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REJOINDER

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Introduction

We feel privileged that a number of renowned researchers in the field of unit root testing have commented on our paper. It is the valuable contributions of researchers such as these that have supplied the stimulation and motivation for the current paper, and we gratefully acknowledge this debt. The five sets of comments not only address those matters concerning unit root testing under uncertainty regarding linear trends and initial conditions, as are directly dealt with in the paper, but also suggest thought-provoking directions in which analyses of these issues might be taken forward. Here, we attempt to respond to some of the issues raised.

Admissibility of Conservative Union of Rejections

In the context of testing in the presence of uncertainty regarding the trend, Ulrich Müller adapts the approach of Elliot and Müller (2006)—which dealt with uncertainty regarding the initial condition—to formally construct a family of unit root tests that maximize weighted asymptotic average power over different values of the local trend parameter, κ, within a Gaussian framework. He identifies a member of this family of tests that behaves in a remarkably similar way to our conservative union of rejections (*UR*) procedure in terms of local asymptotic power across a range of values of $|\kappa| \geq 0$, rendering conservative *UR* as being "close to admissible"; consequently, no test can exist that has substantially better asymptotic power across all values of the local trend. This applies equally to the UR^{μ} and UR^{τ} procedures proposed in the context of uncertainty regarding the initial condition, as Müller also notes. It is also reassuring that this desirable property of the union of rejections approach can be extended to the non-Gaussian case, an issue also raised in the commentaries by Peter Burridge and Zhijie Xiao. Müller's insightful analysis therefore provides a rather more rigorous and compelling positioning of the union of rejections procedures than we attempted in our paper.

In his commentary Patrick Marsh also explores issues relating to the optimality properties of our proposed pre-test (*PT*) and *UR* procedures. Focusing on issues relating to information loss and drawing on recent results in Fraser (2004), Marsh argues that the *PT* tests have a compelling justification in that they are underpinned by statistical optimality theory. Using a setup where the (unknown) trend parameter, γ, is generated according to a Bernoulli probability law, Marsh shows that the *PT* statistics can be written as a weighted average over $DF\text{-}QD^{\mu}$ and $DF\text{-}QD^{\tau}$, where the weights are ancillary for ρ, and that the *PT* tests condition on

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this ancillary statistic. Marsh shows that a somewhat weaker result holds for the *UR* tests.

We draw considerable comfort from the fact that the *UR* procedures come near to exploiting available information on the presence of unit roots in an efficient way when one is uncertain about the trend or initial condition, and, hopefully, this important message will not be overlooked by applied researchers. This property is particularly appealing given the very simple nature of the testing strategy.

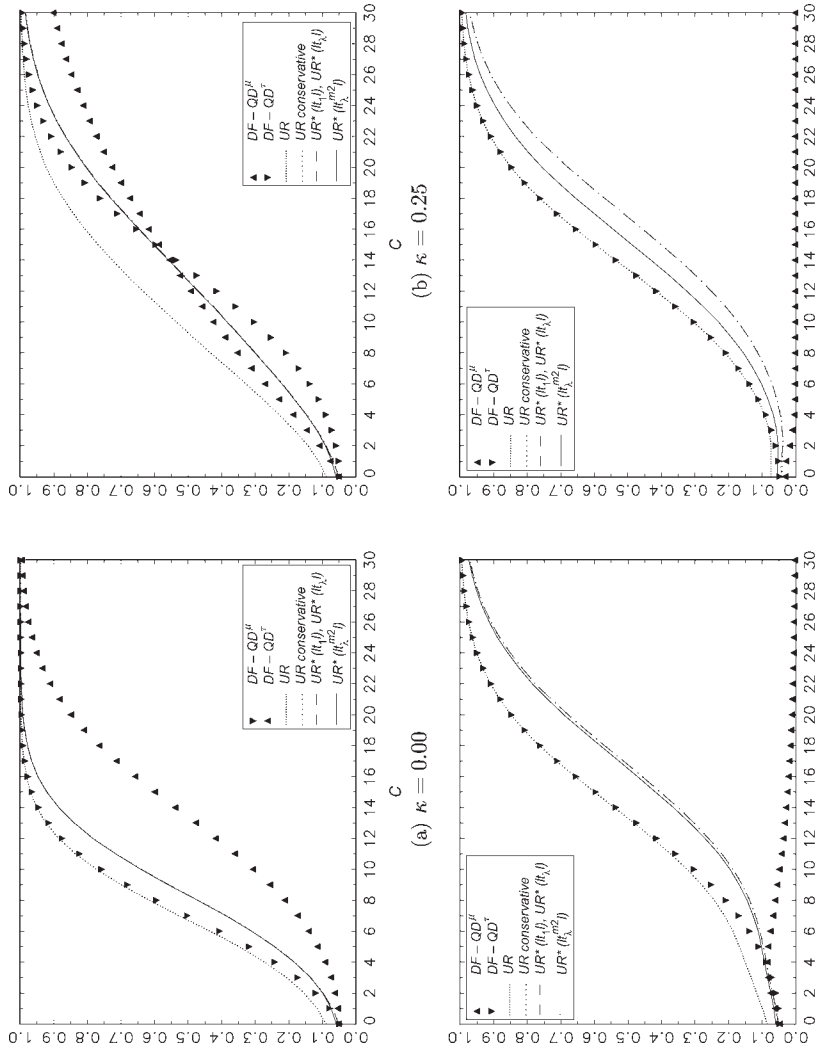
Selection between *UR* and Conservative *UR*

