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The central limit problem for trimmed sums

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1. *Introduction*

Let $X, X_1, X_2, \ldots$ be a sequence of non-degenerate i.i.d. random variables with common distribution function F. For $1 \leqslant j \leqslant n$, let $m_n(j)$ be the number of X_i satisfying either $|X_i| > |X_j|, 1 \leqslant i \leqslant n$, or $|X_i| = |X_j|, 1 \leqslant i \leqslant j$, and let ${}^{(r)}X_n = X_j$ if $m_n(j) = r$. Thus ${}^{(r)}X_n$ is the rth largest random variable in absolute value from amongst $X_1, \ldots, X_n$ with ties being broken according to the order in which the random variables occur. Set ${}^{(r)}S_n = {}^{(r+1)}X_n + \ldots + {}^{(n)}X_n$ and write S_n for ${}^{(0)}S_n$. We will refer to ${}^{(r)}S_n$ as a trimmed sum.

The asymptotic distribution of ${}^{(r_n)}S_n$, appropriately normalized, has been considered by several authors. This problem arises naturally when trying to robustify the estimator $n^{-1}S_n$ by trimming extreme observations. Most of this work has been concerned with the removal of a fixed number of terms, usually referred to as light trimming. The general principle is that light trimming does not improve weak convergence results. For example Maller [**8**], for continuous and symmetric distributions, and Mori [**10**], for general distributions, show that $({}^{(r)}S_n - \delta_n)\gamma_n^{-1} \to N$ if and only if $(S_n - \delta_n)\gamma_n^{-1} \to N$, where N is normal with zero mean and variance one. They also show that $({}^{(r)}S_n - \delta_n)\gamma_n^{-1}$ is stochastically compact, i.e. the sequence is tight and all subsequential limits are non-degenerate, if and only if $(S_n - \delta_n)\gamma_n^{-1}$ is stochastically compact. A necessary and sufficient condition on the underlying distribution for this to hold is due to Feller [**4**]. We will refer to such distributions as belonging to the Feller class. For further work on light trimming see Darling [**2**], Arov and Bobrov [**1**] and Hall [**7**].

In this paper we will consider the case in which $r_n \to \infty$, where the theory is considerably richer. For example, in contrast to the above result, Pruitt [**11**] has shown that if F is continuous and

$$X \text{ is symmetric}, \quad r_n \to \infty, \quad r_n n^{-1} \to 0, \tag{1·1}$$

then there exists a sequence γ_n such that ${}^{(r_n)}S_n \gamma_n^{-1} \to N$ whenever X belongs to the Feller class. This of course raises the question: exactly when is $({}^{(r_n)}S_n - \delta_n)\gamma_n^{-1}$ asymptotically normal? Under assumption (1·1) we will completely answer this question and further solve the following more general central limit problem: find the class of limit laws of normalized trimmed sums and characterize their domains of attraction.

The condition for asymptotic normality is particularly simple to state when the distribution function of X is continuous. Let $a_n(\alpha)$ be any solution of

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$P(|X| > a_n(\alpha)) = n^{-1}(r_n - \alpha r_n^{\frac{1}{2}})$ and assume (1·1); then we have the following special case of Theorem 3·10.

THEOREM 1·1. *There exist γ_n, δ_n such that $({}^{(r_n)}S_n - \delta_n)\gamma_n^{-1} \to N$ if and only if for all $\alpha \in \mathbb{R}$*

$$\frac{EX^2 1(|X| \leqslant a_n(\alpha))}{EX^2 1(|X| \leqslant a_n(0))} \to 1. \tag{1·2}$$

In this case one may take $\gamma_n = (nEX^2 1(|X| \leqslant a_n(0)))^{\frac{1}{2}}$ and $\delta_n = 0$.

The class of limit laws which can arise, in addition to the normal law, is characterized by the following result which is an immediate consequence of Theorems 3·7 and 3·8.

THEOREM 1·2. *Assume (1·1) and let $\mu(d\alpha) = P(N \in d\alpha)$; W is a subsequential limit of $({}^{(r_n)}S_n - \delta_n)\gamma_n^{-1}$ for some distribution X if and only if*

$$E \exp(iuW) = \exp(-iu\delta) \int_{-\infty}^{\infty} \exp(-u^2 h(\alpha)/2)\, \mu(d\alpha) \tag{1·3}$$

for some unique $\delta \in \mathbb{R}$ and some unique function h satisfying

$$h \text{ is non-negative, non-decreasing and convex.} \tag{1·4}$$

A random variable of the form (1·3) is usually referred to as a variance mixture of normals. A necessary and sufficient condition for X to be in the domain of attraction or partial attraction of a given limit law is given in Theorem 3·9. In addition to these results we also obtain in Theorem 3·5 a necessary and sufficient condition for stochastic compactness of the normalized trimmed sum.

With regard to the assumptions in (1·1), $n^{-1}r_n \to 0$ is not essential. In most instances it merely serves to simplify the formulas. We will point out later the modifications needed when $r_n n^{-1} \not\to 0$. The assumption of symmetry on X, however, is not just a technicality. For example, given any sequence $r_n \to \infty$ with $r_n n^{-1} \to 0$, we will exhibit a random variable in the domain of normal attraction of a symmetric stable law such that $({}^{(r_n)}S_n - \delta_n)\gamma_n^{-1} \not\to N$ for any choice of γ_n and δ_n. Thus even a slight perturbation from symmetry can destroy asymptotic normality. The same example also shows that the class of subsequential limit laws is very different when the distribution is no longer symmetric. For a further discussion see Example 4·2.

Another form of trimming often advocated is as follows; let $X_{n1} \leqslant \ldots \leqslant X_{nn}$ be the order statistics of $X_1, \ldots, X_n$ and set $S_n(s_n, r_n) = X_{n, s_n+1} + \ldots + X_{n, n-r_n}$. In Griffin and Pruitt [**6**] we investigate the central limit problem for $S_n(s_n, r_n)$. Since the origin plays no special role for this kind of trimming, there is no need to restrict to symmetric distributions. In particular, we obtain necessary and sufficient conditions for asymptotic normality and characterize the class of subsequential limits in general. For symmetric distributions, one might suspect that the asymptotic behaviour of $S_n(r_n, r_n)$ and ${}^{(2r_n)}S_n$ is essentially the same; however the two problems are fundamentally different. Nevertheless, it is interesting to note that for any symmetric distribution, if there exist γ_n, δ_n such that $(S_n(r_n, r_n) - \delta_n)\gamma_n^{-1} \to N$ then there exist γ_n, δ_n such that $({}^{(2r_n)}S_n - \delta_n)\gamma_n^{-1} \to N$. What is perhaps even more surprising is that if both of these occur, the normalizers need not be of the same order of magnitude. For a further discussion and comparison of the two types of trimming see [**6**].

