Macroeconomic Dynamics, **17**, 2013, 155–171. Printed in the United States of America.
doi:10.1017/S1365100511000071

THE *Z*-TRANSFORM AND COMPARATIVE DYNAMICS IN DISCRETE-TIME MODELS

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This paper develops a general technique for the computation of comparative dynamics in perfect-foresight discrete-time models. The method developed here is both applicable and general; it can be used to analyze the effects of the perturbation of parameters on endogenous variables and the welfare of an economic system derived from more general multisector models. It is neither restricted to the system's dimensions nor restricted by the assumption of distinct eigenvalues in the system.

Keywords: Jordan Decomposition, *Z*-Transform, Multidimensional Dynamic Systems

1. INTRODUCTION

Lucas (2003) emphasizes that it is very important to justify welfare gains (perhaps losses, for some) in the evaluation of the effects of policy change in macroeconomic models. In fact, since the 1980s, many methods for comparative dynamics in perfect foresight models have been developed to analyze welfare gains along the dynamic path in the continuous-time model, such as those of Chamley (1981), Judd (1982, 1996), Turnovsky (2000), and Cui and Gong (2006), among others.

For the discrete-time system, Meijdam and Verhoeven (1998) use the *Z*-transform method to compute the comparative dynamics of a two-dimensional difference system. Cui et al. (2007) extend their method to deal with high-dimensional difference systems. However, their approach is restricted to systems with nonrepeated eigenvalues. Problems arise in solving the linear approximation of the system. In this method the coefficient matrix for the coefficients of the linear approximation has to be inverted. This is done by spectral decomposition of the matrix, which implies diagonalization and by that leads easily to the solution of the model. The diagonalization of a noninvertable matrix is characterized by nonrepeated eigenvalues. If some eigenvalues appear repeatedly, a matrix is then noninvertable, which simply means that the model cannot be solved in this way. In

We are indebted to two anonymous referees for their suggestions and help in revising this paper. All remaining errors are our own. The research reported here was sponsored by the National Science Foundation for Distinguished Young Scholars (70725006). Address correspondence to: Gong Liutang, Guanghua School of Management, Peking University, Beijing, 100871, China; e-mail: ltgong@gsm.pku.edu.cn.

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https://doi.org/10.1017/S1365100511000071 Published online by Cambridge University Press

fact, repeated eigenvalues correspond to high-dimension difference systems, as in the Garman–Ohlson model [Tippett and Warnock (1997)] and the DSGE model [Doan (2010)], among others. Therefore, existing methods cannot be applied directly to an economic system when the Jacobian matrix has repeated eigenvalues. This paper uses the Z-transform method to develop a generalized method for comparative dynamics, which is restricted neither to the system's dimensions nor to the assumption of distinct eigenvalues of the system. The method can be used to deal with economic systems described by more generalized multisector models.

It should be emphasized that even if computational economists use advanced software such as Dynare [see Juilliard (1996)] to simulate their models, the Z-transform allows them to compute the effects of a small but complex unanticipated shock such as a permanent or transitory shock.

Moreover, the approach does not rely on spectral decomposition of the coefficient matrix of the linearized system but on Jordan decomposition, which is available for any matrix. Even though the case of multiple eignvalues does not always occur, economists always assume that the eigenvalues are distinct [see Buiter (1984)]. Hence, the approach discussed here is more general.[1]

The remainder of this paper is organized as follows. Section 2 presents the methodology for steady-state comparison. Section 3 presents an application example, which has two repeated eigenvalues. Section 4 extends the method to deal with a more general system, and Section 5 concludes the paper.

2. METHODOLOGY

2.1. Description of the Economic System

Consider the following system derived by a prefect foresight economic model,

$$x_{t+1}^i = f^i(x_t, y_t, \varepsilon h_t), i = 1, \ldots, n_1, t \geq 0, \tag{1}$$

$$y_{t+1}^j = g^j(x_t, y_t, \varepsilon h_t), j = 1, \ldots, n - n_1, t \geq 0, \tag{2}$$

where $f^i : R^{n_1} \times R^{n-n_1} \times R^m \to R$ and $g^j : R^{n_1} \times R^{n-n_1} \times R^m \to R$ are continuous differentiable functions; $x_t^1, \ldots, x_t^{n_1}$ are predetermined (or backward-looking) variables with the given initial values $x_0^1, \ldots, x_0^{n_1}$; $y_t^1, \ldots, y_t^{n-n_1}$ are nonpredetermined variables (also known as jump or forward-looking variables); ε is a scale parameter, initially equal to zero; and for any $l = 1, \ldots, m$, h_t^l is a bounded and eventually constant function of time. To simplify the notation, we denote $x_t = (x_t^1, \ldots, x_t^{n_1})^T$, $y_t = (y_t^1, \ldots, y_t^{n-n_1})^T$, and $h_t = (h_t^1, \ldots, h_t^m)^T$.

With the time path of x_t and y_t determined by the system (1) and (2), we assume that the social welfare function is given by

$$W = \sum_{t=0}^{\infty} \beta^t v(x_t, y_t), \tag{3}$$

where $0 < \beta < 1$ is the discount rate.

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Suppose that at time $t = 0$, the system is in the corresponding steady state [we denote the steady state $(\bar{x}, \bar{y})$ with $\varepsilon = 0$]; namely, the system satisfies

$$f^i(\bar{x}, \bar{y}, 0) = \bar{x}^i, i = 1, \ldots, n_1, \tag{4}$$

$$g^j(\bar{x}, \bar{y}, 0) = \bar{y}^j, j = 1, \ldots, n - n_1. \tag{5}$$

At $t = 0$, there is an unanticipated change in ε from its initial value, and we are interested in the induced change in social welfare, $dW/d\varepsilon$.

