Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2017.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

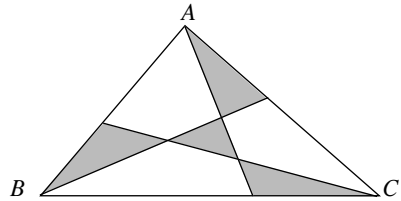
101.E (Marcel Chiriţă)

The triangle ABC has inradius r and circumradius R. The excircle touching side BC has centre I_a and radius r_a with I_b, r_b and I_c, r_c similarly defined. Prove that

$$\frac{I_aB\cos\frac{1}{2}B}{\sqrt{r_b+r_c}}+\frac{I_bC\cos\frac{1}{2}C}{\sqrt{r_a+r_c}}+\frac{I_cA\cos\frac{1}{2}A}{\sqrt{r_b+r_a}}\leqslant\frac{3}{2}\sqrt{\frac{R(2R-r)}{r}}.$$

101.F (Stan Dolan)

Given any triangle ABC, show how to construct three Cevians such that the four shaded triangles in the diagram have equal areas.



101.G (Dao Thanh Oai)

Let ABC be a triangle and let Γ be a circle in the plane of ABC such that two circles through B, C touch Γ at A_1, A_2; two circles through C, A touch Γ at B_1, B_2 and two circles through A, B touch Γ at C_1, C_2.

Show that the three lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

101.H (Mehtaab Sawhney)

Find the smallest positive integer k such that the following inequality holds for all non-negative real numbers a, b, c.

$$(a-b)(a-2b)(ka-c)+(b-c)(b-2c)(kb-a)+(c-a)(c-2a)(kc-b)\geqslant 0.$$

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Solutions and comments on **100.I**, **100.J**, **100.K**, **100.L** (November 2016).

100.I (Stan Dolan)

Let $x_1, \ldots, x_n$ be a sequence of non-zero real numbers such that

$$x_1 - \frac{1}{x_2} = x_2 - \frac{1}{x_3} = \ldots = x_{n-1} - \frac{1}{x_n} = x_n - \frac{1}{x_1}.$$

(a) Prove that either $x_1 = x_2 = \ldots = x_n$ or n is even and $x_1 x_2 \ldots x_n = 1$.

(b) Find the general solution for $x_1, \ldots, x_n$.

Answer: (b) For n even, the solution is either of the form $t, t, \ldots, t$ or $t, \frac{1}{t}, t, \frac{1}{t}, \ldots, t, \frac{1}{t}$ $(t \neq 0)$.

From the impressive variety of solutions submitted, the following by Thomas Kecker, which deal with (a) and (b) simultaneously, caught my eye. For convenience, define $x_{n+i} = x_i$ and let c denote the common value of $x_i - \dfrac{1}{x_{i+1}}$. We consider two cases:

(i) $c = 0$:

In this case $x_i = \dfrac{1}{x_{i+1}}$ for $1 \leqslant i \leqslant n$ and hence $x_i = x_{i+2}$. When n is odd, this implies that $x_1 = x_2 = \ldots = x_n = t$ with $t = \dfrac{1}{t}$ so that $t = \pm 1$. When n is even, we obtain $x_1 = x_3 = \ldots = x_{n-1} = t \neq 0$ and $x_2 = x_4 = \ldots = x_n = \dfrac{1}{t}$ so that $x_1 x_2 \ldots x_n = 1$ in this case.

(ii) $c \neq 0$:

Here we have $x_i = c + \dfrac{1}{x_{i+1}}$ which, starting from $i = n$ and working backwards, produces a continued fraction expression for x_1 with n nested fractions:

$$x_1 = c + \cfrac{1}{c + \cfrac{1}{\ddots + \cfrac{1}{c + \cfrac{1}{x_1}}}}. \qquad (*)$$

Any solution of $x_1 = c + \dfrac{1}{x_1}$ or $x_1^2 - cx_1 - 1 = 0$ is a solution of $(*)$. Conversely, because $(*)$ simplifies to a quadratic in x_1, it has at most two solutions. So the two solutions of $x_1 = c + \dfrac{1}{x_1}$, namely $x_1 = \frac{1}{2}\left(c \pm \sqrt{c^2+4}\right)$ are exactly the solutions of $(*)$. For either choice of x_1, $x_2 = \dfrac{1}{x_1 - c} = x_1$ giving $x_1 = x_2 = \ldots = x_n = t \neq 0$.