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2. *Preliminaries*

We will assume throughout, unless stated otherwise, that r_n is a sequence of integers satisfying $r_n \to \infty$ and $r_n n^{-1} \to 0$. Symmetry of X will not be assumed in this section, since some of the results will be needed in the construction of the example mentioned above. For $x > 0$ define

$$G(x) = P(|X| > x), \quad K(x) = x^{-2} \int_{|y| \leqslant x} y^2 F(dy), \tag{2·1}$$

and let $G(0) = P(|X| > 0), K(0) = 0$. It will be convenient to set $G(0-) = 1$. Let

$$\tilde{G}(u) = \begin{cases} \sup\{x \geqslant 0 : G(x) > u\} & \text{if} \quad 0 \leqslant u < G(0) \\ 0 & \text{if} \quad u \geqslant G(0). \end{cases} \tag{2·2}$$

Note that for $0 < u < G(0)$

$$G(\tilde{G}(u)) \leqslant u \leqslant G(\tilde{G}(u)-). \tag{2·3}$$

The following construction is due to Mori [**10**] and we refer the reader to pages 508–509 of his paper for further details. By the Radon–Nikodým Theorem there exist two Borel functions g^+ and g^- defined on $(0, \infty)$ such that for $x > 0$

$$P(X > x) = \int_{(x, \infty)} g^+(y)\, dh(y), \quad P(X < -x) = \int_{(x, \infty)} g^-(y)\, dh(y),$$

where $h(y) = 1 - G(y)$. Set $g^+(0) = g^-(0) = \frac{1}{2}$. By modifying g^+ and g^- on a set of dh measure zero if necessary, we may assume that for all $y \geqslant 0, 0 \leqslant g^{\pm}(y) \leqslant 1$ and $g^+(y) + g^-(y) = 1$.

Let U be uniformly distributed over $(0, 1)$ and let η satisfy

$$P(\eta = \pm 1 \mid U = u) \overset{\text{a}}{=} g^{\pm}(\tilde{G}(u)) \quad 0 < u < 1.$$

Then $\eta \tilde{G}(U) \overset{d}{=} X$. Let (η_i, U_i) be a sequence of i.i.d. random vectors each distributed as (η, U) and let $U_{n1} \leqslant \ldots \leqslant U_{nn}$ be the order statistics of $U_1, \ldots, U_n$. Note that this ordering is unique and the inequalities are strict except on a null set. Now let ${}^{(i)}Y_n = \eta_k \tilde{G}(U_k)$ if $U_{ni} = U_k$. Then $({}^{(1)}Y_n, \ldots, {}^{(n)}Y_n) \overset{d}{=} ({}^{(1)}X_n, \ldots, {}^{(n)}X_n)$ and so ${}^{(r_n)}S_n \overset{d}{=} {}^{(r_n+1)}Y_n + \ldots + {}^{(n)}Y_n$. Thus we may assume the X_i are given by $\eta_i \tilde{G}(U_i)$, and this we will do. For $0 < u < G(0)$ define

$$H(u) = \tilde{G}(u)^{-2} \int_{U \geqslant u} \tilde{G}(U)^2\, dP. \tag{2·4}$$

A simple computation shows that

$$H(u) = K(\tilde{G}(u)-) + G(\tilde{G}(u)-) - u = K(\tilde{G}(u)) + G(\tilde{G}(u)) - u. \tag{2·5}$$

For $0 < u < 1$, let $X(u)$ have the same distribution as $\eta \tilde{G}(U)$ conditioned by $U \geqslant u$, i.e. for any Borel set B, $P(X(u) \in B) = P(\eta \tilde{G}(U) \in B, U \geqslant u)\,(1-u)^{-1}$. Note that $|X(u)| \leqslant \tilde{G}(u)$; further,

$$P(|X(u)| > x) = (G(x) - u)\,(1-u)^{-1} \vee 0, \tag{2·6}$$

$$P(|X(u)| \geqslant x) = (G(x-) - u)\,(1-u)^{-1} \vee 0. \tag{2·7}$$

Also, for $0 < u < G(0)$,

$$EX^2(u) = \tilde{G}(u)^2\, H(u)\,(1-u)^{-1}. \tag{2·8}$$

Set

$$u_n(\alpha) = ((r_n - \alpha r_n^{\frac{1}{2}})\, n^{-1} \vee 0) \wedge 1, \quad v_n(\alpha) = 1 - u_n(\alpha), \tag{2·9}$$

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$$a_n(\alpha) = \tilde{G}(u_n(\alpha)). \tag{2·10}$$

Observe that $u_n(\,.\,)$, $v_n(\,.\,)$ and $a_n(\,.\,)$ are all monotone, and if $r_n^{\frac{1}{2}} - G(0)\, nr_n^{-\frac{1}{2}} < \alpha < r_n^{\frac{1}{2}}$ then

$$0 < u_n(\alpha) < G(0). \tag{2·11}$$

Let $s_n = r_n^{\frac{1}{2}} \wedge (G(0)\, nr_n^{-\frac{1}{2}} - r_n^{\frac{1}{2}})$; note that $s_n \to \infty$.

