The maximum modulus of a complex polynomial on the unit circle

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Consider a complex polynomial $f(z) = \sum_{r=0}^{n} a_r z^r$. Define

$$M(f) = \sup\{|f(z)| : |z| \leqslant 1\}.$$

By the maximum modulus principle, we have equally

$$M(f) = \sup\{|f(z)| : |z| = 1\}.$$

Also, the maximum value is attained at some point with $|z| = 1$, but we will not need this fact.

We record two elementary facts about $M(f)$. First, if

$$g(z) = f(-z) = \sum_{r=0}^{n} (-1)^r a_r z^r,$$

then $M(g) = M(f)$. Second, if

$$h(z) = z^k f(z) = \sum_{r=0}^{n} a_r z^{r+k},$$

then $M(h) = M(f)$.

Obviously we have

$$M(f) \leqslant \sum_{r=0}^{n} |a_r|. \tag{1}$$

Further, if the coefficients a_r are real and non-negative, then

$$f(1) = \sum_{r=0}^{n} a_r = \sum_{r=0}^{n} |a_r|,$$

so equality holds in (1). Similarly, if $(-1)^r a_r \geqslant 0$ for each r, then $f(-1) = \sum_{r=0}^{n} |a_r|$.

Another case in which equality holds is when $f(z)$ has only two terms. To see this, it is sufficient to consider $f(z)$ of the form $1 + a_n z^n$. If $a_n = re^{i\theta}$, take $z = e^{-i\theta/n}$: then $a_n z^n = r = |a_n|$, so $f(z) = 1 + |a_n|$.

Let us denote $\sum_{r=0}^{n} |a_r|$ by $S(f)$. Our objective here is a comparison between $M(f)$ and $S(f)$ for polynomials f in general. To start with, we describe a simple example where $M(f) < S(f)$.

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Example 1: Let $f(z) = 1 - 2z - z^2$. Then $S(f) = 4$, and, for $|z| = 1$,

$$\begin{aligned} |f(z)|^2 &= (1 - 2z - z^2)(1 - 2\bar{z} - \bar{z}^2) \\ &= (1 + 4 + 1) - 2(z + \bar{z}) - (z^2 + \bar{z}^2) + 2z\bar{z}(z + \bar{z}) \\ &= 6 - z^2 - \bar{z}^2. \end{aligned}$$

Clearly, $|f(z)|^2 \leqslant 8$ for all z with $|z| = 1$, and $|f(i)|^2 = 8$, so $M(f) = 2\sqrt{2}$.

What we need for our comparison is an inequality in the reverse sense to (1). More exactly, let c_n be the least constant such that $S(f) \leqslant c_n M(f)$ for all polynomials f of degree not greater than n. By our remark on polynomials with only two terms, we have $c_1 = 1$. Example 1 shows that $c_2 \geqslant \sqrt{2}$. We show next that this is the exact value.

Theorem 1: $c_2 = \sqrt{2}$.

Proof:

After scalar multiplication, we may assume that $f(z)$ is of the form $1 + a_1 z + a_2 z^2$. Let $|a_1| = r_1$, $|a_2| = r_2$. Choose z_0 with $|z_0| = 1$ such that $a_2 z_0^2 = r_2$. Let $a_1 z_0 = x_1 + iy_1$. Choosing either z_0 or $-z_0$, we can ensure that $x_1 \geqslant 0$. Then $x_1^2 + y_1^2 = r_1^2$ and $f(z_0) = s_2 + x_1 + iy_1$, where $s_2 = 1 + r_2$. So

$$|f(z_0)|^2 = (s_2 + x_1)^2 + y_1^2 = s_2^2 + 2s_2 x_1 + r_1^2 \geqslant s_2^2 + r_1^2.$$

Now $S(f) = s_2 + r_1$, so $S(f)^2 = (s_2 + r_1)^2 \leqslant 2(s_2^2 + r_1^2) \leqslant 2M(f)^2$, hence $c_2 \leqslant \sqrt{2}$.

We move on to the estimation of c_n for general n. We derive an upper bound from the easy finite case of Parseval's identity. This is quite well known, but we include the proof, because it is short (and, in the author's view, elegant). It uses integration and the orthogonality of the functions e^{irt}. Define $M_2(f)$ by

$$M_2(f)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^2 \, dt.$$

Clearly $M_2(f) \leqslant M(f)$.

