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On triangle contractive operators in Hilbert spaces

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Abstract

Let H be a finite dimensional real or complex Hilbert space. We denote by $\Delta(x, y, z)$ the area of the triangle with vertices $x, y, z \in H$. A map $f: H \to H$ is triangle contractive TC if $0 < \alpha < 1$ and for each $x, y, z \in H$ either

$$\text{(i)} \quad \Delta(f(x), f(y), f(z)) \leqslant \alpha \Delta(x, y, z)$$

or

$$\text{(ii)} \quad \|f(x) - f(y)\| \leqslant \alpha \|x - y\|$$

and

$$\|f(y) - f(z)\| \leqslant \alpha \|y - z\|$$

and

$$\|f(z) - f(x)\| \leqslant \alpha \|z - x\|.$$

We prove that if f is TC either there is a fixed point $w = f(w)$ or a fixed line $L \supset f(L)$ We characterize the f which are TC and continuous but have no fixed point.

1. *Introduction*

The concept of a *triangle-contractive* TC operator was recently defined and studied by Daykin and Dugdale. These operators are natural generalizations of contractions in the sense that they decrease triangles instead of line segments. They may be related to Birkhoff's area preserving maps.

Throughout this paper H will be a finite dimensional real or complex Hilbert space. The area of the triangle with vertices at $x, y, z \in H$ is denoted by $\Delta(x, y, z)$, and is calculated as half the base times the perpendicular height. Other familiar formulae may give different values (2). In the abstract above we defined what is meant by a TC operator $f: H \to H$. A point $w \in H$ is a *fixed point* of f if $f(w) = w$, and a line $L \subset H$ is called a *fixed line* of f if $f(L) \subset L$. Our main result is

THEOREM 1. *If $f: H \to H$ is TC it has a fixed point or a fixed line or both.*

This theorem was conjectured by Daykin and Dugdale in (1). The case where H is the real Euclidean plane was proved by Daykin in (4) (see also (3)).

Our Theorem 2 characterizes those TC maps f which are continuous but have no fixed point, and hence only a fixed line. In fact, whenever we assume f is TC we will also assume it is continuous, because it is shown in (1) that if f is TC but not continuous then $f(H)$ is part of a fixed line.

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2. *The Daykin–Dugdale conjecture*

LEMMA 1. *If $f: H \to H$ is continuous but has no fixed point then for $n = 1, 2, \ldots$ there is a point $u_n \in H$ with $\|u_n\| = n$ and $f(u_n) = \lambda_n u_n$, where $\lambda_n > 1$.*

Proof. Let $e: H \to H$ be defined by $e(x) = f(x) - x$. Since f has no fixed point $e(x) \neq 0$ for every $x \to H$. Given n we define $e_n: H \to H$ by

$$e_n(x) = ne(x)/\|e(x)\|.$$

Then e_n is continuous and $\|e_n(x)\| = n$ for every $x \in H$. Thus e_n maps the ball of radius n centred at 0 into itself. Hence by Brouwer's fixed point theorem e_n has a fixed point u_n with $\|u_n\| = n$. The equation $e_n(u_n) = u_n$ is the same as

$$f(u_n) = (1 + n^{-1}\|e(u_n)\|)\, u_n$$

so the lemma is proved.

Let (u_n) be the sequence of Lemma 1. Since $(u_n/\|u_n\|)$ is trivially bounded there is a subsequence (z_n) of (u_n) and a vector v such that

$$f(z_n)/\|f(z_n)\| = z_n/\|z_n\| \to v \quad \text{as} \quad n \to \infty. \tag{1}$$

Let Ψ be the line passing through 0 and having the direction of v. Then we have

LEMMA 2. *Suppose $f: H \to H$ is continuous and TC with constant α but has no fixed point. Let Ψ be the line defined as above. Then if L is a line parallel to Ψ its image $f(L)$ is contained in a line cfL parallel to Ψ. Furthermore, if the distance between two lines L and M parallel to Ψ is δ and if the distance between cfL and cfM is ξ then $\xi \leqslant \alpha\delta$.*

Proof. Choose β arbitrarily in $\alpha < \beta < 1$. Let $x \in L$ and (z_n) be the sequence defined as above, then

$$\begin{aligned}\|f(z_n) - f(x)\| &\geqslant \|f(z_n)\| - \|f(x)\| > \|z_n\| - \|f(x)\| \\ &= \beta\|z_n\| + (1-\beta)\|z_n\| - \|f(x)\|.\end{aligned}$$

Because $\|z_n\| \to \infty$, for n sufficiently large we have

$$(1-\beta)\|z_n\| - \|f(x)\| > \beta\|x\|$$

and hence
$$\|f(z_n) - f(x)\| > \beta(\|z_n\| + \|x\|) \geqslant \beta\|z_n - x\| > \alpha\|z_n - x\|. \tag{2}$$

Since f is TC inequality (2) implies that

$$\Delta(f(x), f(y), f(z_n)) \leqslant \alpha\Delta(x, y, z_n) \tag{3}$$

for $x \in L$, $y \in M$ and n sufficiently large.

