ for every $0 < y < 1$ and, say, $f(1/3) > 2/3$. Choose points a_i's, $i \in \mathbb{Z}$, such that:

$$f^{p-1}(2/3) < a_{2p-1} < a_{2p} < f^p(1/3).$$

In other words the intervals $]a_{2p}, a_{2p+1}[$ are disjoint, and $]a_{2p}, a_{2p+1}[$ contains $[f^p(1/3), f^p(2/3)]$. Make sure that no two different points a_i, a_j are on the same orbit of f. This is certainly true for a generic choice of the a_i's. Finally, choose two sequences (x_p), (x'_p), where p describes $\mathbb{Z}$, such that $x'_p < 0 < x_p$ and $x_p \to +\infty$ when $p \to \pm\infty$ and $x'_p \to -\infty$ when $p \to \pm\infty$. To make things clearer, choose x_p, x'_p not in $\mathbb{Z}$.

For every p, let the transverse compact segment $I_p \subset B$ be defined by $x = x_p$ and $a_{2p} \le y \le a_{2p+1}$ and let the transverse compact segment $I'_p \subset B$ be defined by $x = x'_p$ and $f(a_{2p}) \le y \le f(a_{2p+1})$. See Figure 2.

Perform the following cut-and-paste operations: for each p, the band B is incised along segments I_p and I'_p; the left lip of I_p and the right lip of I'_p are pasted by identifying every point (x_p, y) belonging to the interior of I_p to $(x'_p, f(y)) \in I'_p$; the right lip of I_p and the

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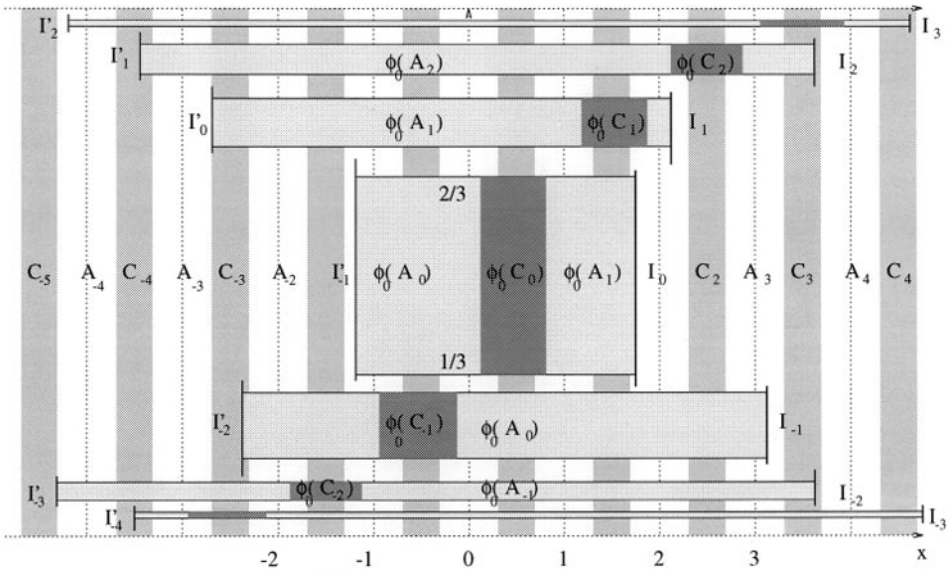


FIGURE 2.

left lip of I'_p are taken off the surface, as well as the extremities of I_p and I'_p. Note that from a topological viewpoint this is like adding infinitely many handles and an isolated flat end. Let M_1 be the resulting foliated surface.

Obviously the natural embedding of H makes M_1 verify (i) and (ii). To prove (iii), regard the flow on M_1 as a discontinuous flow on B, which coincides with $\partial/\partial x$, except when it hits some segment I_p or I'_p. Let us see in detail what happens then. Let (x, y) be the hit point. Keeping to the ideas that we follow the flow downstream:

- if $(x, y) \in I_p$ and is interior to I_p, in other words $x = x_p$ and $a_{2p} < y < a_{2p+1}$, then you jump to $(x'_p, f(y)) \in I'_p$ and continue;
- if $(x, y) \in I_p$ and is extremal, in other words $x = x_p$ and $y = a_{2p}$ or a_{2p+1}, then you stop;
- if $(x, y) \in I'_p$, then you stop.

