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Integral transforms based upon fractional integration

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Abstract. The theory of Fourier transforms

$$g(x) = \int_0^\infty k(ux) f(u)\, du, \quad f(x) = \int_0^\infty k(ux) g(u)\, du$$

can be developed from the functional equation $K(s) K(1-s) = 1$, where $K(s)$ is the Mellin transform of the kernel $k(x)$.

In this paper I show that reciprocities can be obtained which are analogous to the Fourier transforms above but which develop from the much more general functional equation

$$K(s) K(1-s) = \prod_{i=1}^{n} \frac{\Gamma[\alpha_i + (\eta_i + 1 - s)/m_i]\, \Gamma[\alpha_i + (\eta_i + s)/m_i]}{\Gamma[(\eta_i + 1 - s)/m_i]\, \Gamma[(\eta_i + s)/m_i]}.$$

The reciprocities are obtained by using fractional integration. In addition to the reciprocities we have analogues of the Parseval theorem and of the discontinuous integrals usually associated with Fourier transforms.

In order to simplify the analysis I confine myself to the case $n = 1$ and to L_2 space.

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1. *Introduction.* The reciprocity

$$g(x) = \int_0^\infty k(ux) f(u)\, du, \tag{1}$$

$$f(x) = \int_0^\infty k(ux) g(u)\, du, \tag{2}$$

usually known as the Generalized Fourier Integral Transform, can be proved if certain convergence conditions are satisfied and if, in addition, the kernel $k(x)$ satisfies a functional equation. If $K(s)$ denotes the Mellin transform of $k(x)$, i.e.

$$K(s) = \int_0^\infty k(x) x^{s-1} dx, \tag{3}$$

then the functional equation in question is

$$K(s) K(1-s) = 1; \tag{4}$$

see Hardy and Titchmarsh (4), Titchmarsh (7), p. 214.

In a previous paper (3) I attempted to establish reciprocities for the case when the functional equation satisfied by the kernel is more general than (4). The results, however, resemble (1), (2) only vaguely and are somewhat complicated. In this paper

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I shall show that with the help of fractional integration it is possible to establish reciprocities which resemble (1), (2) closely, although the functional equation satisfied by the kernel is much more general than (4).

In the next section, § 2, I shall collect those results in the theory of fractional integration which we shall need here. I use gothic letters for operators: $\mathfrak{J}$, $\mathfrak{K}$, for various operations of fractional integration, and $\mathfrak{M}$ for the Mellin transform. Thus for (3) we may write $\mathfrak{M}\{k(x)\} = K(s)$. In general pairs of functions such as $k(x)$, $K(s)$; $f(x)$, $F(s)$; $g(x)$, $G(s)$, etc., will be considered to be Mellin transforms of each other. If $k(x) \in L_1(0,\infty)$ then $K(s)$ can be defined by means of (3) and similarly for each of the other pairs. If $k(x) \notin L_1(0,\infty)$ then the relationship between $k(x)$ and $K(s)$ must be expressed in some other manner, usually by using the methods of convergence in the mean. This, of course, also applies to all the other pairs of Mellin transforms. This important point will be discussed in further detail later.

2. *Fractional integration.* Various definitions have been given of fractional integration. We use here the definitions given by Kober and generalized by Erdélyi; see (6) and (2). These authors define four types of fractional integration but of these four types the last two are little more than variants of the first two. For this reason we shall deal with the first two types only, these will be denoted by $\mathfrak{J}[f(x)]$ and $\mathfrak{K}[f(x)]$ where, in both cases, $f(x)$ is fractionally integrated. The definitions are as follows:

$$p \geqslant 1, \quad p^{-1}+q^{-1} = 1, \quad m > 0; \tag{5}$$

$$\operatorname{Re}\alpha > 0, \quad \operatorname{Re}\eta > -1/q, \quad \operatorname{Re}\zeta > -1/p, \tag{6}$$

$$f(x) \in L_p(0,\infty), \tag{7}$$

$$\mathfrak{J}[f(x)] = \mathfrak{J}[f(x);\alpha,\eta,m] \tag{8}$$

$$= \frac{m}{\Gamma(\alpha)} x^{-\eta-m\alpha+m-1}\int_0^x (x^m-u^m)^{\alpha-1}u^{\eta}f(u)\,du, \tag{9}$$

$$\mathfrak{K}[f(x)] = \mathfrak{K}[f(x);\alpha,\zeta,m] \tag{10}$$

$$= \frac{m}{\Gamma(\alpha)} x^{\zeta}\int_x^\infty (u^m-x^m)^{\alpha-1}u^{-\zeta-m\alpha+m-1}f(u)\,du. \tag{11}$$

Condition (7) ensures that $\mathfrak{J}[f(x)]$ and $\mathfrak{K}[f(x)]$ both exist and also that both belong to $L_p(0,\infty)$.