2.2. The Linear Approximation System

To derive the impact of the change of ε on the economy, we differentiate (1) and (2) with respect to ε and obtain

$$\begin{pmatrix} \dfrac{dx_{t+1}}{d\varepsilon} \\ \dfrac{dy_{t+1}}{d\varepsilon} \end{pmatrix} = A \begin{pmatrix} \dfrac{dx_t}{d\varepsilon} \\ \dfrac{dy_t}{d\varepsilon} \end{pmatrix} + Bh_t, \tag{6}$$

where $dx_t/d\varepsilon = (dx_t^1/d\varepsilon, \ldots, dx_t^{n_1}/d\varepsilon)^T$ and $dy_t/d\varepsilon = (dy_t^1/d\varepsilon, \ldots, dy_t^{n-n_1}/d\varepsilon)^T$. A is the first $n \times n$ submatrix of the Jacobian matrix of the vector function $(f^1, \ldots, f^{n_1}, g^1, \ldots, g^{n-n_1})^T$ evaluated at the initial steady state $(\bar{x}, \bar{y}, 0)$, and B is the rest of the $n \times m$ submatrix; i.e.,

$$A = \begin{pmatrix} f_1^1 & \cdots & f_n^1 \\ \vdots & \ddots & \vdots \\ f_1^{n_1} & \cdots & f_n^{n_1} \\ g_1^1 & \cdots & g_n^1 \\ \vdots & \ddots & \vdots \\ g_1^{n-n_1} & \cdots & g_n^{n-n_1} \end{pmatrix}, \quad B = \begin{pmatrix} f_{n+1}^1 & \cdots & f_{n+m}^1 \\ \vdots & \ddots & \vdots \\ f_{n+1}^{n_1} & \cdots & f_{n+m}^{n_1} \\ g_{n+1}^1 & \cdots & g_{n+m}^1 \\ \vdots & \ddots & \vdots \\ g_{n+1}^{n-n_1} & \cdots & g_{n+m}^{n-n_1} \end{pmatrix}.$$

The linear approximation (6) is similar to the system described by Blanchard and Kahn (1980). We assume that the steady state of difference system (1) and (2) is saddlepoint stable, and following Blanchard and Kahn (1980), we impose the following assumption.

Assumption 1. *A has as many eigenvalues with absolute values smaller than 1 as there are predetermined endogenous variables, and as many eigenvalues with*

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absolute values larger than 1 as there are anticipated variables: i.e., A has n_1 characteristic roots that are inside the unit circle, and $n - n_1$ characteristic roots that are outside the unit circle.

We will use the *Z*-transform to find the welfare effect of an unanticipated change in ε. First, we provide the definition of, and some important and useful propositions for, the *Z*-transform.

DEFINITION 1. *For any discrete-time sequence x_t, the bilateral or two-sided Z-transform evaluated at s is*

$$X(s) = Z(x_t, s) = \sum_{t=-\infty}^{\infty} x_t s^{-t},$$

where s is generally a complex number.

Alternatively, in cases where x_t is defined only for $t \geq 0$, the *single-sided* or *unilateral Z-transform* evaluated at s is

$$X(s) = Z(x_t, s) = \sum_{t=0}^{\infty} x_t s^{-t}.$$

From this definition, we have some important and elementary properties for the *Z*-transform.

PROPOSITION 1.

(1) *For any two sequences x_t, y_t, $t = 0, 1, 2, \ldots$, and any $a, b \in R$, we have*

$$Z(ax_t + by_t, s) = aZ(x_t, s) + bZ(y_t, s).$$

(2) *For any sequence x_t, $t = 0, 1, 2, \ldots$, and $k > 0$, we have*

$$Z(x_{t+1}, s) = sZ(x_t, s) - sx_0,\ Z(x_{t-k}, s) = s^{-k}Z(x_t, s).$$

Therefore, taking the single-sided *Z*-transform on system (6) yields

$$(sI - A)\begin{pmatrix} Z\left(\dfrac{dx_t}{d\varepsilon}, s\right) \\ Z\left(\dfrac{dy_t}{d\varepsilon}, s\right) \end{pmatrix} = BZ(h_t, s) + \begin{pmatrix} s\dfrac{dx_0}{d\varepsilon} \\ s\dfrac{dy_0}{d\varepsilon} \end{pmatrix}, \tag{7}$$

where $dx_0/d\varepsilon$ and $dy_0/d\varepsilon$ are the initial changes of economic variables induced by ε at time zero, and $Z(h_t, s) = [Z(h_t^1, s), \ldots, Z(h_t^m, s)]^T$ is the *Z*-transform of the vector function h_t.

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If we have derived the expressions for $dx_0/d\varepsilon$ and $dy_0/d\varepsilon$, then from equation (7) we have[2]

$$\begin{pmatrix} Z\left(\frac{dx_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\frac{dy_t}{d\varepsilon}, 1/\beta\right) \end{pmatrix} = \left(\frac{1}{\beta} I - A\right)^{-1} \left(BZ(h_t, 1/\beta) + \frac{1}{\beta} \begin{pmatrix} \frac{dx_0}{d\varepsilon} \\ \frac{dy_0}{d\varepsilon} \end{pmatrix} \right). \quad (8)$$

Then, differentiating equation (3) with respect to ε, we obtain the impact of the change of ε on welfare as follows:

$$\frac{dW}{d\varepsilon} = \begin{pmatrix} \nabla_x v \\ \nabla_y v \end{pmatrix}^T \sum_{t=0}^{\infty} \beta^t \begin{pmatrix} \frac{dx_t}{d\varepsilon} \\ \frac{dy_t}{d\varepsilon} \end{pmatrix} = \begin{pmatrix} \nabla_x v \\ \nabla_y v \end{pmatrix}^T \begin{pmatrix} Z\left(\frac{dx_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\frac{dy_t}{d\varepsilon}, 1/\beta\right) \end{pmatrix}, \quad (9)$$

where $\nabla_x v = [\partial v(\bar{x}, \bar{y})/\partial x_t^1, \ldots, \partial v(\bar{x}, \bar{y})/\partial x_t^{n_1}]^T$ and $\nabla_y v = [\partial v(\bar{x}, \bar{y})/\partial y_t^1, \ldots, \partial v(\bar{x}, \bar{y})/\partial y_t^{n-n_1}]^T$ are the gradients of v with vectors x and y evaluated at the steady state.