The reverse flow has symmetric properties: hitting interior points of I'_p makes you jump to I_p; hitting extremal points of I'_p and all points of I_p makes you stop.

Let $L \subset M_1$ be a leaf. Is it possible that its positive end be an extremity of some I_p, say (x_p, a_i), and that its negative end be an extremity of some I'_q, say $(x'_q, f(a_j))$? No, because a_i and a_j would be in the same orbit of f, but would be different since $a_i \geq f(a_j)$. Thus, we may and do assume that L does not end at the extremity of any I_p, the other case being symmetric and left to the reader. Follow L downstream. There are three cases.

- The positive end of L is some point $(x'_q, y_0) \in I'_q$. Then L must contain the halfline $y = y_0$, $x \leq x'_q$, and meet the nth round for every $n \leq x'_q$.
- The leaf L jumps infinitely many times. Since $f^n(y) \to 1$ as $n \to +\infty$, it follows that L accumulates on the upper component of ∂M_1. Thus L meets every round.
- The leaf L is positively complete and jumps finitely many times. After some point

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(x_0, y_0), it just follows the flow $\partial/\partial x$. It contains the halfline $y = y_0$, $x \geq x_0$, and therefore meets the nth round for every $n \geq x_0$. Thus M_1 does fulfill condition (iii).

For every $y \in [1/3, 2/3]$, since $f^n(y)$ tends to 1 for $n \to +\infty$, it follows that the left through $(0, y)$ accumulates on ∂M_1. Thus we have a centrifugal embedding:

$$\phi_0 : B \to M_1 : (x, y) \mapsto (0, y/2 + 1/3) + x,$$

where $(p, t) \mapsto p + t$ denotes the flow on M_1 considered as an action of the real numbers (Figure 2).

By induction, given a strip-like surface M_k, build a larger one M_{k+1} by taking $\phi_0(M_0)$ off M_1 and pasting M_k in place of it. More precisely, fix an $0 < \epsilon < 1/2$ and partition B into rectangles.

$$A_n =]n - \epsilon, n + \epsilon[\times [0, 1], \quad C_n = [n + \epsilon, n + 1 - \epsilon] \times [0, 1].$$

Let A (respectively C) be the union of the A_n's (respectively C_n's).

Changing if necessary $\phi_0(x, y)$ to $\phi_0(u(x, y), y)$, we can make each $\phi_0(C_n)$ be contained in the region of M_1 defined by $n < x < n + 1$. Then the following properties are obvious (Figure 2):

(a) the restriction $\phi_0|_C$ is proper;
(b) $\phi_0(C)$ does not meet the ladder of M_1;
(c) $\phi_0(A_n)$ meets the zeroth round, for every $n \in \mathbb{Z}$.

Obviously the ladder of M_k extends to a foliation-preserving embedding $(A, \partial A) \to (M_k, \partial M_k)$, still denoted h. Let

$$M_{k+1} = M_k \coprod_{\phi_0 \circ h^{-1}} (M_1 \backslash \phi_0(C))$$

be the sum of M_k and $M_1 \backslash \phi_0(C)$ above the diffeomorphism $\phi_0 \circ h^{-1} : h(A) \to \phi_0(A)$. Let

$$\phi_k : M_k \to M_{k+1} \quad \text{and} \quad \psi : M_1 \backslash \phi_0(C) \to M_{k+1}$$

be the canonical embeddings.

From (a), it follows that M_{k+1} is a foliated manifold with boundary. Now we prove M_{k+1} is strip-like. (b) implies that the ladder of M_1 followed by ψ fulfills conditions (i) and (ii). To verify (iii), let L_{k+1} be any leaf in M_{k+1}. Either $L_{k+1} = \psi(L_1)$ for some leaf $L_1 \subset M_1$, or $L_{k+1} = \phi_k(L_k)$ for some leaf $L_k \subset M_k$.