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Peter Johnson showed that the general solution takes the same form if complex numbers are allowed and remarked that the same problem for elements of a finite field is worth investigating.

Correct solutions were received from: N. Curwen, M. G. Elliott, GCHQ Problem Solving Group, P. F. Johnson, T. Kecker, J. A. Mundie, S. G. Rout, C. Starr and the proposer Stan Dolan.

100.J (Michael Fox)

A problem for 2016.

Find a natural number n for which 2^n begins with the digits 2016.

Answer: The three smallest values of n are 7286, 9422, 11558.

Solvers displayed considerable ingenuity in their hunts for n with wide variations in the amount of computing power deployed and the amount of theory used.

The most popular calculator-based approach went along the following lines: we summarise the details from Gwyn Owen's solution, writing $\log N$ for $\log_{10} N$ throughout. The mantissa of N is given by $\{\log N\} = \log N - \lfloor \log N \rfloor$ and we want $\{\log 2^n\} = \{n \log 2\}$ to be between $\{\log 2016\} = 0.30449$ and $\{\log 2017\} = 0.30470$. Since $\log 2 = 0.30103$, this is equivalent to requiring that

$$0.00346 < \{(n-1)\log 2\} < 0.00367. \qquad (*)$$

Let $\alpha = 10 \log 2$ with $\{\alpha\} = 0.0103$ and $\beta = 3 \log 2$ with $\{\beta\} = 0.9031$ which is equivalent to -0.0969. Then $\{9\alpha + \beta\} = 0.9958$ which is equivalent to -0.0042. Continuing in this way to combine α, β to reduce the size of the contribution nearer and nearer to zero (essentially by mimicking the Euclidean algorithm) shows that $706\alpha + 75\beta = 7285 \log 2$ satisfies $(*)$ so that $n = 7286$ is a solution.

There are, in fact, infinitely many solutions to such problems. If 2^n starts with a block of digits B, then $\{n \log 2\}$ lies in the interval with endpoints $\{\log B\}$ and $\{\log(B+1)\}$. Kronecker's theorem ensures that, because $\log 2$ is irrational, this will happen for infinitely values of n. The key to finding them is a supply of good rational approximations, $\frac{p}{q}$, to $\log 2$. Stan Dolan noted that, if $0 < q \log 2 - p < \log\left(\frac{B+1}{B}\right)$ and $k = \left\lceil \frac{\{\log B\}}{q \log 2 - p} \right\rceil$, then $k(q \log 2 - p)$ lies between $\{\log B\}$ and $\{\log(B+1)\}$ so that kq is a solution.

The proposer, Michael Fox, used continued fraction approximations to $\log 2$. The even convergents and their intermediate values (as discussed in G. Chrystal's *Algebra*) give good approximations decreasing to $\log 2$; the odd convergents and their intermediates give good approximations increasing to $\log 2$. From $2621 \log 2 - 789 = -0.0003814$ and

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$1166 \log 2 - 351 = 0.0009749$ we have

$$7285 \log 2 - 2193 = 4 \times 0.0009749 - 0.0003814 = 0.00352.$$

Thus, from $(*)$, $n = 7286$ is a solution with $2^{7286} = 2.016\ldots \times 10^{2193}$. The subtlety in this approach is that in order to find the smallest solution, intermediate convergents have to be used.

Finally, a number of solvers commented that 2017 gives an easier problem because $2^{393} = 2.017\ldots \times 10^{118}$. Volkhard Schindler observed that there are solutions for 2016 with negative integers: the first three are -3879, -6015, -8151.

Correct solutions were received from: R. G. Bardelang, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, R. Harris, T. Kecker, M. Lukarevski, J. A. Mundie, G. I. Owen, Z. Retkes, S. G. Rout, V. Schindler, G. B. Trustrum and the proposer Michael Fox.

100.K (Muhammed Osman Corbali)

Let the acute-angled triangle ABC have side-lengths a, b, c and circumradius R. Prove that

$$\frac{a^2b^2}{a^2 + b^2 - c^2} + \frac{b^2c^2}{b^2 + c^2 - a^2} + \frac{c^2a^2}{c^2 + a^2 - b^2} \geqslant 9R^2.$$

Solvers used a range of methods such as algebra, trigonometry and Lagrange multipliers to tackle this neat geometrical inequality. The most popular argument was the following one.