Jörg Breitung considers potential improvements to the union of rejections strategy, arguing that conservative *UR* should really only be employed when $\kappa = 0$, because it is correctly sized there but undersized elsewhere, leading to a consequent loss in power. In fact, for large enough κ, it is the uncorrected *UR* procedure that should be applied, because it is correctly sized for κ of moderate to large magnitude and of course has higher power than its conservative counterpart. Consequently, Breitung suggests a novel hybrid approach whereby a trend pre-test is used to determine the critical values to be applied. That is, if a trend pre-test fails to reject, perform conservative *UR*; otherwise perform its uncorrected counterpart.

The essential idea is that using the pre-test in this way leads to a strategy that should never perform more poorly than conservative *UR*, while at the same time achieving the relative power gains of uncorrected *UR* when κ is large. Although pre-tests are not consistent against a local trend, they should possess enough power to detect the presence of a trend for large κ, thereby suggesting uncorrected *UR*.

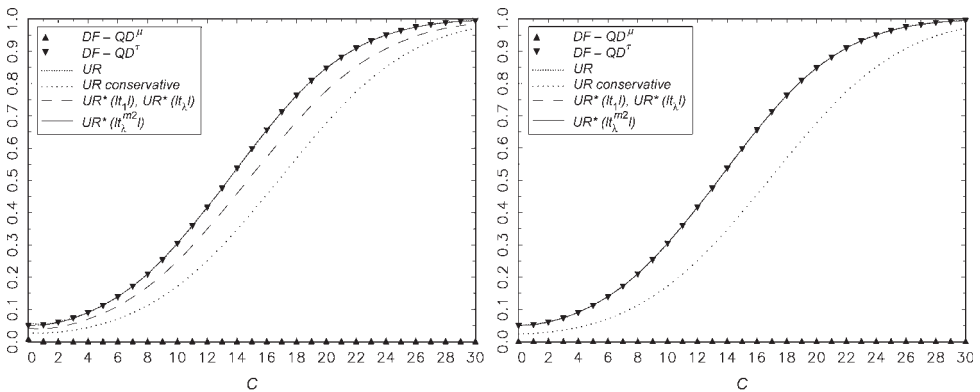
Breitung provides some preliminary small-sample simulations based on the trend pre-test t_1 that clearly demonstrate the potential of this approach. Expanding upon this, in Figures 1 and 2 we show sizes and local powers (at the nominal 0.05 level) for *UR* and conservative *UR*, together with Breitung-type hybrid decision rules based on the pre-tests t_1, t_λ, and t_λ^{m2} given in the paper.[1] We denote these hybrid decision rules as $UR^*(|t_1|)$, $UR^*(|t_\lambda|)$, and $UR^*\left(\left|t_\lambda^{m2}\right|\right)$, respectively. The asymptotic results are given in Figure 1, where we note that the behavior of $UR^*(|t_1|)$ and $UR^*(|t_\lambda|)$ are identical because of the asymptotic equivalence of t_1 and t_λ. Although these strategies are not size-controlled in the manner of conservative *UR*, we first note that the size of $UR^*(|t_1|)$ never exceeds 0.052 across all values of κ considered and the size of $UR^*\left(\left|t_\lambda^{m2}\right|\right)$ never exceeds 0.059 (both occurring when $\kappa = 0.5$). Thus, size distortions are not a significant issue here, particularly for $UR^*(|t_1|)$. As regards power, $UR^*\left(\left|t_\lambda^{m2}\right|\right)$ begins to demonstrate visibly higher power than conservative *UR* once $\kappa \geq 1$, and the same is true of $UR^*(|t_1|)$ once $\kappa \geq 2$, both approaching the same power function as uncorrected *UR* with increasing κ. Finite-sample simulations with $T = 100$ are shown in Figure 2. A very similar qualitative picture arises here, although it is noticeable that the behavior of $UR^*(|t_\lambda|)$ is now more akin to that of the better-performing $UR^*\left(\left|t_\lambda^{m2}\right|\right)$ than $UR^*(|t_1|)$.

