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103.24 Partitions, geometric progressions and a Putnam problem

Introduction: A 1983 Putnam problem

Our starting point is the following problem in the 1983 Putnam Mathematical Competition [1]:

Problem 1983: B2

For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8,\ 4+4,\ 4+2+2,\ 4+2+1+1 \text{ and } 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = \lfloor P(n) \rfloor$ for all positive integers n; here $\lfloor u \rfloor$ denotes the greatest integer less than or equal to u.

Before giving a solution to this Putnam problem, we would like to introduce some terminology in the theory of partitions, the study of which can be dated back to the time of Euler.

A *partition* of a positive integer n is a non-increasing sequence of positive integers (which are called the *parts* of this partition) whose sum equals n. Conventionally, we use $p(n)$ to count the number of partitions of n. We also set $p(0) = 1$. This is helpful when we deal with the generating function of $p(n)$:

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$$\sum_{n \geqslant 0} p(n) q^n = \prod_{k \geqslant 1} \left(1 + q^k + q^{2k} + q^{3k} + \ldots\right)$$

$$= \prod_{k \geqslant 1} \frac{1}{1 - q^k} \qquad \text{with } |q| < 1.$$

Here in the first infinite product, the term q^{mk} can be treated in the sense that the part k appears m times.

We may also restrict the parts of a partition to a prescribed set, i.e. all parts must be chosen from a given set. In the 1983 Putnam problem, this prescribed set is the set of powers of 2, namely, $\{1, 2, 4, 8, \ldots\}$. Noting that each different part may appear at most three times, we have the following generating function identity

$$\sum_{n \geqslant 0} C(n) q^n = \prod_{k \geqslant 0} \left(1 + q^{2^k} + q^{2 \cdot 2^k} + q^{3 \cdot 2^k}\right) = \prod_{k \geqslant 0} \frac{1 - q^{4 \cdot 2^k}}{1 - q^{2^k}}$$

$$= \prod_{k \geqslant 0} \frac{1 - q^{2^{k+2}}}{1 - q^{2^k}} = \frac{1}{(1 - q)(1 - q^2)}$$

$$= \left(1 + q + q^2 + q^3 + \ldots\right)\left(1 + q^2 + q^{2 \cdot 2} + q^{3 \cdot 2} + \ldots\right). \quad (1)$$

Now we only need to count the number of partitions of n where the parts are chosen from $\{1, 2\}$ with no restrictions on the number of appearances of each different part. It is therefore not hard to see that $C(n) = \left\lfloor \frac{1}{2} n + 1 \right\rfloor$.

We remark that this approach is different to the official solution; see [1, p. 130].

Some known generalisations

In her Master Thesis, Rucci [2] generalised this Putnam problem by replacing powers of 2 with powers of an arbitrary integer $m \geqslant 2$.

Theorem 1

For positive integers n, let $C_m(n)$ be the number of representations of n as a sum of non-increasing powers of m, where no power can be used more than $m^2 - 1$ times. Then

$$C_m(n) = \left\lfloor \frac{n}{m} \right\rfloor + 1. \quad (2)$$

Note that Rucci's result can be shown by the same generating function approach. In fact, what Rucci proved states that (see [2, pp. 31-32]).

$$\sum_{n \geqslant 0} C_m(n) q^n = \frac{1}{(1 - q)(1 - q^m)}$$

$$= \left(1 + q + q^2 + q^3 + \ldots\right)\left(1 + q^m + q^{2 \cdot m} + q^{3 \cdot m} + \ldots\right) \quad (3)$$

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which is parallel to (1). On the one hand, it is not hard to see that (3) can be paraphrased as

$C_m(n)$ = the number of partitions of n wherein each part is either 1 or m. (4)

This relation, unlike (2), has more combinatorial meaning. On the other hand, for other generalisations of the Putnam problem, we will have generating function identities similar to (3). However, unlike the last identity of (3) which contains merely two multiplicands, there might be more multiplicands. In such cases, the exact expressions of the number of representations would not be as neat as (2). Hence, in the rest of this Note, we want to shift the reader's attention from exact expressions such as (2) to combinatorial relations like (4).

Motivated by Rucci's generalisation, Flowers et al. [3] recently considered a new partition function $b^*_{m,j}(n)$, which counts the number of partitions of n, where each part is a power of m and each different part appears at most $m^j - 1$ times. Let $b_{m,j}(n)$ enumerate the number of partitions of n with each part being chosen from $\{1, m, m^2, \ldots, m^{j-1}\}$. Flowers et al. proved (both analytically and combinatorially) that

Theorem 2:

For positive integers n,

$$b^*_{m,j}(n) = b_{m,j}(n).$$

We observe the following things from the result of Flowers et al.

