numbers $\{a_1, \ldots, a_n\}$ the polynomial

$$P(x) = \sum_{i=1}^{n} (x - a_i)(x - a_{i+1})$$

with $a_{n+1} = a_1$ has a real root is 4.

The problem analysed here could not have occurred to me had not some of the students committed the blunder in the contest. This explains the choice of the title.

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109.06 With or without L'Hospital's rule

The purpose of this Note is to prove the following:

Proposition: Consider a real-valued continuous function f defined in a neighborhood of $+\infty$ such that $\lim_{x \to \infty} f(x)$ exists and is finite. Then, for any antiderivative F of f, we have

$$\lim_{x \to \infty} \frac{F(x)}{x} = \lim_{x \to \infty} f(x).$$

Proof:

Case 1. Assume that $\lim_{x \to \infty} f(x) = L \neq 0$. Write $F(x) = c + \int_A^x f(t)\,dt$, where $A > 0$ is chosen so that, for $t > A$, $f(t) > \alpha > 0$ if $L > 0$ or $f(t) < \beta < 0$ if $L < 0$. It

follows that lim F (x) = + if L > 0 and lim F (x) = − if L < 0. L'Hospital's rule* (see [1]) says that, if a function F is defined and differentiable in a neighborhood of $+\infty$, $\lim_{x \to +\infty} F(x) = \pm\infty$, and $\lim_{x \to +\infty} F'(x)$ exists, then

$$\lim_{x \to +\infty} \frac{F(x)}{x} = \lim_{x \to +\infty} \frac{F'(x)}{1}.$$

In our case, the conditions of L'Hospital's rule are satisfied, and the conclusion of the Proposition readily follows:

$$\lim_{x \to +\infty} \frac{F(x)}{x} = \lim_{x \to +\infty} f(x).$$

* This is the original 17th-18th century French spelling; in time, the silent 's' has been removed and replaced with the circumflex over the preceding vowel. The former spelling is still used in English where there is no circumflex. Published in 1696, the rule is believed to be, in fact, the work of Johann Bernoulli.

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Case 2. Assume that $\lim_{x \to +\infty} f(x) = 0$. We cannot apply L'Hospital's rule directly, as we did in Case 1, because the condition $\lim_{x \to +\infty} F(x) = \pm\infty$ is not always true as, e.g. the function $f(x) = \frac{1}{x^2}$ shows in this case, for any $c \in \mathbb{R}$, $F(x) = c - \frac{1}{x} \not\to \pm\infty$ as $x \to \infty$. Instead, we proceed as follows. As $\lim_{x \to +\infty} f(x) = 0$ by hypothesis, we can choose $A > 0$ sufficiently large in the domain of f (and of F, of course) such that $|f(x)| < 1$ for $x > A$. Equivalently, we have $F'(x) + 1 > 0$ and $F'(x) - 1 < 0$ for $x > A$, meaning that, for $x > A$, the function $g_1(x) := F(x) + x$ is non-decreasing, whereas the function $g_2(x) := F(x) - x$ is non-increasing. As a consequence of monotonicity, both limits $\lim_{x \to +\infty} g_1(x)$ and $\lim_{x \to +\infty} g_2(x)$ exist. If both limits would be finite, we necessarily deduce from the definition of g_1 that $\lim_{x \to +\infty} F(x) = -\infty$, and from the definition of g_2 that $\lim_{x \to +\infty} F(x) = +\infty$. This contradiction shows that at least one of those limits is infinite, say, $\lim_{x \to +\infty} g_1(x) = +\infty$. We now apply l'Hospital's rule to g_1, to the effect that there exists

$$\lim_{x \to +\infty} \frac{g_1(x)}{x} = \lim_{x \to +\infty} \frac{g_1'(x)}{1} = \lim_{x \to +\infty} \big(f(x) + 1\big) = 1,$$

therefore

$$\lim_{x \to +\infty} \frac{F(x)}{x} = \lim_{x \to +\infty} \frac{g(x)}{x} - 1 = 0,$$

as required. The other case, i.e. $\lim_{x \to +\infty} g_2(x) = -\infty$, is quite similar.

This is the original 17th-18th century French spelling; in time, the silent 's' has been removed and replaced with the circumflex over the preceding vowel. The former spelling is still used in English where there is no circumflex. Published in 1696, the rule is believed to be, in fact, the work of Johann Bernoulli.

Reference

1. https://en.wikipedia.org/wiki/L'Hôpital's rule

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109.07 A proof of the irrationality of π using series of circular functions

Introduction

The irrationality of π is an old and interesting problem. In 1760 J.H. Lambert [1] proved the irrationality of π for the first time in a remarkable

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