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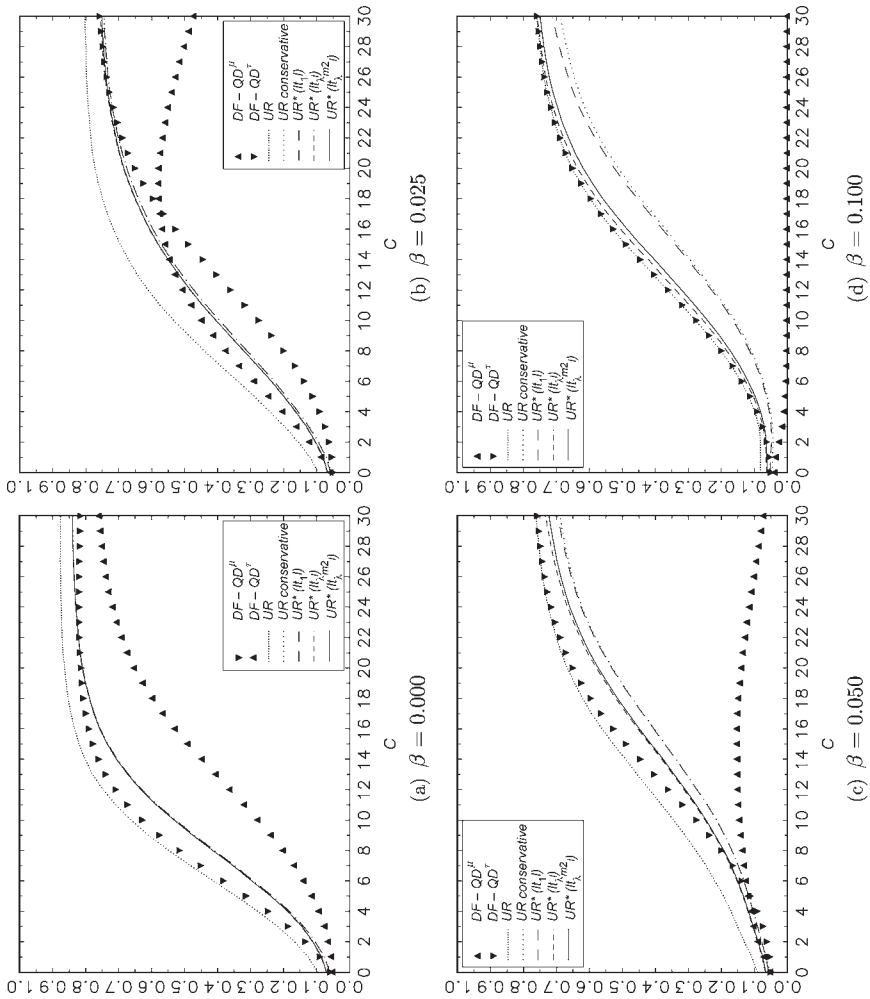
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(e) $\kappa = 2.00$ (f) $\kappa = 4.00$

FIGURE 1. Asymptotic size and local power: $\beta_T = \kappa\omega_\varepsilon T^{-1/2}$.