Because $x_t^1, \ldots, x_t^{n_1}$ are state variables, they cannot jump initially and we have $dx_0/d\varepsilon = 0$. In the following section, we present the method by which to derive $dy_0/d\varepsilon$.

2.3. The Alternative Way to Derive $dy_0/d\varepsilon$

To derive $dy_0/d\varepsilon$, Cui et al. (2007) assume that matrix A has n distinct eigenvalues, which guarantees that A can be diagonalized. This method cannot be adopted directly when A has repeated eigenvalues. Hence, we develop a more general method to derive the expression of $dy_0/d\varepsilon$, which is restricted neither by the dimension of the system nor by the assumption of distinct eigenvalues for matrix A.

Step 1. Jordan decomposition of matrix A. From Assumption 1, let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ be the n eigenvalues of A (counting multiplicities). A can be decomposed as

$$A = V^{-1} J V; \quad (10)$$

the number of eigenvalues larger than one is $n - n_1$, V is an $n \times n$ matrix with rows that are linearly independent left eigenvectors of A, and J is the associated Jordan matrix. As the steady state is saddlepoint stable, the number of eigenvalues inside the unit circle equals the number of predetermined state variables, n_1. Therefore, we can decompose J as

$$J = \begin{pmatrix} J_{n_1} & 0 \\ 0 & J_{n-n_1} \end{pmatrix}, \quad (11)$$

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where

$$J_{n_1} = \begin{pmatrix} J_{m_1}(\mu_1) & & \\ & \ddots & \\ & & J_{m_k}(\mu_k) \end{pmatrix} \quad \text{and}$$

$$J_{n-n_1} = \begin{pmatrix} J_{m_{k+1}}(\mu_{k+1}) & & \\ & \ddots & \\ & & J_{m_{k+l}}(\mu_{k+l}) \end{pmatrix}$$

with

$$J_{m_i} = \begin{pmatrix} \mu_i & 1 & & \\ & \ddots & \ddots & \\ & & \mu_i & 1 \\ & & & \mu_i \end{pmatrix}$$

if the multiplicity of eigenvalue μ_i is m_i, and $J_{m_i} = \mu_i$ if μ_i is not a repeated eigenvalue.

J_{n_1} is the Jordan matrix consisting of the first n_1 rows and n_1 columns of J, and J_{n-n_1} is the Jordan matrix consisting of the last $n - n_1$ rows and columns of J. Therefore, $\sum_{i=1}^{k} m_i = n_1$, $\sum_{i=k+1}^{k+l} m_i = n - n_1$, and $\sum_{i=1}^{k+l} m_i = n$.

Under the decomposition of (10), system (6) can be rewritten as

$$\begin{pmatrix} \frac{dx_{t+1}}{d\varepsilon} \\ \frac{dy_{t+1}}{d\varepsilon} \end{pmatrix} = V^{-1} \begin{pmatrix} J_{n_1} & 0 \\ 0 & J_{n-n_1} \end{pmatrix} V \begin{pmatrix} \frac{dx_t}{d\varepsilon} \\ \frac{dy_t}{d\varepsilon} \end{pmatrix} + Bh_t. \tag{12}$$

Define

$$w_t = V \begin{pmatrix} \frac{dx_t}{d\varepsilon} \\ \frac{dy_t}{d\varepsilon} \end{pmatrix},$$

and equation (12) can be rewritten as

$$\begin{pmatrix} w_{t+1}^1 \\ w_{t+1}^2 \\ \vdots \\ w_{t+1}^n \end{pmatrix} = \begin{pmatrix} J_{n_1} & 0 \\ 0 & J_{n-n_1} \end{pmatrix} \begin{pmatrix} w_t^1 \\ w_t^2 \\ \vdots \\ w_t^n \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \vdots \\ \tilde{B}_n \end{pmatrix} h_t, \tag{13}$$

where $\tilde{B} = VB$, and $\tilde{B}_i$ is the ith row of the matrix $\tilde{B}$.

System (13) consists of n difference equations; we will use it to solve w_t. Because we know that $dx_0/d\varepsilon = 0$, we only need to find $w_0^{n-n_1}, \ldots, w_0^n$.

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Step 2. Solving $w_0^{n-n_1}, \ldots, w_0^n$. If the eigenvalue outside of the unit circle is single, then the subequations of system (13) can be solved recursively. If the eigenvalue outside the unit circle is repeated, then the equation associated with that eigenvalue cannot be solved directly; instead, we will solve it using the recursive structure of the Jordan matrix, with the details presented below.

To explain the method more clearly, we assume that the multiplicty of eigenvalue λ is p, and from equation (13) the subsystem associated with the eigenvalue λ is then

$$\begin{pmatrix} w_{t+1}^1 \\ w_{t+1}^2 \\ \vdots \\ w_{t+1}^{p-1} \\ w_{t+1}^p \end{pmatrix} = \begin{pmatrix} \lambda w_t^1 + w_t^2 + \tilde{B}_1 h_t \\ \lambda w_t^2 + w_t^3 + \tilde{B}_2 h_t \\ \vdots \\ \lambda w_t^{p-1} + w_t^p + \tilde{B}_{p-1} h_t \\ \lambda w_t^p + \tilde{B}_p h_t \end{pmatrix}. \tag{14}$$

The solution of the last equation, $w_{t+1}^p = \lambda w_t^p + \tilde{B}_p h_t$, is

$$w_t^p = \frac{1}{1-\lambda L}\tilde{B}_p h_t = -\frac{1}{\lambda}\sum_{i=0}^{\infty}\frac{1}{\lambda^i}\tilde{B}_p h_{t+i},$$

where L is the lag operator.