1. The parts of partitions counted by $b^*_{m,j}(n)$ are chosen from the union of j pairwise disjoint geometric progressions
$$\{1, m^j, m^{2j}, m^{3j}, \ldots\},$$
$$\{m, m^{j+1}, m^{2j+1}, m^{3j+1}, \ldots\},$$
$$\ldots \ldots \ldots$$
$$\{m^{j-1}, m^{2j-1}, m^{3j-1}, m^{4j-1}, \ldots\}.$$
2. The above j geometric progressions have the same common ratio m^j. Furthermore, each different part of partitions counted by $b^*_{m,j}(n)$ appears at most $m^j - 1$ times, which is 1 less than the common ratio m^j.
3. The parts of partitions counted by $b_{m,j}(n)$ are chosen from $\{1, m, m^2, \ldots, m^{j-1}\}$, which is the set of initial values of the above geometric progressions.

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A new generalisation

Main result

From the result of Flowers et al., we have a new generalisation. Let

$$\{a_1, a_1t, a_1t^2, a_1t^3, \dots\},$$

$$\{a_2, a_2t, a_2t^2, a_2t^3, \dots\},$$

$$\dots\ \dots\ \dots$$

$$\{a_H, a_Ht, a_Ht^2, a_Ht^3, \dots\}$$

be a family $\mathcal{F}$ of H *pairwise disjoint* geometric progressions with the same common ratio $t \in \mathbb{Z}$ with $t \geqslant 2$. Here we may take $H = \infty$, in which case there are infinitely many geometric progressions in $\mathcal{F}$. We also require that all the initial values $a_1, a_2, \dots, a_H$ are distinct positive integers.

Let $p_{\#}(n)$ count the number of partitions of n where each part is chosen from the union of the H geometric progressions and each different part appears at most $t - 1$ times.

Let $p_I(n)$ count the number of partitions of n where each part is chosen from $\{a_1, a_2, \dots, a_H\}$, the set of initial values of the geometric progressions.

Theorem 3:

For positive integers n,

$$p_{\#}(n) = p_I(n).$$

Remark: Our result reduces to the result of Flowers et al. by taking $H = j$, $a_h = m^{h-1}$ for $1 \leqslant h \leqslant j$, and $t = m^j$.

Generating functions

We first prove our main result by showing that the two partition functions share the same generating function.

We have

$$\begin{aligned}
\sum_{n \geqslant 0} p_{\#}(n)\, q^n &= \prod_{h=1}^{H} \prod_{k \geqslant 0} \left(1 + q^{a_h t^k} + q^{2 \cdot a_h t^k} + \dots + q^{(t-1) \cdot a_h t^k}\right) \\
&= \prod_{h=1}^{H} \prod_{k \geqslant 0} \frac{1 - q^{t \cdot a_h t^k}}{1 - q^{a_h t^k}} = \prod_{h=1}^{H} \prod_{k \geqslant 0} \frac{1 - q^{a_h t^{k+1}}}{1 - q^{a_h t^k}} \\
&= \prod_{h=1}^{H} \left(\frac{1 - q^{a_h t}}{1 - q^{a_h}} \frac{1 - q^{a_h t^2}}{1 - q^{a_h t}} \frac{1 - q^{a_h t^3}}{1 - q^{a_h t^2}} \cdots \right) \\
&= \prod_{h=1}^{H} \frac{1}{1 - q^{a_h}},
\end{aligned}$$

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which is also the generating function of $p_I(n)$. Hence we arrive at the desired result.

A bijection

Let $\mathcal{P}_\#(n)$ (resp. $\mathcal{P}_I(n)$) denote the set of partitions counted by $p_\#(n)$ (resp. $p_I(n)$). We now construct a bijection between $\mathcal{P}_\#(n)$ and $\mathcal{P}_I(n)$ to make our result more transparent.

Here the only fact we shall use is that every natural number m can be uniquely represented in base t, i.e. we can uniquely write

$$m = m_0 t^0 + m_1 t^1 + m_2 t^2 + \ldots + m_k t^k$$

with $0 \leqslant m_0, m_1, \ldots, m_k \leqslant t-1$ and $m_k \neq 0$. For $m = 0$, we write $0 = 0 \cdot t^0$.

Given a partition in $\mathcal{P}_I(n)$, say $n = f_1 a_1 + f_2 a_2 + \ldots + f_H a_H$ with f_h counting the frequency of a_h for $1 \leqslant h \leqslant H$, we represent the numbers f_h in base t:

$$f_h = f_{h,0} t^0 + f_{h,1} t^1 + f_{h,2} t^2 + \ldots + f_{h,k_h} t^{k_h} \qquad (\text{for all } h).$$

Then we have

$$n = \sum_{h=1}^{H} f_h a_h = \sum_{h=1}^{H} a_h \sum_{i=0}^{k_h} f_{h,i} t^i = \sum_{h=1}^{H} \sum_{i=0}^{k_h} f_{h,i} \left(a_h t^i\right),$$

yielding a partition in $\mathcal{P}_\#(n)$ since $0 \leqslant f_{h,i} \leqslant t-1$ for all h and i, the consequence of base t representations.