LEMMA 2·1. *Let $\alpha > \beta$ and n be large enough that $|\alpha|, |\beta| < s_n$. Then*

$$P(|X(u_n(\alpha))| \geqslant a_n(\beta)) \geqslant (\alpha - \beta)\, r_n^{\frac{1}{2}}\, n^{-1} v_n^{-1}(\alpha). \tag{2·12}$$

Proof. By (2·3), (2·7), (2·10) and (2·11),

$$\begin{aligned} P(|X(u_n(\alpha))| \geqslant a_n(\beta)) &= (G(a_n(\beta)-) - u_n(\alpha))\, v_n^{-1}(\alpha) \vee 0 \\ &= (G(\tilde{G}(u_n(\beta))-) - u_n(\alpha))\, v_n^{-1}(\alpha) \vee 0 \\ &\geqslant (u_n(\beta) - u_n(\alpha))\, v_n^{-1}(\alpha) \\ &= (\alpha - \beta)\, r_n^{\frac{1}{2}}\, n^{-1} v_n^{-1}(\alpha). \quad \blacksquare \end{aligned}$$

Throughout this paper m will be used to denote the integer $n - r_n$. If we consider a subsequence $n(i)$ then $m(i)$ will denote $n(i) - r_{n(i)}$. Observe that

$$mn^{-1} \to 1. \tag{2·13}$$

For $0 < u < 1$ let $X_k(u)$ be a sequence of i.i.d. random variables each distributed as $X(u)$. For $|\alpha| < s_n$ set

$$V_m(\alpha) = \sum_{k=1}^{m} X_k(u_n(\alpha)),$$

$$\tau_n(\alpha) = (ma_n^2(\alpha)\, H(u_n(\alpha)))^{\frac{1}{2}}.$$

Observe that $\tau_n(\alpha)$ is increasing in α and if X were symmetric then by (2·8)

$$EV_m^2(\alpha) = \tau_n^2(\alpha)\, v_n^{-1}(\alpha). \tag{2·14}$$

LEMMA 2·2. *Let $\alpha > \beta$ and n be large enough that $|\alpha|, |\beta| < s_n$. Then*

$$nH(u_n(\alpha))\, r_n^{-\frac{1}{2}} \leqslant nH(u_n(\beta))\, r_n^{-\frac{1}{2}} + (\alpha - \beta), \tag{2·15}$$

$$\tau_n^2(\alpha)\, \tau_n^{-2}(\beta) \geqslant 1 + (\alpha - \beta)\, r_n^{\frac{1}{2}} (nH(u_n(\beta)))^{-1}. \tag{2·16}$$

Proof. Observe that by (2·4)

$$\begin{aligned} a_n^2(\alpha)\, H(u_n(\alpha)) &= E[\tilde{G}(U)^2; U \geqslant u_n(\alpha)] \\ &= a_n^2(\beta)\, H(u_n(\beta)) + E[\tilde{G}(U)^2; u_n(\alpha) \leqslant U < u_n(\beta)] \\ &\leqslant a_n^2(\beta)\, H(u_n(\beta)) + a_n^2(\alpha)\, (\alpha - \beta)\, r_n^{\frac{1}{2}}\, n^{-1}. \end{aligned} \tag{2·17}$$

Since $a_n(\alpha) \geqslant a_n(\beta)$, (2·15) follows by multiplying both sides by $nr_n^{-\frac{1}{2}} a_n^{-2}(\alpha)$.

A similar calculation to (2·17) gives

$$a_n^2(\alpha)\, H(u_n(\alpha)) \geqslant a_n^2(\beta)\, H(u_n(\beta)) + a_n^2(\beta)\, (\alpha - \beta)\, r_n^{\frac{1}{2}} n^{-1}, \tag{2·18}$$

from which (2·16) follows. $\blacksquare$

COROLLARY 2·3. *(a) If $|\beta| < \lambda$ and $\lambda + 1 < s_n$, then*

$$\tau_n^2(\beta)\, \tau_n^{-2}(\lambda + 1)\, r_n^{\frac{1}{2}} (nH(u_n(\beta)))^{-1} \leqslant 1. \tag{2·19}$$

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(*b*) *For any* $\lambda_1 < \lambda_2$ *and any subsequence* $n(i)$,

$$\liminf_{i\to\infty} \inf_{\lambda_1 \leqslant \beta \leqslant \lambda_2} n(i)\, H(u_{n(i)}(\beta))\, \tau_{n(i)}^{-1} \geqslant \liminf_{i\to\infty} \tau_{n(i)}^2(\lambda_1)\, \tau_{n(i)}^{-2}(\lambda_2+1). \tag{2·20}$$

Proof. By (2·16), if $|\beta| < \lambda$ and $\lambda+1 < s_n$

$$\begin{aligned}\tau_n^2(\lambda+1)\,\tau_n^{-2}(\beta) &\geqslant (\lambda+1-\beta)\, r_n^{\frac{1}{2}}(nH(u_n(\beta)))^{-1} \\ &\geqslant r_n^{\frac{1}{2}}(nH(u_n(\beta)))^{-1},\end{aligned}$$

which proves (2·19). For (2·20) we have, again by (2·16), that if $|\lambda_1|, |\lambda_2|+1 < s_n$ then for any $\lambda_1 \leqslant \beta \leqslant \lambda_2$

$$\begin{aligned}nH(u_n(\beta))\, \tau_n^{-\frac{1}{2}} &\geqslant (\lambda_2+1-\beta)\, \tau_n^2(\beta)\, \tau_n^{-2}(\lambda_2+1) \\ &\geqslant \tau_n^2(\lambda_1)\, \tau_n^{-2}(\lambda_2+1),\end{aligned}$$

from which (2·20) immediately follows. ∎

Define the measures μ_n and μ by

$$\begin{aligned}\mu_n((\alpha,\infty)) &= P(U_{n,r_n} < u_n(\alpha)), \\ \mu((\alpha,\infty)) &= P(N > \alpha),\end{aligned}$$

where N is normal with zero mean and variance one. Our approach to studying the asymptotic distribution of the trimmed sum is based on the following two lemmas.

LEMMA 2·4. *With* μ_n *and* μ *as above we have* $\mu_n \to \mu$.

Proof. Let $N(n, u_n(\alpha)) = \sum_{k=1}^n 1(U_k < u_n(\alpha))$. Then $N(n, u_n(\alpha))$ is binomially distributed and since $nu_n(\alpha)\, v_n(\alpha) \to \infty$, we have that

$$\frac{N(n, u_n(\alpha)) - nu_n(\alpha)}{(nu_n(\alpha)\, v_n(\alpha))^{\frac{1}{2}}} \to N. \tag{2·21}$$

Thus if n is sufficiently large that $|\alpha| < s_n$, then

$$\begin{aligned}P(U_{n,r_n} < u_n(\alpha)) &= P(N(n, u_n(\alpha)) \geqslant r_n) \\ &= P(N(n, u_n(\alpha)) - nu_n(\alpha) \geqslant \alpha r_n^{\frac{1}{2}}) \\ &\to P(N \geqslant \alpha)\end{aligned}$$

by (2·21), since $(nu_n(\alpha)\, v_n(\alpha))^{\frac{1}{2}} \sim r_n^{\frac{1}{2}}$. ∎