When $m = 1$, (9) and (11) reduce to Kober's fractional integration operators. Conversely take $m = 1$ in (9) and (11) and write $u = t^m$, $x = z^m$ and $f(z^m) = g(z)$ and we obtain Erdélyi's operators from Kober's. These slight changes of variables do not affect any of Kober's proofs so that the proofs of all the statements of this section can be found in his work (6).

The Mellin transform of $f(x)$ is denoted either by $\mathfrak{M}\{f(x)\}$ or by $F(s)$, whichever is more convenient. Similarly either $\mathfrak{M}\{g(x)\}$ or $G(s)$ will denote the Mellin transform of $g(x)$ and so on for other functions of x. We write $s = p^{-1}+it$, where p and t are real. If

$$p \geqslant 1, \quad f(x) \in L_p(0,\infty) \tag{12}$$

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then
$$p = 1, \quad \mathfrak{M}\{f(x)\} = F(s) = \int_0^\infty f(x)\, x^{s-1}\, dx, \tag{13}$$

$$p > 1, \quad \mathfrak{M}\{f(x)\} = F(s) = \lim \int_{1/X}^{X} f(x)\, x^{s-1}\, dx, \tag{14}$$

where lim denotes the usual limit in the mean for L_p spaces. When $1 \leqslant p \leqslant 2$ then $F(s) \in L_q(q^{-1} - i\infty, q^{-1} + i\infty)$, where $p^{-1} + q^{-1} = 1$; see (2), p. 220. We shall confine ourselves later to the case $p = q = 2$ in order to simplify the analysis.

If
$$1 \leqslant p \leqslant 2, \quad f(x) \in L_p(0, \infty), \quad \mathfrak{M}\{f(x)\} = F(s), \tag{15}$$

then
$$\mathfrak{M}\{\mathfrak{J}[f(x); \alpha, \eta, m]\} = \frac{\Gamma[(\eta + 1 - s)/m]}{\Gamma[\alpha + (\eta + 1 - s)/m]} F(s) \tag{16}$$

and
$$\mathfrak{M}\{\mathfrak{R}[f(x); \alpha, \eta, m]\} = \frac{\Gamma[(\eta + s)/m]}{\Gamma[\alpha + (\eta + s)/m]} F(s), \tag{17}$$

where $F(s)$ and the two functions of (17) and (18) all belong to $L_q(q^{-1} - i\infty, q^{-1} + i\infty)$; $p^{-1} + q^{-1} = 1$; see (6), p. 202 and (2), p. 222. These results can be extended to the case where $p > 2$ if $f(x) \in M_p(x)$, a subspace of $L_p(0, \infty)$; see (2), p. 221.

Another formula we shall use is the one known as fractional integration by parts. If $f(x) \in L_p(0, \infty)$ and $g(x) \in L_q(0, \infty)$, $p^{-1} + q^{-1} = 1$, then

$$\int_0^\infty \mathfrak{R}[f(x); \alpha, \eta, m]\, g(x)\, dx = \int_0^\infty f(x)\, \mathfrak{J}[g(x); \alpha, \eta, m]\, dx, \tag{18}$$

where the parameters α, η and m are the same for both operators $\mathfrak{J}$ and $\mathfrak{R}$. These parameters must satisfy $\operatorname{Re}\alpha \geqslant 0$, $\operatorname{Re}\eta > \max(-p^{-1}, -q^{-1})$, $m > 0$.

We shall use (18) in the following form. If $f(x)$ and $g(x)$ both belong to $L_2(0, \infty)$, $\operatorname{Re}\alpha \geqslant 0$, $\operatorname{Re}\eta > -\frac{1}{2}$ and $m > 0$ then

$$\int_0^\infty \mathfrak{J}[f(x); \alpha, \eta, m]\, \mathfrak{J}[g(x); \alpha, \eta, m]\, dx = \int_0^\infty \mathfrak{R}\mathfrak{J}[f(x); \alpha, \eta, m]\, g(x)\, dx, \tag{19}$$

where
$$\mathfrak{R}\mathfrak{J}[f(x); \alpha, \eta, m] = \mathfrak{R}\mathfrak{J}[f(x)] \tag{20}$$

is obtained by integrating $f(x)$ fractionally twice, once using the operator $\mathfrak{J}$ with parameters α, η, m and again using the operator $\mathfrak{R}$ with the same parameters. The order in which these operations are performed is immaterial, as we shall see later. The right-hand side of (20) is an abbreviation for the left-hand side.