- In the first case, since L_1 meets infinitely many rounds of M_1, it follows that L_{k+1} also meets infinitely many rounds of M_{k+1}.
- In the second case, for infinitely many n's, L_k meets the nth round of M_k. In other words, $h^{-1}(L_k)$ contains a leaf of A_n, say $L'_n =]n - \epsilon, n + \epsilon[\times \{y_n\}$. Obviously, in M_1 the sequence of segments $\phi_0(L'_n)$ accumulates on at least one component of ∂M_1. Applying ψ, one gets that L_{k+1} accumulates on at least one component of ∂M_{k+1}. In other words, ϕ_k is centrifugal. In particular, L_{k+1} meets every round of M_{k+1}: condition (iii) is verified. The construction of M is achieved.

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2. *The general surface case*

First recall the classification of 2-manifolds [**3**]: each surface S has a finite or infinite genus $\mathrm{G}(S)$, is orientable $(+)$ or not $(-)$, and has a space of ends $\mathrm{E}(S)$ which is compact, metrizable and totally disconnected. This space contains the open subset $\mathrm{OE}(S)$ of orientable ends, containing itself and the open subset $\mathrm{FE}(S)$ of flat ends. If S is orientable then $\mathrm{OE}(S) = \mathrm{E}(S)$. Also, $\mathrm{G}(S) < \infty$ if and only if $\mathrm{FE}(S) = \mathrm{E}(S)$. Conversely, any quintuple (G, $\pm$, E, OE, FE) as above characterizes, up to diffeomorphism, a unique surface. To be short, call $(\mathrm{G}(S), \pm, \mathrm{E}(S), \mathrm{OE}(S), \mathrm{FE}(S))$ the topological characters of S. We endow $\bar{S} = S \cup \mathrm{E}(S)$ with the usual topology.

Let X be a surface of infinite genus. To get the wanted flow on X we shall start from the foliated surface M built in §1 and apply successive cut-and-paste operations to give it the same topological characters as X.

These operations will be of the following type. Let S be a surface whose ends are all orientable, endowed with a flow; and let $I \subset S$ be a transverse properly embedded one-dimensional manifold with boundary, whose connected components I_n are compact. In other words, I_n is a locally finite set of transverse disjoint compact segments. Let p be the finite or infinite number of the I_n's. We shall be interested in the cases $p = 2$ and $p = \infty$.

Choose a tangential orientation for the flow, thus every I_n has an upstream side and a downstream side; choose an oriented connected open set $U \subset S$ containing I and whose complement $S \backslash U$ is compact. Choose a proper embedding ϕ of $(\mathbb{Z}/p\mathbb{Z}) \times [0, 1]$ (if $p < \infty$) or $\mathbb{Z} \times [0, 1]$ (if $p = \infty$) into S, whose image is I, and that gives each I_n the tangential orientation which is compatible with those of the flow and of U.

Incise S along every segment; take off the endpoints; a new foliated surface $S'_+ = \mathrm{CP}(S, I, +)$ is obtained by pasting the upstream lip of every segment to the downstream lip of the next one, using the obvious diffeomorphism $\phi(n, t) \mapsto \phi(n + 1, t)$. In the same way define $S'_- = \mathrm{CP}(S, I, -)$ using the orientation-reversing diffeomorphism $\phi(n, t) \mapsto \phi(n + 1, -t)$.

What are the characters of $S'_\pm$? It is obvious that $\mathrm{G}(S'_\pm) = \mathrm{G}(S) + p$; that S'_+ has the same orientability as S; and that S'_- is not orientable. What about the ends?

If $p < \infty$, then $\mathrm{E}(S'_+)$ equals $\mathrm{E}(S)$ plus two isolated flat orientable ends; and $\mathrm{E}(S'_-)$ equals $\mathrm{E}(S)$ plus one or two isolated flat orientable ends, depending on the parity of p.

If on the contrary $p = \infty$, let $\Lambda = \mathrm{E}(S) \cap \bar{I}$ be the limit set of I. Then $\mathrm{E}(S'_+)$ is homeomorphic to the space $\mathrm{E}(S)$ quotiented by the relation that identifies all points of Λ to one single point. This end is non-flat; it is orientable iff Λ is contained in $\mathrm{OE}(S)$. In the same way, $\mathrm{E}(S'_-)$ equals $\mathrm{E}(S)$ quotiented by the relation that identifies all points of Λ to one single point. This end is non-flat and non-orientable. The flatness and orientability of the other ends have not changed.