LEMMA 2·5. *For any* $\lambda > 0$, *if* n *is large enough that* $\lambda < s_n$, *then for any Borel set* B

$$0 \leqslant P((({}^{(r_n)}S_n - \delta_n)\, \gamma_n^{-1} \in B) - \int_{-\lambda}^{\lambda} P((V_m(\alpha) - \delta_n)\, \gamma_n^{-1} \in B)\, \mu_n(d\alpha) \leqslant \mu_n([-\lambda,\lambda]^c). \tag{2·22}$$

Proof. By lemma 1 of Mori [**10**],

$$\begin{aligned}P((({}^{(r_n)}S_n - \delta_n)\, \gamma_n^{-1} \in B) &= \int_0^1 P((({}^{(r_n)}S_n - \delta_n)\, \gamma_n^{-1} \in B \mid U_{n,r_n} = u)\, P(U_{n,r_n} \in (u, u+du)) \\ &= \int_0^1 P((W_m(u) - \delta_n)\, \gamma_n^{-1} \in B)\, P(U_{n,r_n} \in (u, u+du))\end{aligned}$$

where $W_m(u) = \sum_{k=1}^m X_k(u)$. The result now follows easily from the change of variable $u = u_n(\alpha)$. ∎

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3. *Main results*

For the remainder of the paper, unless stated otherwise, we will assume (1·1) holds. In this section we will prove our main results concerning the asymptotic distribution of $({}^{(r_n)}S_n - \delta_n)\gamma_n^{-1}$. Since our proofs work equally well for subsequences $n(i)$ we state them in this form. (We will use $n(i)$ and n_i interchangeably for subsequences.) We begin by considering tightness.

LEMMA 3·1. *For any subsequence $n(i)$, if $({}^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\gamma_{n_i}^{-1}$ is tight, then $|\delta_{n_i}\gamma_{n_i}^{-1}|$ is bounded and ${}^{(r_{n(i)})}S_{n_i}\gamma_{n_i}^{-1}$ is tight.*

Proof. Suppose $|\delta_{n_i}\gamma_{n_i}^{-1}|$ is unbounded; then along some further subsequence n_j, we may assume that $\delta_{n_j}\gamma_{n_j}^{-1} \to \infty$ ($\delta_{n_j}\gamma_{n_j}^{-1} \to -\infty$ is handled analogously). Fix x; then for sufficiently large n_j

$$\begin{aligned} P(|{}^{(r_{n(j)})}S_{n_j} - \delta_{n_j}|\gamma_{n_j}^{-1} > x) &\geqslant P({}^{(r_{n(j)})}S_{n_j}\gamma_{n_j}^{-1} < -x + \delta_{n_j}\gamma_{n_j}^{-1}) \\ &\geqslant P({}^{(r_{n(j)})}S_{n_j}\gamma_{n_j}^{-1} \leqslant 0) \\ &= \tfrac{1}{2}, \end{aligned}$$

since ${}^{(r_{n(j)})}S_{n_j}$ is symmetric. But this contradicts tightness and so $|\delta_{n_i}\gamma_{n_i}^{-1}|$ is bounded. From this it is immediate that ${}^{(r_{n(i)})}S_{n_i}\gamma_{n_i}^{-1}$ is tight. ∎

THEOREM 3·2. *Given any sequences γ_n, δ_n and any subsequence n_i, then*

$$({}^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\gamma_{n_i}^{-1} \quad \text{is tight} \tag{3·1}$$

if and only if the following two conditions hold:

$$\limsup_{i\to\infty} \tau_{n_i}(\alpha)\gamma_{n_i}^{-1} < \infty \quad \text{for all} \quad \alpha \tag{3·2}$$

$$\limsup_{i\to\infty} |\delta_{n_i}\gamma_{n_i}^{-1}| < \infty. \tag{3·3}$$

Proof. First assume (3·1); then by Lemma 3·1, (3·3) holds. If (3·2) fails then there exists a subsequence n_j and an α_0 such that

$$\tau_{n_j}(\alpha_0)\gamma_{n_j}^{-1} \to \infty. \tag{3·4}$$

Fix $\alpha_0 < \alpha_1 < \alpha_2$. Then by (2·22) for sufficiently large n_j

$$P(|{}^{(r_{n(j)})}S_{n_j}|\gamma_{n_j}^{-1} > x) \geqslant \int_{\alpha_0}^{\alpha_2} P(|V_{m_j}(\alpha)|\gamma_{n_j}^{-1} > x)\,\mu_{n_j}(d\alpha). \tag{3·5}$$

We consider two cases.

$$\liminf_{j\to\infty} \inf_{\alpha_0 \leqslant \alpha \leqslant \alpha_1} m_j H(u_{n_j}(\alpha)) = \infty. \tag{A}$$

By the Berry–Esseen Theorem ([5], p. 542), (2·8) and (2·14), if $|\alpha_0|, |\alpha_1| < s_n$ and $\alpha_0 \leqslant \alpha \leqslant \alpha_1$,

$$\begin{aligned} \sup_y |P(|V_m(\alpha)\tau_n^{-1}(\alpha)v_n^{\frac{1}{2}}(\alpha)| > y) - P(|N| > y)| &\leqslant \frac{6E|X(u_n(\alpha))|^3}{m^{\frac{1}{2}}(E|X(u_n(\alpha))|^2)^{\frac{3}{2}}} \\ &\leqslant \frac{6a_n(\alpha)}{m^{\frac{1}{2}}(E|X(u_n(\alpha))|^2)^{\frac{1}{2}}} \\ &= 6v_n^{\frac{1}{2}}(\alpha)\,(mH(u_n(\alpha)))^{-\frac{1}{2}} \\ &\leqslant 6(mH(u_n(\alpha)))^{-\frac{1}{2}}. \end{aligned}$$

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Thus, letting $y = x\gamma_{n_j}\tau_{n_j}^{-1}(\alpha)\,v_{n_j}^{\frac{1}{2}}(\alpha)$, we have by (3·5) and (A)

$$P(|{}^{(r_{n(j)})}S_{n_j}|\,\gamma_{n_j}^{-1} > x) \geqslant \int_{\alpha_0}^{\alpha_1} P(|N|\,\tau_{n_j}(\alpha)\,v_{n_j}^{-\frac{1}{2}}(\alpha)\,\gamma_{n_j}^{-1} > x)\,\mu_{n_j}(d\alpha) + o(1).$$