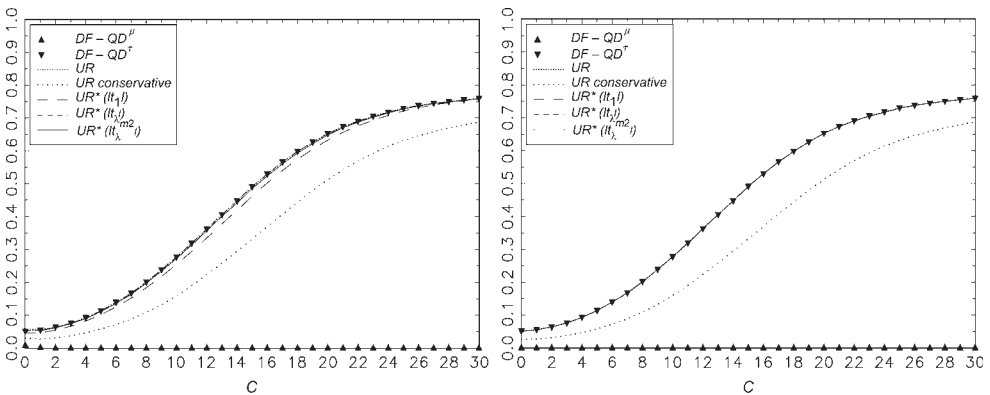


(a) $\beta = 0.000$ (b) $\beta = 0.025$

(c) $\beta = 0.050$ (d) $\beta = 0.100$

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(e) $\beta = 0.200$ (f) $\beta = 0.400$

FIGURE 2. Finite-sample size and power: $T = 100$.

It is important to recognize that these results are not at odds with the asymptotic analysis of Müller discussed in the previous section. Notice that Müller is considering the behavior of a class of tests that leave κ completely unspecified. However, once we use information about the trend, in the form of pre-testing, in conjunction with the *UR* procedure, this is no longer the situation. Moreover, these results are also very much in line with the analysis of Marsh, because these hybrid decision rules are simply exploiting the ancillarity property of the trend pre-test to improve upon the power of the *UR*-type procedure.

Thus, at the additional expense of incurring the calculation of a trend pre-test, for large enough values of a (local) trend, it does seem that worthwhile improvements to the asymptotic and finite-sample power properties of the conservative union of rejections strategy can be made. Of course, it is in the hands of the user to decide as to whether or not the extra computational effort is justified.

Representation of the *UR* Testing Strategies

In the context of testing in the presence of uncertainty regarding the trend, Breitung states that the decision rule associated with *UR* is equivalent to a test based on $\min_\lambda\{WA(\lambda)\}$. In fact, denoting by cv_γ^μ and cv_γ^τ the respective asymptotic null critical values of $DF\text{-}QD^\mu$ and $DF\text{-}QD^\tau$ at a significance level γ, it is clear that

$$\min_\lambda\{WA(\lambda)\} = \min_\lambda\left\{\lambda DF\text{-}QD^\mu + (1-\lambda)\left(\frac{cv_\gamma^\mu}{cv_\gamma^\tau}\right)DF\text{-}QD^\tau\right\}$$

$$= \min\left\{DF\text{-}QD^\mu, \left(\frac{cv_\gamma^\mu}{cv_\gamma^\tau}\right)DF\text{-}QD^\tau\right\}.$$

Consequently, the decision rule associated with *UR* amounts to

$$\text{Reject } H_0 \text{ if } \min\left\{DF\text{-}QD^\mu, \left(\frac{cv_\gamma^\mu}{cv_\gamma^\tau}\right)DF\text{-}QD^\tau\right\} < cv_\gamma^\mu. \tag{1}$$

This representation makes it very straightforward to determine the scaling constant τ_γ to yield the conservative variant of *UR* presented in Section 3.3.2 of the paper. Specifically, setting $c = 0$ and $\kappa = 0$, we can simulate the decision rule in (1) using the (joint) limit distributions of $DF\text{-}QD^\mu$ and $DF\text{-}QD^\tau$ and obtain an asymptotic γ-level critical value from this empirical cumulative distribution function (c.d.f.), say, cv_γ^{UR}. Then τ_γ is obtained simply using $\tau_\gamma = cv_\gamma^{UR}/cv_\gamma^\mu$, which yields the values for τ_γ reported in Table 1A of the paper, without recourse to a grid search procedure. The conservative *UR* decision rule can then be written in a form similar to (1); i.e.,

$$\text{Reject } H_0 \text{ if } \min\left\{DF\text{-}QD^\mu, \left(\frac{cv_\gamma^\mu}{cv_\gamma^\tau}\right)DF\text{-}QD^\tau\right\} < \tau_\gamma\, cv_\gamma^\mu.$$

Needless to say, this approach applies equally to the union of rejections of $DF\text{-}QD^i$ and $DF\text{-}OLS^i$, $i = \mu, \tau$, discussed in Section 4 of the paper, and the

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decision rules can be represented in the following form:

$$\text{Reject } H_0 \text{ if } \min\left\{ DF\text{-}QD^i, \left(\frac{cv_\gamma^{Q,i}}{cv_\gamma^{O,i}}\right) DF\text{-}OLS^i \right\} < cv_\gamma^{Q,i}, \qquad i = \mu, \tau,$$

where $cv_\gamma^{Q,i}$ and $cv_\gamma^{O,i}$ denote the respective asymptotic null critical values of $DF\text{-}QD^i$ and $DF\text{-}OLS^i$, $i = \mu, \tau$, at a significance level γ. This again allows for simpler computation of the scaling constants reported in Table 1B of the paper.