Therefore,

$$w_0^p = -\frac{1}{\lambda}\tilde{B}_p Z(h_t, \lambda). \tag{15}$$

Solving the rest of the equations recursively, we have

$$w_t^{p-m} = -\frac{1}{\lambda}\sum_{i=0}^{\infty}\frac{1}{\lambda^i}\left(w_{t+i}^{p-m+1} + \tilde{B}_{p-m} h_{t+i}\right). \tag{16}$$

Therefore, we have

$$w_0^{p-m} = -\frac{1}{\lambda}\left[Z(w_t^{p-m+1}, \lambda) + \tilde{B}_{p-m} Z(h_t, \lambda)\right] \tag{17}$$

for $m = 1, 2, \ldots, p-1$.

Similarly, we can derive that $w_0^{n-n_1}, \ldots, w_0^n$.

Step 3. The derivation of $dy_0/d\varepsilon$. Substituting $w_0^{n-n_1}, \ldots, w_0^n$ determined by Step 2 into

$$w_t = V\begin{pmatrix} \dfrac{dx_t}{d\varepsilon} \\ \dfrac{dy_t}{d\varepsilon} \end{pmatrix},$$

we have

$$V\begin{pmatrix} 0 \\ \dfrac{dy_0}{d\varepsilon} \end{pmatrix} = w_0. \tag{18}$$

From equation (18), we can derive $dy_0/d\varepsilon$ and w_0^j, $j = 1, \ldots, n_1$.

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Remark. The method for the computation of $dy_0/d\varepsilon$ can be used to compute the effect of the complex shock εh_t first on the transition path and then on social welfare.

3. AN APPLICATION

3.1. The Model

In this section, we use a simple example to explain our method. We consider the following extension of the Tahvonen and Withagen (1996) and Wirl (2000) framework:

$$W = \max_{\{r_t\},\{z_{t+1}\},\{q_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [U(z_t, q_t) - (v + \tau)z_t - C(r_t)], \tag{19}$$

subject to the constraints

$$q_{t+1} = q_t - \delta q_t - r_t, t \geq 0, \tag{20}$$

and

$$z_{t+1} = z_t + \varphi(q_t) - \theta z_t, t \geq 0, \tag{21}$$

with the given initial values of state variables q_0 and z_0.

In line with Wirl (2000), q_t is the firm's effluents or emissions into the environment, and it can be depreciated at the rate δ; z_t is the pollution stock, and suppose that it increases with the effluent but is reduced through indigenous abatement by the environment, i.e., depreciation, and $0 < \theta < 1$ is the depreciation rate. The pollution formation function $\varphi(q_t)$ is

$$\varphi(q_t) = pq_t, \tag{22}$$

where p is a positive constant. For simplify, we assume that $p = 1 - \delta$.

$U(z_t, q_t)$ is the firm's gross surplus from discharging. Suppose that the firm is located on a lake and a higher environmental quality for the lake is perferred and the pollution is penalized at vz_t. The firm can lower its effluents at rate r_t but at a cost $C(r_t)$. For simplicity, we assume that

$$U(z_t, q_t) = aq_t \tag{23}$$

and

$$C(r_t) = r_t + \frac{1}{2}\gamma r_t^2 - \varsigma r_t, \tag{24}$$

where a, γ, and ς are positive constants.

Finally, the fiscal authority taxes at constant rates, τ, and applies a constant subsidy rate, ς.

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Denote by ξ_t and η_t the Lagrangian multiplers associated with constraints (20) and (21), respectively, and using the Lagrangian method, the dynamics of the economy can be derived as

$$q_{t+1} = (1-\delta)q_t + \frac{1-\varsigma+\xi_t}{\gamma}, \tag{25}$$

$$z_{t+1} = pq_t + (1-\theta)z_t, \tag{26}$$

$$\xi_{t+1} = \frac{1}{\beta(1-\delta)}\xi_t - \frac{a}{1-\delta} - \frac{1}{1-\theta}\left(\frac{1}{\beta}\eta_t + v + \tau\right), \tag{27}$$

and

$$\eta_{t+1} = \frac{1}{1-\theta}\left(\frac{1}{\beta}\eta_t + v + \tau\right), \tag{28}$$

with boundary conditions $|\lim_{t\to\infty} q_t| < \infty$, $|\lim_{t\to\infty} z_t| < \infty$, $|\lim_{t\to\infty} \xi_t| < \infty$, $|\lim_{t\to\infty} \eta_t| < \infty$, $q_{t=0} = q_0$, and $z_{t=0} = z_0$.

Equations (25)–(28) determine the dynamics for variables q_t, z_t, ξ_t, and η_t. q_t and z_t are predetermined variables with given initial values of q_0 and z_0; ξ_t and η_t are nonpredetermined variables. After q_t, z_t, ξ_t, and η_t are obtained, the firm's welfare can be derived as

$$W = \sum_{t=0}^{\infty} \beta^t [U(z_t, q_t) - (v+\tau)z_t - C(r_t)].$$

With the specified function forms for $U(z, q)$, $C(r)$, and $\varphi(q)$, the system is linear. In fact, we can deal with more complex problems such as the Lucas (1988) model, but that is left for future research. In the next section, we will use the method developed in Section 2 to derive the effects of policy change on welfare and endogenous variables.