But for $\alpha_0 \leqslant \alpha \leqslant \alpha_1$, $\tau_{n_j}(\alpha)\,v_{n_j}^{-\frac{1}{2}}(\alpha)\,\gamma_{n_j}^{-1} \geqslant \tau_{n_j}(\alpha)\,\gamma_{n_j}^{-1} \geqslant \tau_{n_j}(\alpha_0)\,\gamma_{n_j}^{-1} \geqslant x$ for sufficiently large n_j, independent of α, by (3·4). Thus for large n_j

$$\begin{aligned} P(|{}^{(r_{n(j)})}S_{n_j}|\,\gamma_{n_j}^{-1} > x) &\geqslant P(|N| > 1)\,\mu_{n_j}([\alpha_0, \alpha_1]) + o(1) \\ &\to P(|N| > 1)\,\mu([\alpha_0, \alpha_1]). \end{aligned}$$

Hence ${}^{(r_{n(j)})}S_{n_j}\gamma_{n_j}^{-1}$ is not tight which, by Lemma 3·1, is a contradiction.

$$\liminf_{j\to\infty} \inf_{\alpha_0 \leqslant \alpha \leqslant \alpha_1} m_j\,H(u_{n_j}(\alpha)) < \infty. \tag{B}$$

In this case there exists a subsequence n_k of n_j and a sequence $\beta_{n_k} \in [\alpha_0, \alpha_1]$ such that

$$\sup_k m_k\,H(u_{n_k}(\beta_{n_k})) = \rho < \infty.$$

Fix $\alpha_3 \in (\alpha_1, \alpha_2)$; if k is large enough that $|\alpha_0|, |\alpha_2| < s_{n_k}$, then for any $\alpha \in [\alpha_3, \alpha_2]$

$$\begin{aligned} P(|X(u_{n_k}(\alpha))| > 2x\gamma_{n_k}) &= P(|X(u_{n_k}(\alpha))| > \tau_{n_k}(\beta_{n_k})\,2x\gamma_{n_k}\tau_{n_k}^{-1}(\beta_{n_k})) \\ &\geqslant P(|X(u_{n_k}(\alpha))| > \rho^{\frac{1}{2}}a_{n_k}(\beta_{n_k})\,2x\gamma_{n_k}\tau_{n_k}^{-1}(\beta_{n_k})) \\ &\geqslant P(|X(u_{n_k}(\alpha))| > a_{n_k}(\alpha_1)\,2x\rho^{\frac{1}{2}}\gamma_{n_k}\tau_{n_k}^{-1}(\alpha_0)). \end{aligned}$$

Thus if in addition k is large enough that $\gamma_{n_k}\tau_{n_k}^{-1}(\alpha_0) < (2x\rho^{\frac{1}{2}})^{-1}$, which happens by (3·4), then by (2·12) for $\alpha_3 \leqslant \alpha \leqslant \alpha_2$

$$\begin{aligned} P(|X(u_{n_k}(\alpha))| > 2x\gamma_{n_k}) &\geqslant P(|X(u_{n_k}(\alpha))| \geqslant a_{n_k}(\alpha_1)) \\ &\geqslant (\alpha - \alpha_1)\,r_{n_k}^{\frac{1}{2}}\,n_k^{-1}\,v_{n_k}^{-1}(\alpha) \\ &\geqslant (\alpha_3 - \alpha_1)\,r_{n_k}^{\frac{1}{2}}\,n_k^{-1}. \end{aligned}$$

Consequently, for sufficiently large k, if $\alpha_3 \leqslant \alpha \leqslant \alpha_2$, then by Lévy's inequality

$$\begin{aligned} 2P(|V_{m_k}(\alpha)| > x\gamma_{n_k}) &\geqslant P(\max_{1\leqslant q\leqslant m_k} |V_q(\alpha)| > x\gamma_{n_k}) \\ &= 1 - P(\max_{1\leqslant q\leqslant m_k} |V_q(\alpha)| \leqslant x\gamma_{n_k}) \\ &\geqslant 1 - P(\max_{1\leqslant q\leqslant m_k} |X_q(u_{n_k}(\alpha))| \leqslant 2x\gamma_{n_k}) \\ &= 1 - (1 - P(|X(u_{n_k}(\alpha))| > 2x\gamma_{n_k}))^{m_k} \\ &\geqslant 1 - \exp(-m_k\,n_k^{-1}(\alpha_3 - \alpha_1)\,r_{n_k}^{\frac{1}{2}}) \\ &= 1 - o(1), \end{aligned}$$

where we have abused notation slightly and let $V_q(\alpha) = \Sigma_{i=1}^{q} X_i(u_{n_k}(\alpha))$. Hence by (3·5), for large k,

$$\begin{aligned} P(|{}^{(r_{n(k)})}S_{n_k}|\,\gamma_{n_k}^{-1} > x) &\geqslant \int_{\alpha_3}^{\alpha_2} P(|V_{m_k}(\alpha)| > x\gamma_{n_k})\,\mu_{n_k}(d\alpha) \\ &\geqslant 2^{-1}(1 - o(1))\,\mu_{n_k}([\alpha_3, \alpha_2]) \\ &\to 2^{-1}\mu([\alpha_3, \alpha_1]). \end{aligned}$$

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Thus ${}^{(\tau_{n(k)})}S_{n_k}\gamma_{n_k}^{-1}$ is not tight, which again contradicts Lemma 3·1.

Now assume that (3·2) and (3·3) hold; then for any $\lambda > 0$ if $|\alpha| < \lambda$ and $\lambda < s_{n_i}$

$$\begin{aligned} P(|V_{m_i}(\alpha)| > x\gamma_{n_i}) &\leqslant EV_{m_i}^2(\alpha)\,(x\gamma_{n_i})^{-2} \\ &= \tau_{n_i}^2(\alpha)\,v_{n_i}^{-1}(\alpha)\,(x\gamma_{n_i})^{-2} \\ &\leqslant (\tau_{n_i}(\lambda)\,\gamma_{n_i}^{-1})^2\,v_{n_i}^{-1}(-\lambda)\,x^{-2} \\ &\leqslant cx^{-2}, \end{aligned}$$

where c depends only on λ. Thus by (2·22), for sufficiently large i,

$$\begin{aligned} P(|{}^{(\tau_{n(i)})}S_{n_i}\gamma_{n_i}^{-1}| > x) &\leqslant \int_{-\lambda}^{\lambda} P(|V_{m_i}(\alpha)| > x\gamma_{n_i})\,\mu_{n_i}(d\alpha) + \mu_{n_i}([-\lambda,\lambda]^c) \\ &\leqslant cx^{-2}\mu_{n_i}([-\lambda,\lambda]) + \mu_{n_i}([-\lambda,\lambda]^c) \\ &\to cx^{-2}\mu([-\lambda,\lambda]) + \mu([-\lambda,\lambda]^c). \end{aligned}$$

Since λ was arbitrary, this proves ${}^{(\tau_{n(i)})}S_{n_i}\gamma_{n_i}^{-1}$ is tight and consequently by the boundedness of $|\delta_{n_i}\gamma_{n_i}^{-1}|$, $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1}$ is tight. ∎

Our next aim is to characterize when $({}^{(\tau_n)}S_n - \delta_n)\,\gamma_n^{-1}$ has no degenerate subsequential limits.