If $f(x) \in L_2(0, \infty)$ then $\mathfrak{J}[f(x)] \in L_2(0, \infty)$ and so $\mathfrak{R}\mathfrak{J}[f(x)]$ exists. From (17) and (18) we have

$$\mathfrak{M}\{\mathfrak{R}\mathfrak{J}[f(x)]\} = \frac{\Gamma[(\eta + s)/m]\, \Gamma[(\eta + 1 - s)/m]}{\Gamma[\alpha + (\eta + s)/m]\, \Gamma[\alpha + (\eta + 1 - s)/m]} F(s). \tag{21}$$

This formula holds if $\operatorname{Re}\alpha \geqslant 0$, $\operatorname{Re}\eta > -\frac{1}{2}$, $m > 0$ and $f(x) \in L_2(0, \infty)$ or, what is equivalent, $F(s) \in L_2(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty)$. For discussions of the fractional integration by parts formula see Kober (6), p. 200 and Erdélyi (2), p. 221.

3. *Reciprocities with fractional integration.* We can now illustrate the new type of integral transform to be discussed in this paper.

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Consider the equation

$$g(x) = \int_0^\infty k(ux) f(u)\, du, \tag{22}$$

where $g(x)$ and $k(x)$ are given and $f(x)$ is to be determined. If $K(s)$ is the Mellin transform of $k(x)$ and $K(s)K(1-s) = 1$ then we can solve (22) by means of equation (2). We shall prove here that if

$$K(s)K(1-s) = \frac{\Gamma[\alpha+(\eta+1-s)/m]\,\Gamma[\alpha+(\eta+s)/m]}{\Gamma[(\eta+1-s)/m]\,\Gamma[(\eta+s)/m]}, \tag{23}$$

then the solution of (22) is

$$f(x) = \int_0^\infty \mathfrak{J}[k(ux);\alpha,\eta,m]\,\mathfrak{J}[g(u);\alpha,\eta,m]\, du. \tag{24}$$

Evidently, when $\alpha = 0$ then (22), (24) and (23) reduce to the generalized Fourier integral transform given by (1), (2) and (4).

By means of (18), the fractional integration by parts formula, the right-hand side of (24) can be expressed in several different forms. Although these forms are all equivalent some are easier to compute than others.

Associated with (22) and (24) we also have results analogous to the Parseval theorem and to the discontinuous integrals which are satisfied by Fourier kernels.

An example is given by

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty u^\alpha x^\alpha \cos(ux - \tfrac{1}{2}\alpha\pi) f(u)\, du, \tag{25}$$

where we are required to solve for $f(x)$. The kernel $k(x)$ is given by

$$k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} x^\alpha \cos(x - \tfrac{1}{2}\alpha\pi) \tag{26}$$

and its Mellin transform by

$$K(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Gamma(\alpha+s) \cos\left(\frac{\pi s}{2}\right). \tag{27}$$

It is easily shown that $K(s)$ satisfies the functional equation (23) and the solution of (25) is then given by

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos(ux)\, \mathfrak{R}[g(u);\alpha,0,1]\, du. \tag{28}$$

Equations (25) and (28) reduce to the cosine transform when $\alpha = 0$.

We shall obtain our results not quite in the forms given by (24) and (28) but in a form appropriate to L_2-space theory.

4. *Preliminary lemmas.* In the complex s-plane we write $s = \sigma + it$, σ and t real, and we shall always be concerned with the line $\sigma = \frac{1}{2}$.