In particular the surfaces $S'_\pm$ do not depend, up to diffeomorphism, on the choices of the orientations, nor on the choices of U, ϕ.

Now we set a frame in which applying the above cut-and-paste operations will keep the foliation without minimal subset.

Definitions. Let V be a manifold (of any dimension) endowed with a flow. Call a subset *uniform* if it meets every neighborhood of every end. A *separatrix* is a properly embedded

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half leaf. A *non-separatrix* is a leaf no half of which is properly embedded. Recall that a leaf is *wandering* if it meets the interior of a transverse codimension one ball that meets every leaf at most once. Let $\mathrm{WNS}(V) \subset V$ be the interior of the set of wandering non-separatrices. The flow is *wandering enough* if $\mathrm{WNS}(V)$ is uniform in V.

Coming back to dimension two, if a foliated surface S is wandering enough, then $\mathrm{WNS}(S)$ contains a locally finite, uniform set of transverse compact segments A_k, such that every leaf meets at most one of them, and at most once—call it an *admissible segments system*.

Assume the foliated manifold is both wandering enough and without minimal subset, and I is contained in an admissible segments system. Then the induced flow on $S'_{\pm}$ is obviously also wandering enough. Also, since our cut-and-paste operation has support in $\mathrm{WNS}(S)$, since it permutes half leaves without changing their limit set, and since this limit set is non-empty and contained in the complement $\mathrm{S}\backslash\mathrm{WNS}(S)$, it follows that $S'_{\pm}$ also contains no minimal subset.

We shall obtain X as the last of five successive foliated surfaces, all without minimal set, all wandering enough. All but maybe the last one will have only orientable ends.

First step: back to the surface M of §1. To understand its topology, consider M_1. It is orientable and has infinite genus. To every couple of segments (I_p, I'_p) corresponds an end e_p of M_1, which is isolated in $\mathrm{E}(M_1)$ and flat. Also, to the two ends of B corresponds a single end e_∞ in M_1. It is not flat. The sequence e_p tends to e_∞ as $p \to \pm\infty$. It follows easily that M is also orientable; every end of M, but one, corresponds to some couple (I_p, I'_p) in some M_k, and thus is flat, orientable and isolated.

Let us choose the a_i's more carefully than in §1, to make sure M is wandering enough. Let $\eta > 0$ be so small that $f(1/3 - \eta) > 2/3 + \eta$. If the a_i's are taken such that $]a_{2p}, a_{2p+1}[$ contains $[f^p(1/3 - \eta), f^p(2/3 + \eta)]$ then in M_1 the leaf through $(0, y)$ is, for every $f^p(1/3 - \eta) \leq y < f^p(1/3)$, a wandering non-separatrix. All are disjoint from M_0, and they accumulate, when y tends to $f^p(1/3)$, on the end of M_1 associated to the segments I_p, I'_p. Thus M is wandering enough.

Second step: fixing the orientability. If X is orientable, set $M' = M$. If not, choose in M an admissible segments system; pick up two of them A_0, A_1; and let $M' = \mathrm{CP}(M, A_0 \cup A_1, -)$. Thus in all cases M' has the same orientability as X.

Third step: identifying ends. Let A' be an admissible segments system in M'. Thus $M'' = \mathrm{CP}(M', A', +)$ has the same orientability as X and only one end, which is of course non-flat.

Fourth step: fixing the space of ends. Let A'' be an admissible segments system in M''. Fix in $\mathrm{E}(X)$ a non-flat point e_0. Since any compact metrizable totally discontinuous space is embeddable into the interval, it is easy to find a topological embedding $h : \mathrm{E}(X) \to \bar{M}''$ (recall that $\bar{S} = S \cup \mathrm{E}(S)$) such that $h(e_0)$ is the unique end of M'' and such that $h(\mathrm{E}(X)\backslash\{e_0\})$ is contained in A''.

Obviously $M''' = M''\backslash h(\mathrm{E}(X)\backslash\{e_0\})$ is a surface with the same orientability as X, and there is a homeomorphism $h : \mathrm{E}(X) \to \mathrm{E}(M'')$. All its ends are flat, except $h(e_0)$.