3.2. Effects of Policy Change on the Economy

Suppose that at time zero the fiscal authority has been taxing at constant rates, τ, and constant subsidy rate, ς, and that the economy has reached the corresponding steady state, $(q^*, z^*, \xi^*, \eta^*)$. Then, the government announces that at time t, $t \geq 0$, τ and ς will change by εh_t^{τ} and $\varepsilon h_t^{\varsigma}$, respectively, where $h_t \equiv (h_t^{\tau}, h_t^{\varsigma})^T$ are eventually constant functions of time, and ε is a scale parameter, initially equal to zero.

Substituting the new policies into the dynamic system (25)–(28) yields

$$q_{t+1} = (1-\delta)q_t + \frac{1-\varsigma-\varepsilon h_t^{\varsigma}+\xi_t}{\gamma}, \tag{25$'$}$$

$$z_{t+1} = pq_t + (1-\theta)z_t, \tag{26$'$}$$

$$\xi_{t+1} = \frac{1}{\beta(1-\delta)}\xi_t - \frac{a}{1-\delta} - \frac{1}{1-\theta}\left(\frac{1}{\beta}\eta_t + v + \tau + \varepsilon h_t^{\tau}\right), \tag{27$'$}$$

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and

$$\eta_{t+1} = \frac{1}{1-\theta}\left(\frac{1}{\beta}\eta_t + v + \tau + \varepsilon h_t^{\tau}\right). \tag{28'}$$

Differentiating system (25′)–(28′) with respect to ε, evaluated at $\varepsilon = 0$, we have

$$\begin{pmatrix} \frac{dq_{t+1}}{d\varepsilon} \\ \frac{dz_{t+1}}{d\varepsilon} \\ \frac{d\xi_{t+1}}{d\varepsilon} \\ \frac{d\eta_{t+1}}{d\varepsilon} \end{pmatrix} = A \begin{pmatrix} \frac{dq_t}{d\varepsilon} \\ \frac{dz_t}{d\varepsilon} \\ \frac{d\xi_t}{d\varepsilon} \\ \frac{d\eta_t}{d\varepsilon} \end{pmatrix} + Bh_t, \tag{29}$$

where the coefficient matrices A and B are as follows:

$$A = \begin{pmatrix} 1-\delta & 0 & \frac{1}{\gamma} & 0 \\ 1-\delta & 1-\theta & 0 & 0 \\ 0 & 0 & \frac{1}{\beta(1-\delta)} & -\frac{1}{\beta(1-\theta)} \\ 0 & 0 & 0 & \frac{1}{\beta(1-\theta)} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{1}{\gamma} \\ 0 & 0 \\ -\frac{1}{1-\theta} & 0 \\ \frac{1}{1-\theta} & 0 \end{pmatrix}. \tag{30}$$

Equation (29) is equivalent to equation (6) in Section 2. In the following section, we derive the effects of policy change on welfare and endogenous variables.

Step 1. Jordan decomposition of the matrix A. It is easy to derive that the four eigenvalues of the matrix A are $1-\delta$, $1-\theta$, $1/\beta(1-\delta)$, and $1/\beta(1-\theta)$. As δ, θ, and $\beta \in (0, 1)$, two eigenvalues are outside of the unit circle, denoted by $\mu_1 = 1/\beta(1-\delta)$ and $\mu_2 = 1/\beta(1-\theta)$, and two eigenvalues are inside the unit circle, denoted by $\omega_1 = 1-\delta$ and $\omega_2 = 1-\theta$, respectively; the steady state is saddlepoint stable.

Taking Jordan decomposition for matrix A, we can obtain

$$A = V^{-1} J V,$$

where V is an invertible matrix associated with the Jordan matrix J of coefficient matrix A.

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Then, premultiplying both sides of system (29) by V, the system can be represented by

$$V\begin{pmatrix} \frac{dq_{t+1}}{d\varepsilon} \\ \frac{dz_{t+1}}{d\varepsilon} \\ \frac{d\xi_{t+1}}{d\varepsilon} \\ \frac{d\eta_{t+1}}{d\varepsilon} \end{pmatrix} = \begin{pmatrix} \omega_1 & \phi & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \mu_1 & \phi \\ 0 & 0 & 0 & \mu_2 \end{pmatrix} V\begin{pmatrix} \frac{dq_t}{d\varepsilon} \\ \frac{dz_t}{d\varepsilon} \\ \frac{d\xi_t}{d\varepsilon} \\ \frac{d\eta_t}{d\varepsilon} \end{pmatrix} + V\begin{pmatrix} 0 & -\frac{1}{\gamma} \\ 0 & 0 \\ -\frac{1}{1-\theta} & 0 \\ \frac{1}{1-\theta} & 0 \end{pmatrix} h_t, \tag{31}$$

where ϕ in the Jordan matrix are defined as

$$\phi = \begin{cases} 0, & \text{if } \theta \neq \delta \\ 1, & \text{if } \theta = \delta. \end{cases}$$

Letting $V(dq_{t+1}/d\varepsilon, dz_{t+1}/d\varepsilon, d\xi_{t+1}/d\varepsilon, d\eta_{t+1}/d\varepsilon)^T = (w^1_{t+1}, w^2_{t+1}, w^3_{t+1}, w^4_{t+1})^T = w_{t+1}$, equation (31) can be rewritten as

$$\begin{pmatrix} w^1_{t+1} \\ w^2_{t+1} \\ w^3_{t+1} \\ w^4_{t+1} \end{pmatrix} = \begin{pmatrix} \omega_1 & \phi & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \mu_1 & \phi \\ 0 & 0 & 0 & \mu_2 \end{pmatrix}\begin{pmatrix} w^1_t \\ w^2_t \\ w^3_t \\ w^4_t \end{pmatrix} + V\begin{pmatrix} 0 & -\frac{1}{\gamma} \\ 0 & 0 \\ -\frac{1}{1-\theta} & 0 \\ \frac{1}{1-\theta} & 0 \end{pmatrix} h_t. \tag{32}$$