LEMMA 1. *If, along the line $s = \frac{1}{2}+it$, $K(s)$ is bounded and satisfies the functional equation* (23) *then, on this line, $K(s) = O(s^\alpha)$ for large s.*

Proof. Write

$$K(s) = \frac{\Gamma[\alpha+(\eta+s)/m]}{\Gamma[(\eta+s)/m]}\, L(s). \tag{29}$$

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From the functional equation (23) we then have

$$L(s)\,L(1-s) = 1. \tag{30}$$

On the line $\sigma = \frac{1}{2}$ we have $K(s)$ is bounded, hence from (29) $L(s)$ is also bounded and so, from (30), $L(s) = O(1)$. Consequently the order of $K(s)$ on $\sigma = \frac{1}{2}$ is determined by the orders of the two γ-functions in (29). On using the asymptotic expansion

$$\log\{\Gamma(s+a)\} = (s+a-\tfrac{1}{2})\log s - s + \tfrac{1}{2}\log(2\pi) + O(s^{-1}), \tag{31}$$

where $-\pi < \arg s < \pi$, and s is large (Whittaker and Watson (9), p. 272) it follows easily that on $\sigma = \frac{1}{2}$ we must have $K(s) = O(s^{\alpha})$.

The next lemma is associated with L_2 theory. The transform inverse to (3) is

$$k(x) = \frac{1}{2\pi i}\int_C K(s)\,x^{-s}\,ds, \tag{32}$$

where C denotes the contour of integration. Here the contour C will always be the line $s = \frac{1}{2}+it$. On integrating (32) formally with respect to x and writing

$$k_1(x) = \int_0^x k(x)\,dx \tag{33}$$

we then have

$$\frac{k_1(x)}{x} = \frac{1}{2\pi i}\int_C \frac{K(s)}{(1-s)}\,x^{-s}\,ds. \tag{34}$$

The conditions to be used later will not ensure the existence of the integral of (32). We shall have, however, on the line $\sigma = \frac{1}{2}$, $K(s)/(1-s) \in L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ so that the integral of (34) exists in mean square. Hence $k_1(x)/x$ exists and belongs to $L_2(0,\infty)$ and so we can then define $k(x)$ as the derivative of $k_1(x)$ by (32); see (7), p. 221.

LEMMA 2. *If* (i) $0 \leqslant \operatorname{Re}\alpha < \frac{1}{2}$, *and* (ii) $K(s)$ *satisfies the functional equation* (23) *then* (*a*)

$$K(s)/(1-s) \in L_2(\tfrac{1}{2}-i\infty, \tfrac{1}{2}+i\infty)$$

and (*b*) *there exists a function* $k_1(x)/x$, *belonging to* $L_2(0,\infty)$, *whose Mellin transform is* $K(s)/(1-s)$.

Proof. From Lemma 1, on the line $\sigma = \frac{1}{2}$, we have $K(s)/(1-s) = O(s^{\alpha-1})$. Hence (*a*) is true if $1 - \operatorname{Re}\alpha > \frac{1}{2}$ or $\operatorname{Re}\alpha < \frac{1}{2}$.

The statements in (*b*) follow then from the remarks made previous to the lemma.

5. *Reciprocities with fractional integration.*

THEOREM 1. *Assumptions*:

(i) $$0 \leqslant \operatorname{Re}\alpha < \tfrac{1}{2}, \quad \operatorname{Re}\eta > -\tfrac{1}{2}, \quad m > 0;$$

(ii) $$K(s)\,K(1-s) = \frac{\Gamma[\alpha+(\eta+1-s)/m]\,\Gamma[\alpha+(\eta+s)/m]}{\Gamma[(\eta+1-s)/m]\,\Gamma[(\eta+s)/m]}$$

and on the line $s = \frac{1}{2}+it$, *the function* $K(s)$ *is bounded for all finite values of* s;

(iii) $$f(x) \in L_2(0,\infty).$$

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Conclusions:

$$\int_0^x g(u)\,du = \int_0^\infty k_1(ux)\,f(u)\frac{du}{u}, \tag{35}$$

where $g(u) \in L_2(0,\infty)$, *and*

$$\int_0^x f(u)\,du = \int_0^\infty \Re\Im[k_1(ux)/u;\,\alpha,\eta,m]\,g(u)\,du, \tag{36}$$

$$\int_0^x f(u)\,du = \int_0^\infty \Im[k_1(ux)/u;\,\alpha,\eta,m]\,\Im[g(u);\,\alpha,\eta,m]\,du, \tag{37}$$

$$\int_0^x f(u)\,du = \int_0^\infty \Re[k_1(ux)/u;\,\alpha,\eta,m]\,\Re[g(u);\,\alpha,\eta,m]\,du, \tag{38}$$

$$\int_0^x f(u)\,du = \int_0^\infty \frac{k_1(ux)}{u}\,\Re\Im[g(u);\,\alpha,\eta,m]\,du. \tag{39}$$

Although these forms are all equivalent some are easier to evaluate than others. When $\alpha = 0$ all these results reduce to the generalized Fourier transform.

