105.20 Revisiting even and odd square-free numbers

An integer is *square-free* if none of its prime factors appears to a power greater than 1 (this includes 1, since it has no prime factors). Denote by $F(x)$ the number of square-free positive integers not greater than x. It is well known that $F(x) \sim \dfrac{6}{\pi^2}x$ as $x \to \infty$, where the notation $f(x) \sim g(x)$ means $\dfrac{f(x)}{g(x)} \to 1$ as $x \to \infty$ (see, for example, [1, Theorem 333]).

In response to a conjecture in [2], the author showed in [3] that, asymptotically, two thirds of the square free numbers are odd and one third even. This was done by modifying the proof of the result for $F(x)$, using concepts like the Möbius function and Dirichlet convolutions. Here, hoping to spare any future readers unnecessary effort, I offer a much more elementary proof.

Let $F_1(x)$ be the number of even square-free integers not greater than x, and $F_2(x)$ the number of odd ones. There is a very obvious relationship. An even square-free number n is necessarily of the form $4k + 2$: then $\frac{1}{2}n = 2k + 1$ is odd and square-free. The converse is obviously true as well. Consequently, $F_1(x) = F_2\left(\frac{1}{2}x\right)$. Since $F(x) = F_1(x) + F_2(x)$, we have

$$F(x) = F_2(x) + F_2\left(\frac{x}{2}\right). \tag{1}$$

Now $F(x) \sim cx$, where $c = \dfrac{6}{\pi^2}$. Suppose we know that $F_1(x) \sim ax$ and $F_2(x) \sim bx$. Then (1) implies that $F(x) \sim \frac{3}{2}bx$, hence $\frac{3}{2}b = c$, so $b = \frac{2}{3}c$ and $a = \frac{1}{3}c$. Apparently game over!

Of course, the snag is that we don't know, until we have proved it, that $F_2(x)/x$ tends to any limit as $x \to \infty$. The fact that $F_1(x) + F_2(x) \sim cx$, without further information, certainly does not imply that $F_1(x)/x$ and $F_2(x)/x$ tend to limits, even for positive, increasing functions F_1 and F_2.

The matter will be resolved by inverting (1) to express $F_2(x)$ in terms of $F(x)$. To start, we have $F(x/2) = F_2(x/2) + F_2(x/4)$, hence, with (1),

$$F(x) - F\left(\frac{x}{2}\right) = F_2(x) - F_2\left(\frac{x}{4}\right).$$

So for each $r \geqslant 1$,

$$F\left(\frac{x}{2^{2r}}\right) - F\left(\frac{x}{2^{2r+1}}\right) = F_2\left(\frac{x}{4^r}\right) - F_2\left(\frac{x}{4^{r+1}}\right).$$

Add these identities for $0 \leqslant r \leqslant k - 1$. By cancellation on the right-hand side, we obtain

$$F(x) - F\left(\frac{x}{2}\right) + F\left(\frac{x}{4}\right) - \ldots - F\left(\frac{x}{2^{2k-1}}\right) = F_2(x) - F_2\left(\frac{x}{4^k}\right). \tag{2}$$

Clearly, $F_2(t) = 0$ when $t < 1$, so when $4^k > x$, the right-hand side of (2) is simply $F_2(x)$, and we have indeed expressed $F_2(x)$ in terms of $F(x)$.

https://doi.org/10.1017/mag.2021.63 Published online by Cambridge University Press

However, for our purposes, we will apply (2) with another choice of k.

We now choose $\varepsilon > 0$ and let the definition of a limit do the work. There exists x_0 such that $(c - \varepsilon)x \leqslant F(x) \leqslant (c + \varepsilon)x$ for all $x \geqslant x_0$. We will show that for all large enough x, $F_2(x)$ lies between $\left(\frac{2}{3}c - 4\varepsilon\right)x$ and $\left(\frac{2}{3}c + 4\varepsilon\right)x$. We deal with the upper bound first. Let k be the largest integer such that $x/2^{2k-1} \geqslant x_0$. Then $x/2^{2k} < 2x_0$, and for $r \leqslant k - 1$, we have

$$F\left(\frac{x}{2^{2r}}\right) \leqslant (c + \varepsilon)\frac{x}{2^{2r}},$$

$$F\left(\frac{x}{2^{2r+1}}\right) \geqslant (c - \varepsilon)\frac{x}{2^{2r+1}}.$$

So the left-hand side of (2) is not greater than

$$cx\left(1 - \frac{1}{2} + \frac{1}{2^2} - \ldots - \frac{1}{2^{2k-1}}\right) + \varepsilon x\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{2k-1}}\right).$$

By the geometric series, the first bracket equals $\frac{2}{3}\left(1 - \frac{1}{2^{2k}}\right)$ and the second bracket is less than 2. So

$$F_2(x) \leqslant \left(\frac{2}{3}c + 2\varepsilon\right)x + F_2\left(\frac{x}{4^k}\right).$$

Now $x/4^k \leqslant 2x_0$ and obviously $F_2(t) \leqslant t$ for all t, so for $x > x_0/\varepsilon$, we have

$$F_2(x) \leqslant \left(\frac{2}{3}c + 2\varepsilon\right)x + 2x_0 < \left(\frac{2}{3}c + 4\varepsilon\right)x.$$

as required. With minor modifications, a similar proof establishes the lower bound.

References

1. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (5th edn.), Oxford Univ. Press (1979).
2. J. A. Scott, Square-free integers once again, *Math. Gaz.* **92** (March 2008) pp. 70-71.
3. G. J. O. Jameson, Even and odd square-free numbers, *Math. Gaz.* **94** (March 2010) pp. 123–127.

10.1017/mag.2021.63

G. J. O. JAMESON
13 Sandown Road, Lancaster LA1 4LN
e-mail: *pgjameson@talktalk.net*

© The Mathematical Association 2021

https://doi.org/10.1017/mag.2021.63 Published online by Cambridge University Press