LEMMA 3·3. *Assume that for some subsequence n_i and some α_0, we have $\tau_{n_i}(\alpha_0)\,\gamma_{n_i}^{-1} \to 0$. Then for all $x > 0$*

$$\liminf_{i\to\infty} P(|{}^{(\tau_{n(i)})}S_{n_i}\gamma_{n_i}^{-1}| \leqslant x) \geqslant \mu((-\infty, \alpha_0]). \tag{3·6}$$

Proof. Assume that $|\alpha_0| < \lambda < s_{n_i}$; then for $\alpha \in [-\lambda, \alpha_0]$

$$\begin{aligned} P(|V_{m_i}(\alpha)| > x\gamma_{n_i}) &\leqslant EV_{m_i}^2(\alpha)\,(x\gamma_{n_i})^{-2} \\ &= \tau_{n_i}^2(\alpha)\,v_{n_i}^{-1}(\alpha)\,(x\gamma_{n_i})^{-2} \\ &\leqslant (\tau_{n_i}(\alpha_0)\,\gamma_{n_i}^{-1})^2\,v_{n_i}^{-1}(-\lambda)\,x^{-2} \\ &= o(1) \end{aligned}$$

uniformly in α. Thus by (2·22) for large i

$$\begin{aligned} P(|{}^{(\tau_{n(i)})}S_{n_i}|\,\gamma_{n_i}^{-1} \leqslant x) &\geqslant \int_{-\lambda}^{\lambda} P(|V_{m_i}(\alpha)| \leqslant x\gamma_{n_i})\,\mu_{n_i}(d\alpha) \\ &\geqslant (1 - o(1))\,\mu_{n_i}([-\lambda, \alpha_0]) \\ &\to \mu([-\lambda, \alpha_0]). \end{aligned}$$

The result follows by letting $\lambda \to \infty$. ∎

THEOREM 3·4. *For any sequences γ_n, δ_n and any subsequence n_i, if $|\delta_{n_i}\gamma_{n_i}^{-1}|$ is bounded then the following are equivalent:*

$$\textit{all subsequential limits of } ({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1} \textit{ are non-degenerate,} \tag{3·7}$$

$$\liminf_{i\to\infty} \tau_{n_i}(\alpha)\,\gamma_{n_i}^{-1} > 0 \textit{ for some } \alpha. \tag{3·8}$$

Remarks. 1. In (3·7), we include the possibility that there are no subsequential limits.

2. Without the boundedness condition on $|\delta_{n_i}\gamma_{n_i}^{-1}|$, the result is false since we can

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always choose γ_n increasing fast enough that (3·8) fails and then choose δ_n so that $\delta_n \gamma_n^{-1} \to \infty$. In this case (3·7) would hold since there are no subsequential limits.

3. If (3·8) holds for $\alpha = \alpha_0$ then it holds for all $\alpha \geqslant \alpha_0$ by monotonicity of $\tau_n(\,.\,)$.

Proof. Assume (3·8) fails. Then by monotonicity of $\tau_n(\,.\,)$ we can find a subsequence n_j of n_i such that $\tau_{n_j}(j)\,\gamma_{n_j}^{-1} \to 0$. In particular, $\tau_{n_j}(\alpha)\,\gamma_{n_j}^{-1} \to 0$ for all α. Thus by (3·6), ${}^{(\tau_{n(j)})}S_{nj}\,\gamma_{n_j}^{-1} \xrightarrow{P} 0$, and so by the boundedness of $|\delta_{n_i}\gamma_{n_i}^{-1}|$, (3·7) fails.

Now assume (3·8) holds for some $\alpha = \alpha_0$ and by way of contradiction assume that (3·7) fails. Then by symmetry and the boundedness of $|\delta_{n_i}\gamma_{n_i}^{-1}|$, we can find a subsequence n_j of n_i such that ${}^{(\tau_{n(j)})}S_{n_j}\gamma_{n_j}^{-1} \xrightarrow{P} 0$. In particular ${}^{(\tau_{n(j)})}S_{n_j}\gamma_{n_j}^{-1}$ is tight. Let $\lambda > |\alpha_0|$. Then by (2·20), (3·2) and (3·8),

$$\liminf_{j\to\infty} \inf_{\alpha_0 \leqslant \alpha \leqslant \lambda} n_j H(u_{n_j}(\alpha))\, r_{n_j}^{-\frac{1}{2}} > 0. \tag{3·9}$$

Thus we can apply the Berry–Esseen theorem as in Theorem 3·2, case A. Hence for all x if j is sufficiently large

$$\begin{aligned} P(|{}^{(\tau_{n(j)})}S_{n_j}\gamma_{n_j}^{-1}| > x) &\geqslant \int_{\alpha_0}^{\lambda} P(|V_{m_j}(\alpha)|\,\gamma_{n_j}^{-1} > x)\,\mu_{n_j}(d\alpha) \\ &= \int_{\alpha_0}^{\lambda} P(|N|\,\tau_{n_j}(\alpha)\,v_{n_j}^{-\frac{1}{2}}(\alpha)\,\gamma_{n_j}^{-1} > x)\,\mu_{n_j}(d\alpha) + o(1) \\ &\geqslant \int_{\alpha_0}^{\lambda} P(|N| > x\gamma_{n_j}\tau_{n_j}^{-1}(\alpha_0))\,\mu_{n_j}(d\alpha) + o(1) \end{aligned}$$

since $\tau_{n_j}(\alpha) \geqslant \tau_{n_j}(\alpha_0)$ and $v_{n_j}(\alpha) \leqslant 1$. Now for large j, $\gamma_{n_j}\tau_{n_j}^{-1}(\alpha_0) \leqslant c < \infty$ by (3·8), and so for all $x > 0$

$$\liminf_{j\to\infty} P(|{}^{(\tau_{n(j)})}S_{n_j}\gamma_{n_j}^{-1}| > x) \geqslant P(|N| > cx)\,\mu([\alpha_0, \lambda])$$

which is a contradiction since ${}^{(\tau_{n(j)})}S_{n_j}\gamma_{n_j}^{-1} \xrightarrow{P} 0$. ∎

Combining the previous two Theorems we obtain necessary and sufficient conditions for stochastic compactness.