Lemma 1: If $f(z) = \sum_{r=0}^{n} a_r z^r$, then

$$M_2(f) = \sqrt{\sum_{r=0}^{n} |a_r|^2}. \tag{2}$$

Proof: We have

$$\begin{aligned} |f(e^{it})|^2 &= \left(\sum_{r=0}^{n} a_r e^{irt}\right)\left(\sum_{s=0}^{n} \bar{a}_s e^{ist}\right) \\ &= \sum_{r=0}^{n} |a_r|^2 + \sum_{r \neq s} a_r \bar{a}_s e^{i(r-s)t}. \end{aligned}$$

If $r \neq s$, then $\int_{-\pi}^{\pi} e^{i(r-s)t} dt = 0$, so (2) follows.

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Note: Of course, the coefficients a_r are the (complex) Fourier coefficients of the function $F(t) = f(e^{it})$. The normal way to present Parseval's identity is in terms of the Fourier coefficients of a given function, usually an infinite sequence.

So we have $M(f) \geqslant \sqrt{\sum_{r=0}^{n} |a_r|^2}$. Now, by the Cauchy-Schwarz inequality, if $b_j > 0$ for $1 \leqslant j \leqslant m$, then

$$\sum_{j=1}^{m} b_j = \sum_{j=1}^{m} 1.b_j \leqslant \sqrt{m}\sqrt{\sum_{j=1}^{m} b_j^2}.$$

Combining these statements, we can deduce our desired estimation without further proof:

Theorem 2: If $f(z) = \sum_{r=0}^{n} a_r z^r$ and m is the number of non-zero coefficients a_r, then

$$S(f) \leqslant \sqrt{m} M(f). \tag{3}$$

In particular, for any polynomial f of degree n, we have

$$S(f) \leqslant \sqrt{n+1} M(f), \tag{4}$$

so $c_n \leqslant \sqrt{n+1}$.

Of course, (3) and (4) actually apply with $M(f)$ replaced by $M_2(f)$. Equality in (3) holds trivially in the case $m = 1$, but it can never hold when $m > 1$. For this would require that $M_2(f) = M(f)$, which only happens if $|f(z)|$ is constant on $|z| = 1$. To show that this does not occur for polynomials with more than one term, suppose that $f(z) = \sum_{r=0}^{n} a_r z^r$, with a_0 and a_n non-zero. Then for $|z| = 1$, since $\bar{z} = z^{-1}$,

$$|f(z)|^2 - \sum_{r=0}^{n} |a_r|^2 = \sum_{r \neq s} a_r \bar{a}_s z^{r-s}.$$

Denote the expression on the right-hand side by $R(z)$. It contains negative powers of z, so is not a polynomial. However, $z^n R(z)$ is a polynomial, with at least two non-zero terms. If $|f(z)|$ is constant on $|z| = 1$, then $R(z) = C$ (say) for $|z| = 1$, so the polynomial $z^n R(z) - Cz^n$ is zero for all such z, which of course is not possible.

We now turn to the problem of giving a lower estimate for c_n. We will show that $\sqrt{n+1}$ is at least asymptotically the right value. This, perhaps, is rather less well known than Theorem 2. More exactly, we will prove:

Theorem 3: Let $n_k = 2^k - 1$. For each k, there is a polynomial g_k of degree n_k such that $S(g_k) = 2^k$ and $M(g_k) \leqslant \sqrt{2^{k+1}}$. Hence

$$c_{n_k} \geqslant \sqrt{2^{k-1}} = \sqrt{\frac{n_k + 1}{2}}.$$

These polynomials are obtained by an ingenious construction introduced by H. S. Shapiro in his master's thesis [1]: they are known as the Shapiro, or Rudin-Shapiro, polynomials. The method has been generalised for various purposes in later publications.

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We actually define two sequences of polynomials g_k, h_k recursively as follows. Put $g_0(z) = h_0(z) = 1$, then for $k \geqslant 0$,

$$g_{k+1}(z) = g_k(z) + z^{2^k} h_k(z), \tag{5}$$

$$h_{k+1}(z) = g_k(z) - z^{2^k} h_k(z). \tag{6}$$

We see that the next two cases are $g_1(z) = 1 + z$, $h_1(z) = 1 - z$, then

$$g_2(z) = 1 + z + z^2(1 - z) = 1 + z + z^2 - z^3,$$

$$h_2(z) = 1 + z - z^2(1 - z) = 1 + z - z^2 + z^3.$$

It is easily verified by induction that $g_k(z)$ and $h_k(z)$ are of the form

$$g_k(z) = \sum_{r=0}^{n_k} \delta_r z^r, \qquad h_k(z) = \sum_{r=0}^{n_k} \varepsilon_r z^r,$$

where all the numbers δ_r, ε_r are either 1 or -1 (just note that $n_k + 2^k = n_{k+1}$). So g_k and h_k are polynomials of degree n_k, with $S(g_k) = S(h_k) = n_k + 1 = 2^k$.