Scope and Limitations of the Analysis

In their commentaries, both Burridge and Xiao point to other important sources of uncertainty, aside from those involving the linear trend and initial condition. These include uncertainty over the innovation distribution; regression model specification (an issue also mentioned in Marsh's commentary), such as the potential presence of higher order trends; variance shifts; structural breaks; etc. All of these are well known to have the potential to bias inference from unit root tests unless their presence is recognized and, where necessary, appropriately modeled. Of course, as less and less is assumed about the absence of such auxiliary factors affecting the data generating process underlying a series, it must become progressively harder for any procedure that needs to be robust to such factors to be capable of identifying departures from the unit root at the same time. The *UR* strategies are no different in this regard, and further research into the impact of these factors would be very welcome, as would extensions of the approaches we develop in the paper to a multivariate (cointegration testing) setting.

Given such constraints, we might, however, ask what then are the factors we should consider to be of most importance? Few would argue against trends needing to be detected where present in the data, simply because of the devastating effects on size and power on unit root tests such as $DF\text{-}QD^\mu$ and $DF\text{-}OLS^\mu$. It is also hard to argue that the power losses suffered when trends are incorporated unnecessarily are insignificant.

The case for focusing on the initial condition may at first sight seem rather less compelling, but it is actually rather easy to highlight its important role at the empirical level—which is, of course, the setting of primary concern to the practitioner. By way of an example, we considered an important and frequently studied U.K. macroeconomic time series: monthly long-term interest rates (the 10-year bond rate) for the period from January 1979 to December 2001. This provides a fairly long sample of 276 observations. For the full period, $DF\text{-}QD^\tau = -3.48$ and $DF\text{-}OLS^\tau = -3.68$, so that both tests reject the unit root at the asymptotic 0.05 level (the critical values are -2.85 and -3.42, respectively).[2] If instead we assume that the data start just four months later, in May 1979 (272 observations), $DF\text{-}QD^\tau = -2.48$ and $DF\text{-}OLS^\tau = -3.63$, such that $DF\text{-}QD^\tau$ now no longer rejects the unit root null. Starting a further four months later, in September 1979 (268 observations), $DF\text{-}QD^\tau = -3.08$ and $DF\text{-}OLS^\tau = -2.48$, and so $DF\text{-}QD^\tau$

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changes inference yet again, rejecting the unit root, but now $DF\text{-}OLS^{\tau}$ fails to reject. Whatever else the properties of these data might be, this clearly shows the sensitivity of both tests to something as simple as where the sample starts; small changes in this (relative to the total sample size) can completely overturn the inference regarding the presence or otherwise of a unit root. We very much doubt that this is a state of affairs in which an applied modeler could take much comfort.

This example shows that we certainly cannot afford to downplay the possible influence of the initial condition on unit root test inference at a practical level and, other forms of uncertainty notwithstanding, test procedures that retain decent power irrespective of the initial condition would seem to be a requisite for robust analysis, an issue also discussed by Marsh in his commentary. Rather interestingly, if we calculated the size-corrected union of rejections of $DF\text{-}QD^{\tau}$ and $DF\text{-}OLS^{\tau}$, UR^{τ} (using the 0.05 level scaling constant of 1.058 from Table 1B), for the January, May, and September 1979 starting points used previously, the unit root is *always* rejected.