Conversely, for any partition in $\mathcal{P}_\#(n)$, say

$$n = \sum_{h=1}^{H} \sum_{i \geqslant 0} f_{h,i} \left(a_h t^i\right),$$

we have

$$n = \sum_{h=1}^{H} \left(\sum_{i \geqslant 0} f_{h,i} t^i \right) a_h,$$

which is a partition in $\mathcal{P}_I(n)$.

We therefore obtain a bijection between $\mathcal{P}_\#(n)$ and $\mathcal{P}_I(n)$.

Example

Suppose that the family $\mathcal{F}$ contains three geometric progressions with initial values $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, and common ratio $t = 4$, then the partitions in $\mathcal{P}_I(8)$ and $\mathcal{P}_\#(8)$ are listed in Table 1, where the two partitions in the same row correspond to each other by the bijection mentioned above. Here $f * (i)$ means that the part i appears f times. For $1 * (3) + 5 * (1) \in \mathcal{P}_I(8)$, we have $5 = 1 \times 4^0 + 1 \times 4^1$, and hence $5 * (1) = 1 * (1) + 1 * (4)$, which gives the corresponding partition in $\mathcal{P}_\#(8)$.

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$\mathcal{P}_I(8)$	$\mathcal{P}_\#(8)$
$2 * (3) + 1 * (2)$	$2 * (3) + 1 * (2)$
$2 * (3) + 2 * (1)$	$2 * (3) + 2 * (1)$
$1 * (3) + 2 * (2) + 1 * (1)$	$1 * (3) + 2 * (2) + 1 * (1)$
$1 * (3) + 1 * (2) + 3 * (1)$	$1 * (3) + 1 * (2) + 3 * (1)$
$1 * (3) + 5 * (1)$	$1 * (4) + 1 * (3) + 1 * (1)$
$4 * (2)$	$1 * (8)$
$3 * (2) + 2 * (1)$	$3 * (2) + 2 * (1)$
$2 * (2) + 4 * (1)$	$1 * (4) + 2 * (2)$
$1 * (2) + 6 * (1)$	$1 * (4) + 1 * (2) + 2 * (1)$
$8 * (1)$	$2 * (4)$

TABLE 1: Partitions in $\mathcal{P}_I(8)$ and $\mathcal{P}_\#(8)$

Closing Remarks

Finally, we show how our result may recover a classical result due to Euler (see [4, Corollary 1.2]).

Let $a_h = 2h - 1$ for $h \geqslant 1$ and $t = 2$. The family $\mathcal{F}$ becomes

$$\{1\cdot 2^0,\ 1\cdot 2^1,\ 1\cdot 2^2,\ 1\cdot 2^3,\ \dots\},$$

$$\{3\cdot 2^0,\ 3\cdot 2^1,\ 3\cdot 2^2,\ 3\cdot 2^3,\ \dots\},$$

$$\{5\cdot 2^0,\ 5\cdot 2^1,\ 5\cdot 2^2,\ 5\cdot 2^3,\ \dots\},$$

$$\dots\ \dots\ \dots$$

We notice that these geometric progressions are pairwise disjoint and their union is the set of positive integers. Consequently, $p_\#(n)$ counts the number of partitions where each different part appears at most once, whereas $p_I(n)$ counts the number of partitions into odd parts. It follows from our result that

Theorem 4: (Euler)

The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

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103.25 An extension of Thébault's second problem

One of the things that amazes me about geometry is how *much* of it there is. Of course, one can say that about any subfield of mathematics, but geometry has been available to both the amateur and professional for such a long time, and has been pored over by so many, many eyes, that one might think there just isn't anything simple left undiscovered. So, when something elementary, yet still *new*, grabs my attention, it's always a delightful surprise to me. This Note provides another such nugget, for your enjoyment.

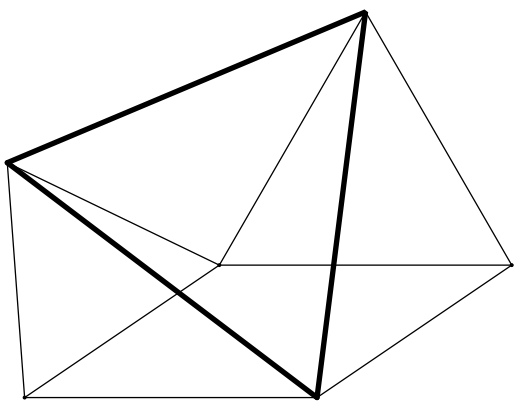


FIGURE 1

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