Step 2. Solving w^3_0 and w^4_0. From the last equation of (32), we have

$$w^4_t = \frac{1}{\mu_2}\sum_{i=0}^{\infty}\left\{\frac{1}{\mu_2^i}\left[\frac{V_{41}}{\gamma}h^{\varsigma}_{t+i} + \left(\frac{V_{43}-V_{44}}{1-\theta}\right)h^{\tau}_{t+i}\right]\right\}. \tag{33}$$

Therefore,

$$w^4_0 = \frac{1}{\mu_2}\left[\frac{V_{41}}{\gamma}Z\left(h^{\varsigma}_t, \mu_2\right) + \left(\frac{V_{43}-V_{44}}{1-\theta}\right)Z\left(h^{\tau}_t, \mu_2\right)\right]. \tag{34}$$

Similarly, we can derive the solution of w^3_t,

$$w^3_t = \frac{1}{\mu_1}\sum_{i=0}^{\infty}\left\{\frac{1}{\mu_1^i}\left[\frac{V_{31}}{\gamma}h^{\varsigma}_{t+i} + \left(\frac{V_{33}-V_{34}}{1-\theta}\right)h^{\tau}_{t+i} + \phi w^4_{t+i}\right]\right\}, \tag{35}$$

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and

$$w_0^3 = \frac{1}{\mu_1}\left[\frac{V_{31}}{\gamma} Z\left(h_t^{\varsigma}, \mu_1\right) + \left(\frac{V_{33} - V_{34}}{1-\theta}\right) Z\left(h_t^{\tau}, \mu_1\right) + \phi Z\left(w_t^4, \mu_1\right)\right]. \quad (36)$$

Step 3. Derivation of $d\xi_0/d\varepsilon$ and $d\eta_0/d\varepsilon$. As effluent discharge and pollution stock are both state variables that cannot jump, the initial effects of the unanticipated policy shock on these state variables are null; i.e., $dq_0/d\varepsilon = dz_0/d\varepsilon = 0$.

Therefore, to evaluate the relationship $V(dq_{t+1}/d\varepsilon, dz_{t+1}/d\varepsilon, d\xi_{t+1}/d\varepsilon, d\eta_{t+1}/d\varepsilon)^T = (w_{t+1}^1, w_{t+1}^2, w_{t+1}^3, w_{t+1}^4)^T$ at time 0, we have

$$\begin{pmatrix} V_{33} & V_{34} \\ V_{43} & V_{44} \end{pmatrix} \begin{pmatrix} \dfrac{d\xi_0}{d\varepsilon} \\ \dfrac{d\eta_0}{d\varepsilon} \end{pmatrix} = \begin{pmatrix} w_0^3 \\ w_0^4 \end{pmatrix}. \quad (37)$$

From equation (37), we obtain

$$\frac{d\xi_0}{d\varepsilon} = \frac{V_{44} w_0^3 - V_{34} w_0^4}{V_{33} V_{44} - V_{34} V_{43}}, \quad (38)$$

and

$$\frac{d\eta_0}{d\varepsilon} = \frac{V_{33} w_0^4 - V_{43} w_0^3}{V_{33} V_{44} - V_{34} V_{43}}. \quad (39)$$

Step 4. Derivation of the effect of policy change on welfare, $dW/d\varepsilon$. Equations (38) and (39) present the initial effects of tax on pollution and subsidies on investment in the economy. Similarly to equation (9), we have the effect of policy change on welfare,

$$\frac{dW}{d\varepsilon} = \begin{pmatrix} a \\ -(\upsilon + \tau) \\ \xi^* \\ 0 \end{pmatrix}^T \begin{pmatrix} Z\left(\dfrac{dq_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\dfrac{dz_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\dfrac{d\xi_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\dfrac{d\eta_t}{d\varepsilon}, 1/\beta\right) \end{pmatrix}, \quad (40)$$

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where

$$\begin{pmatrix} Z\left(\frac{dq_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\frac{dz_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\frac{d\xi_t}{d\varepsilon}, 1/\beta\right) \\ Z\left(\frac{d\eta_t}{d\varepsilon}, 1/\beta\right) \end{pmatrix} = \left(\frac{1}{\beta}I - A\right)^{-1} \left[BZ(h_t, 1/\beta) + \frac{1}{\beta} \begin{pmatrix} 0 \\ 0 \\ \frac{d\xi_0}{d\varepsilon} \\ \frac{d\eta_0}{d\varepsilon} \end{pmatrix} \right], \quad \textbf{(41)}$$

with $d\xi_0/d\varepsilon$ and $d\eta_0/d\varepsilon$ are determined by equations (38) and (39).

Equations (38), (39), and (40) analytically present the effects of a small shock on the economy. We can use the relationships to analyze these effects theoretically. However, to illustrate our results more intuitively, we use numerical solutions to explain them.

3.3. Numerical Illustration of Our Results

We consider two cases: repeated eigenvalues and distinct eigenvalues outside the unit circle. For simplicity, we focus on analyzing the effects of tax change; therefore, $h_t^{\varsigma} = 0$.

Case 1: $\theta = \delta$ (repeated eigenvalues outside the unit circle). We choose parameters $\delta = 0.5$, $p = 1$, $\nu = 0.45$, $\tau = 0.2$, $\varsigma = 0.1$, $\gamma = 1$, $\beta = 0.5$, and $\theta = 0.5$. The coefficient matrix A in equation (30) can be rewritten as

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Obviously, the eigenvalues are $\omega_1 = \omega_2 = 1/2$, $\mu_1 = \mu_2 = 4$. Hence, there are two repeated eigenvalues outside the unit circle.