THEOREM 3·5. *There exist γ_n, δ_n such that $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1}$ is stochastically compact if and only if there exists an α_0 such that for all $\alpha \geqslant \alpha_0$*

$$\limsup_{i\to\infty} \tau_{n_i}(\alpha)\,\tau_{n_i}^{-1}(\alpha_0) < \infty. \tag{3·10}$$

In this case one may take $\gamma_{n_i} = \tau_{n_i}(\alpha_0)$ and $\delta_{n_i} = 0$.

Proof. Assume that $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1}$ is stochastically compact. Then by Theorem 3·2, $|\delta_{n_i}\gamma_{n_i}^{-1}|$ is bounded and so by (3·8) there exists an α_0 such that

$$\liminf_{i\to\infty} \tau_{n_i}(\alpha_0)\,\gamma_{n_i}^{-1} > 0.$$

Furthermore, by (3·2), for all $\alpha \geqslant \alpha_0$

$$\limsup_{i\to\infty} \tau_{n_i}(\alpha)\,\gamma_{n_i}^{-1} < \infty.$$

Hence (3·10) holds.

Now assume (3·10) holds and set $\gamma_{n_i} = \tau_{n_i}(\alpha_0)$ and $\delta_{n_i} = 0$. Then (3·2), (3·3) and

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(3·8) all hold, hence by Theorems 3·2 and 3·4, $({}^{(r_{n(i)})}S_{n_i}-\delta_{n_i})\gamma_{n_i}^{-1}$ is stochastically compact. ∎

Observe that (3·10) is a condition on truncated second moments, namely

$$\limsup_{i\to\infty}\frac{E[\tilde{G}(U)^2: U\geqslant u_{n_i}(\alpha)]}{E[\tilde{G}(U)^2: U\geqslant u_{n_i}(\alpha_0)]}<\infty. \tag{3·11}$$

In the case of a continuous distribution, this just becomes

$$\limsup_{i\to\infty}\frac{EX^2 1(|X|\leqslant a_{n_i}(\alpha))}{EX^2 1(|X|\leqslant a_{n_i}(\alpha_0))}<\infty. \tag{3·12}$$

Our next object is to obtain a characterization of the possible subsequential limits of $({}^{(r_n)}S_n-\delta_n)\gamma_n^{-1}$.

LEMMA 3·6. *Assume that* $({}^{(r_{n(i)})}S_{n_i}-\delta_{n_i})\gamma_{n_i}^{-1}$ *is tight; then every subsequence contains a further subsequence* n_j *such that* $\tau_{n_j}(\alpha)v_{n_j}^{-\frac{1}{2}}(\alpha)\gamma_{n_j}^{-1}\to\tau(\alpha)$ *uniformly on compact sets (u.c.) for some function* τ *such that*

$$\tau \text{ is continuous, non-negative, non-decreasing and } \tau^2 \text{ is convex.} \tag{3·13}$$

Proof. First observe that $v_{n_i}(\alpha)\to 1$ u.c., so it suffices to show that $\tau_{n_j}(\alpha)\gamma_{n_j}^{-1}\to\tau(\alpha)$ u.c. Fix $\lambda>0$, then for $-\lambda\leqslant\beta<\alpha\leqslant\lambda$ we have by (2·17) and (2·19), for sufficiently large i depending only on λ

$$\begin{aligned}(\tau_{n_i}^2(\alpha)-\tau_{n_i}^2(\beta))\gamma_{n_i}^{-2} &\leqslant m_i a_{n_i}^2(\alpha)(\alpha-\beta)r_{n_i}^{\frac{1}{2}}n_i^{-1}\gamma_{n_i}^{-2}\\ &= (\alpha-\beta)\tau_{n_i}^2(\alpha)\gamma_{n_i}^{-2}r_{n_i}^{\frac{1}{2}}(n_i H(u_{n_i}(\alpha)))^{-1}\\ &\leqslant (\alpha-\beta)\tau_{n_i}^2(\lambda+1)\gamma_{n_i}^{-2}\\ &\leqslant c(\alpha-\beta)\end{aligned}$$

by (3·2) where c depends only on λ. Thus $\tau_{n_i}(\alpha)\gamma_{n_i}^{-1}$ is an equicontinuous family of functions on $[-\lambda,\lambda]$ and by (3·2) it is pointwise bounded. Thus by the Arzela–Ascoli theorem, we can find a subsequence along which $\tau_{n_i}(\alpha)\gamma_{n_i}^{-1}\to\tau(\alpha)$ uniformly on $[-\lambda,\lambda]$ for some continuous function τ, which is necessarily non-negative and non-decreasing since τ_{n_i} is. To see that τ^2 is convex, observe that by (2·17) and (2·18) if $-\lambda\leqslant\gamma<\beta<\alpha\leqslant\lambda$ and i is sufficiently large then

$$(\tau_{n_i}^2(\beta)-\tau_{n_i}^2(\gamma))(\beta-\gamma)^{-1}\leqslant(\tau_{n_i}^2(\alpha)-\tau_{n_i}^2(\beta))(\alpha-\beta)^{-1}.$$

Thus $\tau_{n_i}^2$ is convex on $[-\lambda,\lambda]$ and consequently so is τ^2. Since this can be done for any $\lambda>0$, the result now follows by a standard diagonal argument. ∎

Remark. As was pointed out above, $v_n(\alpha)\to 1$ u.c., thus

$$\tau_{n_j}(\alpha)v_{n_j}^{-\frac{1}{2}}(\alpha)\gamma_{n_j}^{-1}\to\tau(\alpha)\text{ u.c. iff }\tau_{n_j}(\alpha)\gamma_{n_j}^{-1}\to\tau(\alpha)\text{ u.c.} \tag{3·14}$$

This will be used several times in the upcoming proof.