Let O denote the circumcentre of triangle ABC, Δ the area of triangle ABC and Δ_a, Δ_b, Δ_c the areas of triangles OBC, OAC, OAB so that $\Delta = \Delta_a + \Delta_b + \Delta_c$. From the cosine rule,

$$\frac{b^2c^2}{b^2 + c^2 - a^2} = \frac{bc}{2\cos A} = \frac{bc \sin A}{\sin 2A} = \frac{\Delta R^2}{\Delta_a} = \frac{(\Delta_a + \Delta_b + \Delta_c)R^2}{\Delta_a}.$$

Then

$$\sum \frac{b^2c^2}{b^2 + c^2 - a^2} = (\Delta_a + \Delta_b + \Delta_c)\left(\frac{1}{\Delta_a} + \frac{1}{\Delta_b} + \frac{1}{\Delta_c}\right)R^2 \geqslant 9R^2$$

on using the standard AM-HM inequality, $(x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) \geqslant 9$ for positive real numbers x_1, x_2, x_3. There is equality if, and only if, triangle ABC is equilateral.

Correct solutions were received from: R. G. Bardelang, M. Bataille, N. Curwen, P. De, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, T. Kecker, P. Kitchenside, M. Lukarevski, J. A. Mundie, P. Nüesch, Z. Retkes, S. G. Rout, V. Schindler, A. Tee, G. B. Trustrum, L. Wimmer and the proposer Muhammed Osman Corbali.

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100.L (Robert M. Young)

A problem for Boxing Day, "Waste not, want not".

Starting with the unit square, an open-top box is formed in the standard way by removing four equal squares from the corners and then folding up the flaps to form the sides. From these deleted squares, four new boxes are then formed in the same way. At the next stage, sixteen new boxes are formed, and so on *ad infinitum*. How should the successive squares be removed so that the sum of the volumes of all the boxes is a maximum?

Answer: At the nth stage, the squares removed should have side-length $\cos^n 80°$.

Let x_n denote the side-length of the squares removed at the nth stage. The V, the total volume of all the boxes formed is given by

$$V = x_1(1 - 2x_1)^2 + 4x_2(x_1 - 2x_2)^2 + 16x_3(x_2 - 2x_3)^2 + \ldots$$

where, for sense, $0 \leqslant x_1 \leqslant \frac{1}{2}$ and $0 \leqslant x_{n+1} \leqslant \frac{1}{2}x_n$ for all $n \geqslant 1$. As Robert Young, the proposer, observed, since $V(x_1, x_2, \ldots)$ is a continuous function on a compact subset of the Hilbert space ℓ^2 it attains its maximum value. If $x = x_1$ gives the maximum volume V^* then, by scaling, $V^* = x(1 - 2x)^2 + 4x^3V^*$ so that V^* is the maximum value of $f(x) = \dfrac{x(1-2x)^2}{1-4x^3}$ for $0 \leqslant x \leqslant \frac{1}{2}$. But $f'(x) = 0$ gives $(1 - 2x)(8x^3 - 6x + 1) = 0$ so the solution sought satisfies $8x^3 - 6x + 1 = 0$; setting $x = \cos\theta$ shows that $\cos 3\theta = -\frac{1}{2}$ and $x = \cos 80° \approx 0.17365$ with

$$V^* = f(\cos 80°) = \tfrac{2}{3}\cos 80°(1 - 2\cos 80°) \approx 0.075561.$$

By scaling, the squares removed at the nth stage have $x_n = \cos^n 80°$. (Notice that this is not the same as replicating at each stage the optimal solution for just the first stage: maximising $x(1 - 2x)^2$ requires $x = \frac{1}{6}$, but $f(\frac{1}{6}) = \frac{4}{53} \approx 0.075472$.)

Jonny Griffiths commented that he also proposed extending the classical box problem along the lines of **100.L** as a problem in *Mathematical Spectrum*: his solution appears in Volume 36 (January 2004).

Correct solutions were received from: N. Curwen, M. V. Channakeshava, S. Dolan, M. G. Elliott, S. García-Alonso & Á. Plaza, GCHQ Problem Solving Group, J. Griffiths, R. Harris, T. Kecker, J. A. Mundie, Z. Retkes, S. G. Rout, V. Schindler, C. Starr and his A level set, G. B. Trustrum, L. Wimmer and the proposer Robert M. Young

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