Proof. We first establish (35) and (36).

Define $H(s)$ by means of the equation

$$H(s) = \frac{\Gamma[(\eta+1-s)/m]\,\Gamma[(\eta+s)/m]}{\Gamma[\alpha+(\eta+1-s)/m]\,\Gamma[\alpha+(\eta+s)/m]}\,K(s). \tag{40}$$

From (31) and Lemma 1 it follows that on the line $s = \frac{1}{2}+it$ we have $H(s) = O(s^{-\alpha})$ for large s and so $H(s)/(1-s) \in L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. Consequently there exists a function $h_1(x)/x$, belonging to $L_2(0,\infty)$, whose Mellin transform is $H(s)/(1-s)$.

From (40) and condition (ii) we see that $H(s)$ is bounded on the line $\sigma = \frac{1}{2}$ and also that

$$K(s)H(1-s) = 1. \tag{41}$$

With these properties of $H(s)$, in particular (41), we associate the following: $K(s)/(1-s)$ is bounded on $\sigma = \frac{1}{2}$ and belongs to $L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$, condition (ii) and Lemma 2, and also $f(x) \in L_2(0,\infty)$, condition (iii). These form a set of conditions sufficient to ensure the validity of Watson's theorem on generalized Fourier transforms with unsymmetrical kernels; see Titchmarsh (7), p. 226. This gives us

$$\int_0^x g(u)\,du = \int_0^\infty k_1(ux)\,f(u)\frac{du}{u}, \tag{42}$$

where $g(u) \in L_2(0,\infty)$ and

$$\int_0^x f(u)\,du = \int_0^\infty \frac{h_1(ux)}{u}\,g(u)\,du. \tag{43}$$

Hence (35) is established and it remains to show that (43) and (36) are identical.

To show this we first note that $x^{1-s}K(s)/(1-s) \in L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and, from (34), that it is the Mellin transform of $k_1(ux)/u$, considered as a function of u. Now multiply (40) by $x^{1-s}/(1-s)$ and compare the result with (21). It follows that

$$\frac{x^{1-s}H(s)}{(1-s)} = \mathfrak{M}\{\Re\Im[k_1(ux)/u;\,\alpha,\eta,m]\}, \tag{44}$$

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where $\mathfrak{M}$ denotes the Mellin transform and the operators $\mathfrak{K}$ and $\mathfrak{J}$ both use the same parameters α, η and m. From the properties of $H(s)$ obtained above the left-hand side of (44) is the Mellin transform of $h_1(ux)/u$, considered as a function of u. Hence, from (44), we have

$$h_1(ux)/u = \mathfrak{K}\mathfrak{J}[k_1(ux)/u; \alpha, \eta, m] \tag{45}$$

almost everywhere. This establishes (36) since, by (45), (36) and (43) are identical.

(37), (38) and (39) are now easily deducible from (36) by means of the fractional integration by parts formulae (18) and (19). These formulae can be applied since, considered as functions of u, each of the factors in the right-hand integrand of (36) belongs to $L_2(0, \infty)$.

(37) follows from (36) by (19). Since $k_1(ux)/u$ and $g(u)$ both belong to $L_2(0, \infty)$ it follows from § 2 that the factors in the integrand on the right of (37) also belong to $L_2(0, \infty)$. Hence we may apply (18) to (37) to deduce (39).

From (21) it is evident that $\mathfrak{K}$ and $\mathfrak{J}$ are commutative if both operators use the same parameters α, η and m, which is the case here. On interchanging $\mathfrak{K}$ and $\mathfrak{J}$ in (36) and applying (18) we can then deduce (38).

This completes the proof of Theorem 1.

6. *The analogue of the Parseval theorem.*

THEOREM 2. *Assumptions*:

(i) *the assumptions of Theorem* 1;

(ii) $f(x)$ *and* $g(x)$ *are related as in* (35);

(iii) $p(x) \in L_2(0, \infty)$;

(iv) $p(x)$ *and* $q(x)$ *are related as* $f(x)$ *and* $g(x)$ *are in Theorem* 1; *i.e.*

$$\int_0^x q(u)\,du = \int_0^\infty k_1(ux)\,p(u)\frac{du}{u}. \tag{46}$$

Conclusions:

$$\int_0^\infty p(u)\,f(u)\,du = \int_0^\infty \mathfrak{K}[q(u); \alpha, \eta, m]\,\mathfrak{K}[g(u); \alpha, \eta, m]\,du \tag{47}$$

and

$$\int_0^\infty p(u)\,\mathfrak{K}[g(u); \alpha, \eta, m]\,du = \int_0^\infty \mathfrak{K}[q(u); \alpha, \eta, m]\,f(u)\,du. \tag{48}$$

(47) and (48) can be put into various different forms by means of the fractional integration by parts formula of § 2.