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Fifth step: fixing the flatness of the ends. Let A''' be an admissible segment system for M'''. Since the A'''_k's are uniform in M''', we can extract a subfamily whose accumulation set in $\mathrm{E}(M''')$ coincides with $\mathrm{E}(M''')\backslash h(\mathrm{FE}(X))$. After reindexing these segments, we have a two-parameter family, $A'''_{i,j}$, $i, j \in \mathbb{N}$, such that for every fixed j the sequence $A'''_{i,j}$ converges, when $i \to +\infty$, to some end $e_j \in \mathrm{E}(M''')\backslash h(\mathrm{FE}(X))$; and such that the sequence of these limit ends (e_j) is dense in $\mathrm{E}(M''')\backslash h(\mathrm{FE}(X))$. For each fixed j, perform the cut-and-paste operation $\mathrm{CP}(M''', \{A_{i,j}\}_{i\in\mathbb{N}}, +)$ to get a surface M'''' with the same orientability as X, such that there is a homeomorphism $h : (\mathrm{E}(X) \to \mathrm{E}(M''''))$ carrying $\mathrm{FE}(X)$ onto $\mathrm{FE}(M'''')$.

Sixth step: fixing the orientability of the ends. The same as the fifth step, but change M''' to M''''', FE to OE and $\mathrm{CP}(\cdot,\cdot,+)$ to $\mathrm{CP}(\cdot,\cdot,-)$. You get a surface M''''' with the same characters as X, therefore diffeomorphic to X.

3. *The higher-dimensional case*

In this third section we show that every non-compact manifold M of dimension $n \geq 3$ carries a flow $\mathcal{F}_M$ without minimal set. In fact, an induction on n will prove that M carries a *wandering enough* flow without minimal set (see §2).

Since M is non-compact, it admits a non-singular flow. Embedding infinitely many Wilson's plugs [**6**] changes it to a Smale flow $\mathcal{S}$, i.e. the limit set is a locally finite uniform union of periodic orbits C_k, and every other orbit accumulates on at least one of them.

Now build a transverse hypersurface $T \subset M$ that meets every leaf. To this end, as usual choose for each k, a transverse, embedded, $(n-1)$-dimensional compact ball $T_k \subset M$ that meets C_k in its center c_k, and is otherwise disjoint from the C_ℓ's. We shall join them together to make a connected hypersurface T, but we must take some care because we want every component of ∂T to be a compact $(n-2)$-ball. For every pair of neighbouring balls T_k, T_ℓ, from their union release two balls, that is a $\mathbb{B}^{n-1} \times \mathbb{S}^0$; then on the boundary $\mathbb{S}^{n-2} \times \mathbb{S}^0$ paste a $\mathbb{B}^{n-3} \times \mathbb{B}^1$ transverse to the flow (see Figure 3). Get a connected manifold T whose boundary ∂T is a union of compact $(n-2)$-spheres ∂T_i.

In case $n = 3$, we can make sure that T has infinite genus by adding a handle minus a disk to every T_k (see Figure 3).

Let N be the $(n-1)$-manifold without boundary you get by pasting a $(n-1)$-ball B_i on each ∂T_i. By induction, N carries a wandering enough flow $\mathcal{F}_N$ without minimal subset. It is not difficult to extract from $\mathrm{WNS}(N)$ an open saturated subset which is uniform in N, and such that its complement $\Lambda_N = N\backslash\Omega_N$ is also uniform in N. Notice that every leaf closure meets Λ_N. Since both Λ_N and Ω_N are uniform, applying if necessary to $\mathcal{F}_N$ an isotopy of N we can assume that every c_k belongs to Λ_N and that $\Lambda_N \subset T$. Here we use this lemma: in a manifold, two disjoint discrete subsets can be sent into two disjoint uniform subsets by some isotopy. The details are left to the reader.