Let δ_n be the distance of y from the line L_n passing through x and z_n. Similarly let ρ_n be the distance of $f(y)$ from the line L_n' passing through $f(x)$ and $f(z_n)$. Let L' and M' be the lines parallel to Ψ and passing through $f(x)$ and $f(y)$ respectively. Further let ρ be the distance between L' and M' Now z_n and $f(z_n)$ tend to infinity in the direction of Ψ as $n \to \infty$. By Lemma 3 below $\delta_n \to \delta$ and $\rho_n \to \rho$ as $n \to \infty$. Also (2) tells us that $\beta\|z_n - x\| < \|f(z_n) - f(x)\|$ while (3) says that $\frac{1}{2}\rho_n\|f(z_n) - f(x)\| < \frac{1}{2}\alpha\delta_n\|z_n - x\|$. We conclude that $\beta\rho_n < \alpha\delta_n$ and hence in the limit $\beta\rho \leqslant \alpha\delta$. Because β is arbitrary we can also take the limit $\beta \to 1$ to find that $\rho \leqslant \alpha\delta$. Since f is continuous $\rho \to 0$ as $\delta \to 0$. Therefore for any $y \in L$ we have $f(y) \in L'$. So L' is the line cfL and the lemma is proved.

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LEMMA 3. *Let L, L_n, L', L'_n be the lines in the proof of Lemma 2, then $L_n \to L$ and $L'_n \to L'$ as $n \to \infty$.*

Proof. Let $x_n = (z_n - x)/\|z_n - x\|$. Then the line L_n which by definition passes through x has the direction of x_n. Since $\|z_n - x\| \to \infty$ using (1) we have

$$x_n = (z_n/\|z_n\|)(\|z_n\|/\|z_n - x\|) - (x/\|z_n - x\|) \to v$$

thus $L_n \to L$. The second part is similar, so this ends the proof.

Proof of Theorem 1. We assume that f is TC and continuous but has no fixed point. Let G be the hyperplane passing through 0 and orthogonal to Ψ. For any $x \in G$ let L_x denote the line parallel to Ψ passing through x. We define a map $g: G \to G$ as follows. Given $x \in G$ by Lemma 2 the image $f(L_x)$ of the line L_x is contained in a line cfL_x passing through $f(x)$ and parallel to Ψ. The unique point at which cfL_x meets G is to be $g(x)$.

Now if $x, y \in G$ the distance between L_x and L_y is $\|x - y\|$. By definition of g the distance between cfL_x and cfL_y is $\|g(x) - g(y)\|$. So from Lemma 2 we have

$$\|g(x) - g(y)\| \leqslant \alpha \|x - y\|.$$

In other words g is α-contractive and hence has a unique fixed point w. Again by the definition of $g(x)$ we obtain $f(L_w) \subset L_w$, so L_w is a fixed line of f and the theorem is proved.

3. *Conditions for the existence of fixed points*

If u is a unit vector of H let H_u be the subspace of H orthogonal to u and let

$$p_u: H \to H_u$$

be the orthogonal projection into H_u. Given u and $f: H \to H$ we define $h: H \to H$ by $h(x) = (f(x), u)\, u$ for $x \in H$, or simply $h = (f, u)\, u$. Then if $w \in H$ we define $h_w: H \to H$ by $h_w(x) = h(x) + w$ for $x \in H$, or simply $h_w = h + w$. With this notation we have

THEOREM 2. *Suppose $f: H \to H$ is continuous and TC with constant α. Then f has no fixed point if and only if there exists a unit vector u in H such that*

$$f = g + h,$$

where $g: H \to H_u$ is the α-contractive map $g = p_u f p_u$ with fixed point w while $h = (f, u)\, u$ and $h_w = h + w$ has no fixed point.

In Theorem 2 when f has no fixed point the existence of u and so on comes from the proof of Theorem 1. The other implication in Theorem 2 follows immediately from

LEMMA 4. *Let $f: H \to H$ be any operator in H and let u be a unit vector in H. Suppose $f = g + k$ where $g = p_u f p_u$ is α-contractive with fixed point w while $k(x) = \|k(x)\|\, u$ for every $x \in H$. Suppose further that $k_w = k + w$ has no fixed point. Then*

(i) *If L is a line with direction u then $f(L)$ is contained in a line cfL with direction u.*

(ii) *If the distance between two lines L and M of direction u is δ and if the distance between cfL and cfM is ξ then $\xi \leqslant \alpha\delta$.*

(iii) *The line L_w of direction u passing through w is the unique fixed line of f.*

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Proof. Parts (i) and (ii) are straightforward so we deal with part (iii) only. The line L_w is clearly a fixed line. To prove it is unique we suppose that N is a different fixed line and derive a contradiction. Now $p_u g = g$ and $p_u k = 0$ so $p_u f = g$. More generally $p_u f^n = g^n$ for $n = 1, 2, \ldots$ by the induction argument

$$p_u f^n = p_u f f^{n-1} = g f^{n-1}, \quad p_u f p_u f^{n-1} = p_u f g^{n-1} = g g^{n-1} = g^n.$$

If x is any point of N then $f^n(x) \in N$ for $n = 1, 2, \ldots$ and $p_u f^n(x) = g^n(x) \to w$ as $n \to \infty$. Therefore the lines L_w and N intersect in some point. This point is a fixed point of f and hence of k_w, a contradiction which completes the proof.

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