We assume that at time 0, say, the government increases the tax rate on pollution, which it maintains at the higher level until time T:

$$h_t^{\tau} = \begin{cases} 1, & 0 \le t \le T; \\ 0, & t > T. \end{cases}$$

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TABLE 1. Effects of an increase in tax rate on pollution

	$T=2$	$T=5$	$T=10$	$T=\infty$
$\frac{dr_0}{d\varepsilon}$	1.4661	1.4815	1.4819	1.4819
$\frac{dW}{d\varepsilon}$	−0.2839	−0.4869	−0.5175	−0.5185

Therefore, we have

$$Z\left(h_t^{\tau},\mu\right)=\frac{1-(1/\mu)^{T+1}}{1-1/\mu}$$

and

$$w_0^3=\frac{1}{\mu_1}\left\{\left(\frac{V_{33}-V_{34}}{1-\theta}\right)\left[\frac{1-(1/\mu_1)^{T+1}}{1-1/\mu_1}\right]+\phi Z\left(w_t^4,\mu_1\right)\right\},$$
$$w_0^4=\frac{V_{43}-V_{44}}{\mu_2(1-\theta)}\left[\frac{1-(1/\mu_2)^{T+1}}{1-1/\mu_2}\right],$$

where

$$w_t^4=\begin{cases}\dfrac{V_{43}-V_{44}}{\mu_2(1-\theta)}\dfrac{1-(1/\mu_2)^{T+1-t}}{1-1/\mu_2}, & t\le T;\\ 0, & t>T.\end{cases}$$

Using the above equations, we can obtain numerical solutions for short-run effects based on the above calculation.

To investigate the effect of an increase in the tax rate on pollution exclusively, let T change from 2, 5, and 10 to infinity. The numerical solutions are presented in Table 1.

Therefore, an increase in the tax rate on pollution offers a negative incentive and disencourages investment in lowering effluents. Table 1 also presents the negative effects of policy change on the firm's welfare; this conclusion is the same as those of many existing studies, where an increase in pollution tax harms the firm's welfare and lowers its effluents.

Case 2: $\theta\neq\delta$ (distinct eigenvalues outside the unit circle). Let us change $\delta=0.6$ and leave all other parameters as in Case 1. Coefficient matrix A in equation (30) can then be derived as

$$A=\begin{pmatrix}0.4 & 0 & 1 & 0\\ 0.4 & 0.5 & 0 & 0\\ 0 & 0 & 5 & -4\\ 0 & 0 & 0 & 4\end{pmatrix}.$$

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TABLE 2. Effects of an increase in tax rate on pollution

	$T=2$	$T=5$	$T=10$	$T=\infty$
$\frac{dr_0}{d\varepsilon}$	1.2973	1.3326	1.3333	1.3333
$\frac{dW}{d\varepsilon}$	−0.4527	−0.6361	−0.6657	−0.6667

The eigenvalues are $\omega_1 = 0.5$, $\omega_2 = 0.4$, $\mu_1 = 4$, and $\mu_2 = 5$. Hence, there are two distinct eigenvalues outside the unit circle.

Similarly, let

$$h_t^\tau = \begin{cases} 1, & 0 \le t \le T \\ 0, & t > T \end{cases}$$

and we obtain

$$w_0^3 = \frac{V_{33} - V_{34}}{\mu_1(1-\theta)} Z\left(h_t^\tau, \mu_1\right), \; w_0^4 = \frac{V_{43} - V_{44}}{\mu_2(1-\theta)} Z\left(h_t^\tau, \mu_2\right).$$

Table 2 shows the effects of the tax on the economy.

From Table 2, we also know that an increase in pollution tax initially lowers the firm's effluents and and harms its welfare. However, the quantitative difference is smaller than the case $\theta = \delta$.

4. AN EXTENSION TO A MORE GENERAL SYSTEM

The method developed here can be used to deal with all systems that can be linearized as system (6). For the more generalized system,

$$F^i(x_{t+1}, y_{t+1}, x_t, y_t, \varepsilon h_t) = 0, \; i = 1, \ldots, n_1, \tag{1'}$$

$$G^j(x_{t+1}, y_{t+1}, x_t, y_t, \varepsilon h_t) = 0, \; j = 1, \ldots, n - n_1, \tag{2'}$$

where $F^i : R^{n_1} \times R^{n-n_1} \times R^{n_1} \times R^{n-n_1} \times R^m \to R$ and $G^j : R^{n_1} \times R^{n-n_1} \times R^{n_1} \times R^{n-n_1} \times R^m \to R$ are continuous differentiable functions. ε and h_t are assumed to be as before.

Linearizing systems (1′) and (2′) around the steady state yields

$$\Phi \begin{pmatrix} \frac{dx_{t+1}}{d\varepsilon} \\ \frac{dy_{t+1}}{d\varepsilon} \end{pmatrix} = \Psi \begin{pmatrix} \frac{dx_t}{d\varepsilon} \\ \frac{dy_t}{d\varepsilon} \end{pmatrix} + \Omega h_t, \tag{42}$$

where Φ is the first $n \times n$ submatrix of the Jacobian matrix of the vector function $(F^1, \ldots, F^{n_1}, G^1, \ldots, G^{n-n_1})^T$ evaluated at steady state; Ψ and Ω are the rest of the $n \times n$ and $n \times m$ submatrices.

As the matrix Φ may not be invertible, we cannot use Jordan decomposition to deal with it, and we use Schur decomposition instead. Hence, there are square

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matrices Q and Z and upper triangular matrices Λ and Π, such that

$$Q'\Lambda Z' = \Phi,\ Q'\Pi Z' = \Psi, \tag{43}$$

where Q and Z are unitary matrices that satisfy $QQ' = ZZ' = I$.