The critical fact about these polynomials is the identity seen in the next Lemma.

Lemma 2: For all $k \geqslant 0$ and all z with $|z| = 1$, we have

$$|g_k(z)|^2 + |h_k(z)|^2 = 2^{k+1}. \tag{7}$$

Proof: To start, it is clear that $|g_0(z)|^2 + |h_0(z)|^2 = 2$. Assume that (7) holds for a certain k. Recall the parallelogram identity:

$$|a + b|^2 + |a - b|^2 = 2|a|^2 + 2|b|^2$$

for all complex numbers a, b. So by (5) and (6), we have at once

$$|g_{k+1}(z)|^2 + |h_{k+1}(z)|^2 = 2|g_k(z)|^2 + 2|h_k(z)|^2 = 2^{k+2}.$$

In particular, disregarding $h_k(z)$ and invoking the maximum modulus principle, we have $M(g_k) \leqslant \sqrt{2^{k+1}}$. This completes the proof of Theorem 3.

We remark that this also shows that $M(g_k) \leqslant \sqrt{2} M_2(g_k)$. An easy deduction from the Theorem is:

Corollary: For all n, we have $c_n \geqslant \frac{1}{2}\sqrt{n}$.

Proof: From the way we defined it, it is clear that $c_n \geqslant c_m$ for $n > m$. For $n_k \leqslant n < n_{k+1}$, we have $c_n \geqslant c_{n_k} \geqslant \sqrt{\frac{2^k}{2}} > \frac{1}{2}\sqrt{n}$, since $2^k > \frac{n}{2}$.

The polynomials g_k and h_k have delivered the result we wanted, but they are much too interesting for us to abandon them without some further investigation (readers who don't share this view can stop here!). Let us identfy the values at 1.

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Lemma 3: For all k, we have

$$g_{2k}(1) = h_{2k}(1) = 2^k, \tag{8}$$

$$g_{2k+1}(1) = 2^{k+1},\ h_{2k+1}(1) = 0. \tag{9}$$

Proof: For this purpose, write just g_k, h_k for $g_k(1)$ and $h_k(1)$. By (5) and (6), we have $g_{k+1} = g_k + h_k$ and $h_{k+1} = g_k - h_k$. Clearly, $g_0 = h_0 = 1$ and $g_1 = 2$, $h_1 = 0$. Now assume (8) for some $k \geqslant 1$. Applying the recursion twice, we first deduce (9), then $g_{2k+2} = h_{2k+2} = 2^{k+1}$, so that (8) holds for $k + 1$.

The reader may care to verify that

$$g_{2k}(-1) = 2^k, \qquad h_{2k}(-1) = -2^k, \qquad g_{2k+1}(-1) = 0, \qquad h_{2k+1}(-1) = 2^{k+1}.$$

Lemma 3 shows that for all odd k, $M(g_k)$ actually equals the value $\sqrt{2^{k+1}}$ given as an upper estimate in Theorem 3. However, $M(g_k)$ may have a smaller value when k is even: trivially, $M(g_0) = 1$ (not $\sqrt{2}$). We will not attempt an evaluation for even k in general, but we finish by showing that $M(g_2) < \sqrt{2^3}$, so that c_3 is in fact greater than the lower bound $\sqrt{2}$ given by Theorem 3. This also gives another illustration of the kind of methods that are effective for the evaluation of $M(f)$.

Example 2: We evaluate $M(g_2)$. Recall that $g_2(z) = 1 + z + z^2 - z^3$. For $|z| = 1$,

$$|g_2(z)|^2 = g_2(z)\overline{g_2(z)} = \left(1 + z + z^2 - z^3\right)\left(1 + z^{-1} + z^{-2} - z^{-3}\right).$$

We see that the z^2 and z^{-2} terms cancel, giving

$$|g_2(z)|^2 = 4 + \left(z + z^{-1}\right) - \left(z^3 + z^{-3}\right) = 4 + 2\cos t - 2\cos 3t,$$

where $z = e^{it}$. Now $\cos t - \cos 3t = 4x - 4x^3$, where $x = \cos t$. The maximum value for $|x| \leqslant 1$ is $8/(3\sqrt{3})$, occurring when $x = \frac{1}{\sqrt{3}}$. Hence $M(g_2)^2 = 4 + \frac{16}{3\sqrt{3}} \approx 7.079$.

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Reference

1. H. S. Shapiro, *Extremal problems for polynomials and power series*, Thesis for master's degree, Massachusetts Institute of Technology (1951).

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