Form a plug as follows. Endow the product $P = T \times [-1, +1]$ with coordinates $p \in T$ and $s \in [-1, +1]$. Let X_N be any integrable non-singular vector field tangent to $\mathcal{F}_N$. Let $\chi : T \times [-1, 1] \to [0, 1]$ be a smooth function such that $\chi^{-1}(1) = \Lambda_N \times \{1/2\}$ and that $\chi(p, -s) = -\chi(p, s)$, and that $\chi = 0$ on a neighborhood of $\partial(T \times [-1, 1])$. Define two

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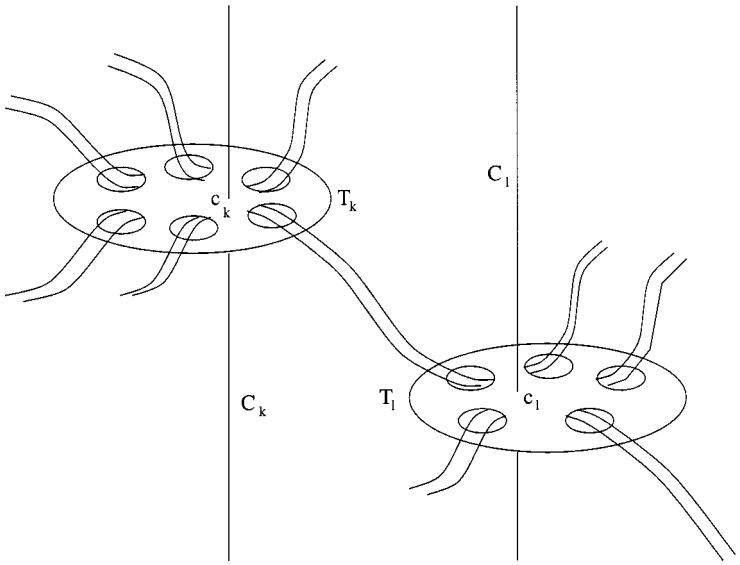


FIGURE 3.

vector fields in P: the vertical one $V_P = \partial/\partial s$ and

$$X_P = \chi X_N + (1 - \chi^2) V_P.$$

Obviously X_P is smooth; it coincides with V_P on a neighborhood of ∂P; and there are two foliation-preserving embeddings

$$\Lambda_N \to \Lambda_P = \Lambda_N \times \{-\tfrac{1}{2}, +\tfrac{1}{2}\} : p \mapsto (p, \pm\tfrac{1}{2}).$$

Exactly as in Schweitzer's plug [**4**], the dynamics of X_P result easily from its definition. For every $p \in \Lambda_N$, the half leaf through $(p, -1)$ is trapped in P, it does not meet $T \times \{1\}$. Every half leaf trapped in P accumulates on Λ_P. For every $p \in T \backslash \Lambda_N$, the half left through $(p, -1)$ is not trapped in P, and quits P on the opposite point $(p, 1)$ [Wilson's matching ends property].

We now modify $\mathcal{S}$ in a neighborhood of T. Obviously there is a proper embedding $\phi : P \to M$ such that $\phi(p, 0) = p$ and $\phi_* V_P = \mathcal{S}$. Change $\mathcal{S}$ to $\phi_* V_P = \mathcal{S}$. Change $\mathcal{S}$ to $\phi_* X_P$ in $\phi(P)$, to get on M a new flow $\mathcal{F}_M$. Denote $\Lambda_M = \phi(\Lambda_P)$.

We claim that the closure of every half leaf L of $\mathcal{F}_M$ meets Λ_M.

This is certainly true if L is trapped in $\phi(P)$. For example, denote L_k the left of $\mathcal{F}_M$ that coincide with C_k outside $\phi(P)$. Since $c_k \in \Lambda_N$, the leaf L_k is trapped in $\phi(P)$ and accumulates on Λ_M.

If L is not trapped, then because of Wilson's matching ends property, L coincides outside $\phi(P)$ with a half leaf of $\mathcal{S}$, say L'. Since $\mathcal{S}$ is a Smale flow, L' accumulates on some C_k, thus L accumulates on L_k, thus on Λ_M.

From the claim follow easily the demanded properties for $\mathcal{F}_M$: on one side, since $\mathcal{F}_N$ had no minimal subset, Λ_N does not contain any, thus Λ_P does not either, thus $\mathcal{F}_M$ does

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not either; on the other side, obviously $M \setminus \Lambda_M$ is wandering and contains no separatrix, so $\mathcal{F}_M$ is wandering enough.

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