Dealing Jointly with Trend and Initial Condition Uncertainty

Both Burridge and Xiao also draw attention to our remark made in the concluding section of the paper concerning the use of a four-way union of rejections to deal with trend and initial condition uncertainty in a joint fashion. This would be a natural extension of the current analysis, to assume no knowledge of either. Letting $cv_{\gamma}^{Q,i}$ and $cv_{\gamma}^{O,i}$ denote the respective asymptotic null critical values of $DF\text{-}QD^{i}$ and $DF\text{-}OLS^{i}$, $i = \mu, \tau$, for individual tests conducted at a significance level γ, the decision rule "reject the $I(1)$ null if any of the four tests rejects" can be written as

$$
\begin{aligned}
UR_4 := {} & DF\text{-}QD^{\mu}\mathbb{I}(DF\text{-}QD^{\mu} < \tau_{\gamma}\, cv_{\gamma}^{Q,\mu}) \\
& + DF\text{-}QD^{\tau}\mathbb{I}(DF\text{-}QD^{\mu} \geq \tau_{\gamma}\, cv_{\gamma}^{Q,\mu})\mathbb{I}(DF\text{-}QD^{\tau} < \tau_{\gamma}\, cv_{\gamma}^{Q,\tau}) \\
& + DF\text{-}OLS^{\mu}\mathbb{I}(DF\text{-}QD^{\mu} \geq \tau_{\gamma}\, cv_{\gamma}^{Q,\mu})\mathbb{I}(DF\text{-}QD^{\tau} \geq \tau_{\gamma}\, cv_{\gamma}^{Q,\tau}) \\
& \times \mathbb{I}(DF\text{-}OLS^{\mu} < \tau_{\gamma}\, cv_{\gamma}^{O,\mu}) \\
& + DF\text{-}OLS^{\tau}\mathbb{I}(DF\text{-}QD^{\mu} \geq \tau_{\gamma}\, cv_{\gamma}^{Q,\mu}) \\
& \times \mathbb{I}(DF\text{-}QD^{\tau} \geq \tau_{\gamma}\, cv_{\gamma}^{Q,\tau})\mathbb{I}(DF\text{-}OLS^{\mu} \geq \tau_{\gamma}\, cv_{\gamma}^{O,\mu}),
\end{aligned}
$$

where a rejection of H_0 is recorded when $UR_4 = DF\text{-}QD^{\mu} < \tau_{\gamma}\, cv_{\gamma}^{Q,\mu}$ or $UR_4 = DF\text{-}QD^{\tau} < \tau_{\gamma}\, cv_{\gamma}^{Q,\tau}$ or $UR_4 = DF\text{-}OLS^{\mu} < \tau_{\gamma}\, cv_{\gamma}^{O,\mu}$ or $UR_4 = DF\text{-}OLS^{\tau} < \tau_{\gamma}\, cv_{\gamma}^{O,\tau}$. Paralleling the discussion in the "Representation of the UR Testing Strategies" section of this rejoinder, we can also represent the decision rule associated with UR_4 in the equivalent form

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$$\text{Reject } H_0 \text{ if } \min\left\{DF\text{-}QD^{\mu}, \left(\frac{cv_{\gamma}^{Q,\mu}}{cv_{\gamma}^{Q,\tau}}\right) DF\text{-}QD^{\tau},\right.$$

$$\left.\left(\frac{cv_{\gamma}^{Q,\mu}}{cv_{\gamma}^{O,\mu}}\right) DF\text{-}OLS^{\mu}, \left(\frac{cv_{\gamma}^{Q,\mu}}{cv_{\gamma}^{O,\tau}}\right) DF\text{-}OLS^{\tau}\right\} < \tau_{\gamma}\, cv_{\gamma}^{Q,\mu}.$$

Setting the critical value scaling constant τ_γ equal to 1 gives rise to an unadjusted procedure. Such a procedure lacks asymptotic size control, the maximum size being obtained when $\kappa = 0$. For individual 0.05 level tests we find this size to be 0.145, which, notably, is somewhat lower than the Bonferroni upper bound of 0.20. To yield a conservative procedure with maximum size 0.05 we need to set $\tau_\gamma = 1.157$.[3]

In this rejoinder we stop short of an examination of the power properties of UR_4. To be informative, these would need to be gauged relative to other feasible procedures, such as unit root tests based on trend pre-testing, and this in itself raises new questions as to the effect of the initial condition on the behavior of trend pre-tests. These are interesting avenues for research that we are actively investigating in separate work.

NOTES

1. The simulation model and parameter settings underlying Figures 1 and 2 are exactly the same as those used to produce Figures 2–3 and 4–5 of the paper, respectively.

2. For transparency, we abstract from the trend issue by only considering unit root tests incorporating both a constant and linear trend term here. Lag selection is performed as in the paper, using the procedure of Ng and Perron (2001) and Perron and Qu (2007).

3. For the benefit of the reader interested in applying UR_4, we also note that for a conservative procedure with asymptotic size 0.10 (0.01) the corresponding value is $\tau_\gamma = 1.197$ (1.113).

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