Thus, we have

$$Q'\Lambda Z'\begin{pmatrix} \frac{dx_{t+1}}{d\varepsilon} \\ \frac{dy_{t+1}}{d\varepsilon} \end{pmatrix} = Q'\Pi Z'\begin{pmatrix} \frac{dx_t}{d\varepsilon} \\ \frac{dy_t}{d\varepsilon} \end{pmatrix} + \Omega h_t. \tag{44}$$

We premultiply both sides of equation (44) by $\Lambda^{-1}Q$ and obtain

$$Z'\begin{pmatrix} \frac{dx_{t+1}}{d\varepsilon} \\ \frac{dy_{t+1}}{d\varepsilon} \end{pmatrix} = \Lambda^{-1}\Pi Z'\begin{pmatrix} \frac{dx_t}{d\varepsilon} \\ \frac{dy_t}{d\varepsilon} \end{pmatrix} + \Lambda^{-1}Q\Omega h_t. \tag{45}$$

Define

$$\begin{pmatrix} \frac{dp_{t+1}}{d\varepsilon} \\ \frac{dq_{t+1}}{d\varepsilon} \end{pmatrix} = Z'\begin{pmatrix} \frac{dx_{t+1}}{d\varepsilon} \\ \frac{dy_{t+1}}{d\varepsilon} \end{pmatrix};$$

then

$$\begin{pmatrix} \frac{dp_{t+1}}{d\varepsilon} \\ \frac{dq_{t+1}}{d\varepsilon} \end{pmatrix} = \Lambda^{-1}\Pi\begin{pmatrix} \frac{dp_{t+1}}{d\varepsilon} \\ \frac{dq_{t+1}}{d\varepsilon} \end{pmatrix} + \Lambda^{-1}Q\Omega h_t, \tag{46}$$

which is the same as the linearized system (6) described in Section 2. The methods developed in Section 2 can be also used to deal with this generalized system.

5. CONCLUDING REMARKS

With the aid of the Z-transform, this paper presents a generalized method for computing comparative dynamics in discrete-time models. Using this method, we can analytically derive the effects of policy change on social welfare and endogenous variables. Our method is restricted neither to the system's dimensions nor to the assumption of distinct eigenvalues of the system. It can be used to deal with the difference systems raised by any multisector model. For example, we could use it to analyze the effects of fiscal policies on the economy in the Uzawa–Lucas model, as in Ortigueira (1998).

However, the methodology developed here can be only used to analyze a saddle-point stable system, where the eigenvalues of coefficient matrix A do not equal 1.

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Unit eigenvalues appear in models and make simulation difficult. Therefore, further researches should consider more general systems with unit eigenvalues [see Laffargue (2004)].

Furthermore, the methodology developed here can be used only for a certainty shock. The stochastic shocks in many economic models may also be important and interesting. Therefore, further research should extend this method to deal with more general systems with stochastic shocks.

NOTES

1. We would like to thank one of the referees for pointing this out.
2. Assume that $1/\beta$ is not an eigenvalue of matrix A.

REFERENCES

Blanchard, O. and C.M. Kahn (1980) The solution of linear difference models under rational expectations. *Econometrica* 48, 1305–1313.

Buiter, W. (1984) Saddle-point problems in continuous time rational expectations models: A general method and some macroeconomic examples. *Econometrica* 52, 665–680.

Chamley, C. (1981) The welfare cost of capital income taxation in a growing economy. *Journal of Political Economy* 89, 468–496.

Cui, X. and L. Gong (2006) Laplace transform methods for linearizing multidimensional systems. *Economics Letters* 90, 176–182.

Cui, X., Liutang Gong, Xiaojun Zhao, and Heng-fu Zou (2007) Z-Transform Method for Linearizing Multidimensional Discrete-Time Systems. Working paper, Peking University.

Doan, T.A. (2010) Practical issues with state-space models with mixed stationary and nonstationary dynamics. Technical Paper No. 2010-1.

Judd, K. (1982) An alternative to steady-state comparisons in perfect foresight models. *Economics Letters* 10, 55–59.

Judd, K. (1996) Approximation, perturbation and projection methods in economic analysis. In H. Amman, D. Kendrick, and J. Rust (eds.), *Handbook of Computational Economics*, pp. 509–585. Elsevier: North-Holland.

Juilliard, M. (1996) Dynare: A Program for the Resolution and Simulation of Dynamic Models with Forward Variables through the Use of a Relaxation Algorithm. CEPREMAP working paper 9602.

Laffargue, J.P. (2004) A sufficient condition for the existence and the uniqueness of a solution in macroeconomic models with perfect foresight. *Journal of Economic Dynamics and Control* 28, 1955–1975.

Lucas, Robert E. (2003) Macroeconomic priorities. *American Economic Review* 93, 1–14.

Meijdam, L. and M. Verhoeven (1998) Comparative dynamics in perfect-foresight models. *Computational Economics* 12, 115–124.

Ortigueira, S. (1998) Fiscal policy in an endogenous growth model with human capital accumulation. *Journal of Monetary Economics* 42, 323–355.

Tahvonen, O. and C. Withagen (1996) Optimality of irreversible pollution accumulation. *Journal of Economic Dynamics and Control* 20, 1775–1795.

Tippett, M. and T. Warnock (1997) The Garman–Ohlson structure system. *Journal of Business Finance and Accounting* 24 (7), 1075–1099.

Turnovsky, S. (2000) *Methods of Macroeconomic Dynamics*. Boston: MIT Press.

Wirl, F. (2000) Optimal accumulation of pollution: Existence of limit cycles for the social optimum and the competitive equilibrium. *Journal of Economic Dynamics and Control* 24, 297–306.

https://doi.org/10.1017/S1365100511000071 Published online by Cambridge University Press