Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is not necessary to submit solutions to both. Solutions should arrive by May 20th 2019. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the July 2019 edition.

The MA and the *Gazette* comply fully with the provisions of the 2018 GDPR legislation. Submissions **must** be accompanied by the SPC permission form which is available on the Mathematical Association website

https://www.m-a.org.uk/the-mathematical-gazette

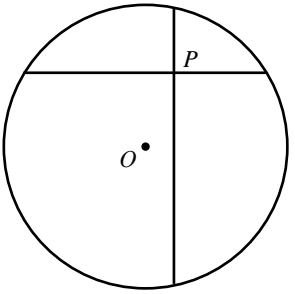
Note that if permission is not given, a pupil ***may still participate and will be eligible for a prize in the same way as others****.*

Problem 2019.1 **(Paul Stephenson)**

A unit circle has centre at the origin O. Two perpendicular chords of the circle intersect at the point $P(x, y)$, where x and y are both positive.

One of the four regions into which the chords divide the circle has area $\frac{\pi}{2}$.

Find the area of the diagonally opposite region in terms of x and y.



Problem 2019.2 **(Chris Starr)**

For which integer values of n is

$$\left(2\cos\frac{\pi}{7}\right)^n + \left(2\cos\frac{3\pi}{7}\right)^n + \left(2\cos\frac{5\pi}{7}\right)^n$$

an integer?

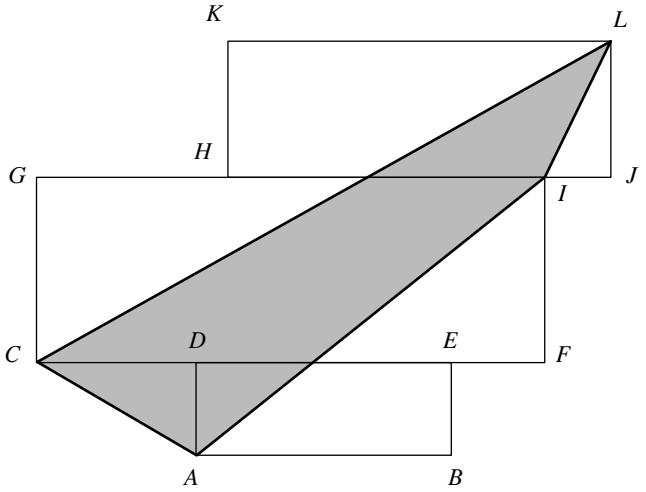
Solutions to 2018.5 and 2018.6

Both problems were solved by Andrew Ng (Tonbridge School). Problem 2018.6 was solved by Ngo Hang Chan (Charterhouse), Sae Koyama (King's College London Mathematics School) and a student for whom no GDPR form was submitted.

https://doi.org/10.1017/mag.2019.45 Published online by Cambridge University Press

Problem 2018.5 **(Stan Dolan)**

The diagram shows three similar rectangles, *ABED*, *CFIG* and *HJLK*. Find, with proof, a trapezium with the same area as the shaded quadrilateral *AILC*.



Solution

Andrew gave a geometric solution using similar triangles. However, this argument can be greatly simplified once one has drawn in line segment *CI*.

CI and *HL* are parallel and so triangles *CIL* and *CIH* have equal areas. Similarly, triangles *CIA* and *CIE* have equal areas.

Trapezium *CEIH* therefore has the same area as quadrilateral *AILC*.

Problem 2018.6 **(Paul Stephenson)**

Consider the two infinite series

$$1+\frac{1}{3}+\frac{1}{6}+\ \dots\ +\frac{2}{n(n+1)}+\ \dots \qquad \text{(Reciprocals of the triangle numbers)}$$

$$1+\frac{1}{2}+\frac{1}{4}+\ \dots\ +\frac{1}{2^{n-1}}+\ \dots \qquad \text{(Reciprocals of powers of 2)}$$

Precisely when is a sum of successive terms of the first series equal to the sum of successive terms of the second series?

An example of two such sums is

$$\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\frac{1}{28}=\frac{1}{2}+\frac{1}{4}.$$

https://doi.org/10.1017/mag.2019.45 Published online by Cambridge University Press

Solution

Andrew, Hang Chan and Sae all used the method of differences to show that the sum of the first $2^n - 1$ terms of the first series is

$$2 - \frac{2}{2^n},$$

This is the sum of the first n terms of the geometric progression.

The terms of the first series can therefore be divided up into successive blocks of $1, 2, 4, \ldots$ terms such that the sum of the terms in the i th block is the i th term of the geometric progression. The example given in the statement of the problem is of this form:

$$\left(\frac{1}{3} + \frac{1}{6}\right) + \left(\frac{1}{10} + \frac{1}{15} + \frac{1}{21} + \frac{1}{28}\right) = \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right).$$

As noted by Andrew, this also gives solutions involving infinite sums. The sum of the terms of the first series after the $(2^n - 1)$ th term will be the same as the sum of the terms of the second series after the n th term.

However, there are other, more complicated, ways of obtaining equal sums. The following solution is a combination of the methods of Andrew and Sae.

Equate the sum of terms of the first series from the a th to the $(b - 1)$ th to the sum of terms of the second series from the $(c + 1)$ th to the $(c + d)$ th. We require

$$2\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{2^d - 1}{2^{c+d-1}}$$
$$\Rightarrow (b - a)2^{c+d} = (2^d - 1)ab.$$

Any odd factor of a must be a factor of $b - a$ and therefore of b. Similarly, any odd factor of b must be a factor of a. Then $a = 2^A X$ and $b = 2^{A+B} X$, where X is odd and so

$$(2^B - 1)2^{c+d} = (2^d - 1)2^{A+B} X$$
$$\Rightarrow X = \frac{2^B - 1}{2^d - 1}, \qquad A + B = c + d.$$

For $2^B - 1$ to be a multiple of $2^d - 1$ we require B to be a multiple of d.

The full solution is therefore as follows. Choose a non-negative integer A and positive integers d and e arbitrarily. Then

$$a = 2^A\left(\frac{2^{de} - 1}{2^d - 1}\right), b = 2^{A+de}\left(\frac{2^{de} - 1}{2^d - 1}\right), c = A + (e - 1)d.$$

The first prize of £25 is awarded to Andrew Ng. The second prize of £20 is awarded to Sae Koyama.

10.1017/mag.2019.45 STAN DOLAN

Note from the Editor

See *www.m-a.org.uk/the-mathematical-gazette* for the SPC questions.

https://doi.org/10.1017/mag.2019.45 Published online by Cambridge University Press