Proof. The proof depends upon showing that (38) can be regarded as a generalized Fourier transform with symmetrical kernels and then using the associated Parseval theorem.

Let

$$M(s) = \frac{\Gamma[(\eta+s)/m]}{\Gamma[\alpha+(\eta+s)/m]}\,K(s). \tag{49}$$

Immediate deductions from (49) are, from (31) and Lemma 1, $M(s)$ is bounded and $O(1)$ along the line $\sigma = \frac{1}{2}$ and, from condition (ii), Theorem 1,

$$M(s)\,M(1-s) = 1. \tag{50}$$

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Consequently $x^{1-s}M(s)/(1-s) \in L_2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$ and is the Mellin transform of a function $m_1(ux)/u$, considered as a function of u. On multiplying (49) by $x^{1-s}/(1-s)$ and comparing with (17) it follows that

$$\frac{m_1(ux)}{u} = \mathfrak{K}[k_1(ux)/u;\ \alpha, \eta, m]. \tag{51}$$

Equation (50) shows that we may use Watson's theorem, with $m_1(u)/u$ as the symmetrical kernel, in a generalized Fourier transform, Titchmarsh (7), p. 221; i.e. if $a(u) \in L_2(0, \infty)$ then

$$\int_0^x b(u)\,du = \int_0^\infty \frac{m_1(ux)}{u}\, a(u)\,du, \tag{52}$$

where $b(u) \in L_2(0, \infty)$, $a(u)$ and $b(u)$ can be interchanged in (52), and also we have the Parseval formula

$$\int_0^\infty \{a(u)\}^2\,du = \int_0^\infty \{b(u)\}^2\,du. \tag{53}$$

From (51), equation (52) is identical with (38). Now add (or subtract) (35) and (46) and write $a(u) = p(u) \pm f(u)$. Then we have, from (38) or (52),

$$b(u) = \mathfrak{K}[q(u);\ \alpha, \eta, m] \pm \mathfrak{K}[g(u);\ \alpha, \eta, m],$$

where the two $+$ (or the two $-$) signs are taken together. On inserting these results in (53) and subtracting we immediately obtain (47).

Since $a(u)$ and $b(u)$ can be interchanged in (52) we may also interchange $f(u)$ and $\mathfrak{K}[g(u)]$ in (38). If we now write

$$a(u) = p(u) \pm \mathfrak{K}[g(u);\ \alpha, \eta, m]$$

then we should have $$b(u) = \mathfrak{K}[q(u);\ \alpha, \eta, m] \pm f(u),$$

where again the two $+$ or two $-$ signs are taken together. On inserting these in (53) and subtracting, (48) is then immediately obtained.

This completes the proof of Theorem 2, and I just note that (47) and (48) cannot be obtained from each other by fractional integration by parts and represent quite different results.

7. *The use of successive fractional integration.* It is easy to establish theorems analogous to Theorems 1 and 2 above when the functional equation of condition (ii), Theorem 1 is replaced by the more general equation

$$K(s)\,K(1-s) = \prod_{i=1}^{n} \frac{\Gamma[\alpha_i + (\eta_i + 1 - s)/m_i]\,\Gamma[\alpha_i + (\eta_i + s)/m_i]}{\Gamma[(\eta_i + 1 - s)\,m_i]\,\Gamma[(\eta_i + s)\,m_i]}. \tag{54}$$

We must now use successive fractional integration, for example, in Theorem 1 (36) instead of the operator $\mathfrak{K}\mathfrak{J}$ we should have a product of $2n$ operators of the type $\mathfrak{K}_i\mathfrak{J}_i$ $(i = 1, 2, \ldots, n)$, where $\mathfrak{K}_i$ and $\mathfrak{J}_i$ use the parameters α_i, η_i and m_i. The proofs of these more general results are very similar to the ones given here.

In conclusion I wish to express my thanks to the referee for valuable suggestions which led to a considerable shortening of the proofs I originally submitted.

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