THEOREM 3·7. *If* $({}^{(r_{n(i)})}S_{n_i}-\delta_{n_i})\gamma_{n_i}^{-1}\to W$ *then there exists a unique function* τ *satisfying* (3·13) *and a unique* $\delta\in\mathbb{R}$ *such that*

$$E\exp(iuW)=\exp(-iu\delta)\int_{-\infty}^{\infty}\exp(-u^2\tau^2(\alpha)/2)\,\mu(d\alpha), \tag{3·15}$$

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where μ is the standard normal distribution. Furthermore, the two conditions

$$\tau_{n_i}(\alpha)\,\gamma_{n_i}^{-1} \to \tau(\alpha) \quad \textit{for all} \quad \alpha, \tag{3·16}$$

$$\delta_{n_i}\,\gamma_{n_i}^{-1} \to \delta \tag{3·17}$$

are necessary and sufficient for $({}^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1} \to W$, where W is given by (3·15).

Proof. We first prove uniqueness of the representation for W. Thus let δ_1, τ_1 and δ_2, τ_2 satisfy (3·15). Then for all u

$$\exp(-iu(\delta_1-\delta_2)) = \left(\int_{-\infty}^{\infty} \exp(-u^2\tau_2^2(\alpha)/2)\,\mu(d\alpha)\right)\left(\int_{-\infty}^{\infty} \exp(-u^2\tau_1^2(\alpha)/2)\,\mu(d\alpha)\right)^{-1}.$$

Since the RHS is real, we must have $\delta_1 = \delta_2$. Now define the measures ν_1 and ν_2 on $[0, \infty)$ by $\nu_1(A) = \mu\{\alpha : \tau_1^2(\alpha) \in A\}$ and $\nu_2(A) = \mu\{\alpha : \tau_2^2(\alpha) \in A\}$. Then for all $u \in \mathbb{R}$

$$\int_0^{\infty} \exp(-u^2x/2)\,\nu_1(dx) = \int_0^{\infty} \exp(-u^2x/2)\,\nu_2(dx).$$

Thus by the uniqueness of Laplace transforms $\nu_1 = \nu_2$. This in turn implies $\tau_1(\alpha) = \tau_2(\alpha)$ for all α by the regularity conditions on τ_1 and τ_2.

Now assume $({}^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1} \to W$. We will prove the existence of δ and τ satisfying (3·15)–(3·17). First observe that by Lemmas 3·1 and 3·6, given any subsequence of n_i, there exists a further subsequence n_j such that $\delta_{n_j}\gamma_{n_j}^{-1} \to \delta$ and $\tau_{n_j}(\alpha)\,v_{n_j}^{-\frac{1}{2}}(\alpha)\,\gamma_{n_j}^{-1} \to \tau(\alpha)$ u.c. for some δ and some τ satisfying (3·13). Thus if we can prove (3·15) for this δ and τ then by Lemma 3·6, uniqueness and (3·14) we will have proved (3·16) and (3·17). Let

$$\beta = \sup\{\alpha : \tau(\alpha) = 0\}.$$

Let $\lambda > 0$ and for convenience let $\lambda > |\beta|$ if $-\infty < \beta < \infty$. If $|\beta| < \infty$, let $\epsilon \in (0, \lambda - |\beta|)$ and write

$$\begin{aligned}\int_{-\lambda}^{\lambda} P((V_{m_j}(\alpha)-\delta_{n_j})\,\gamma_{n_j}^{-1} > x)\,\mu_{n_j}(d\alpha) &= \int_{-\lambda}^{\beta-\epsilon} + \int_{\beta-\epsilon}^{\beta+\epsilon} + \int_{\beta+\epsilon}^{\lambda} \\ &= I_1 + I_2 + I_3.\end{aligned}$$

If $\beta = \infty$ or $\beta = -\infty$ then, as will become clear, there is no need to break up the integral. In the former case the integral from $-\lambda$ to λ is handled in the same way as I_1, while in the latter case it is handled in the same way as I_3. We consider I_3 first. Using (3·14), observe that $\tau_{n_j}(\beta+\epsilon)\,\tau_{n_j}^{-1}(\lambda+1) \to \tau(\beta+\epsilon)\,\tau^{-1}(\lambda+1)$. Thus by (2·20),

$$\liminf_{j\to\infty} \inf_{\beta+\epsilon \leqslant \alpha \leqslant \lambda} n_j H(u_{n_j}(\alpha))\,r_{n_j}^{-\frac{1}{2}} > 0.$$

Hence we can apply the Berry–Esseen theorem as before to obtain

$$\begin{aligned}I_3 &= \int_{\beta+\epsilon}^{\lambda} P(N\tau_{n_j}(\alpha)\,v_{n_j}^{-\frac{1}{2}}(\alpha)\,\gamma_{n_j}^{-1} - \delta_{n_j}\gamma_{n_j}^{-1} > x)\,\mu_{n_j}(d\alpha) + o(1) \\ &= I_4 + o(1).\end{aligned}$$

Now for any $\eta > 0$, if j is sufficiently large

$$\int_{\beta+\epsilon}^{\lambda} P(N\tau(\alpha)-\delta > x+\eta)\,\mu_{n_j}(d\alpha) \leqslant I_4 \leqslant \int_{\beta+\epsilon}^{\lambda} P(N\tau(\alpha)-\delta > x-\eta)\,\mu_{n_j}(d\alpha)$$

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by uniform convergence. Thus by weak convergence and then by bounded convergence, if we let $j \to \infty$ and then let $\eta \downarrow 0$ we obtain

$$I_4 \to \int_{\beta+\epsilon}^{\lambda} P(N\tau(\alpha) - \delta > x)\,\mu(d\alpha).$$

Next if $-\lambda \leqslant \alpha \leqslant \beta - \epsilon$ and j is sufficiently large, then for any $\eta > 0$

$$\begin{aligned} P(|V_{m_j}(\alpha)| > \eta\gamma_{n_j}) &\leqslant EV_{m_j}^2(\alpha)\,(\eta\gamma_{n_j})^{-2} \\ &\leqslant \tau_{n_j}^2(\beta-\epsilon)\,v_{n_j}^{-1}(-\lambda)\,(\eta\gamma_{n_j})^{-2} \\ &\to \tau^2(\beta-\epsilon)\,\eta^{-2} = 0 \end{aligned}$$

uniformly in α. From this it is easy to see that

$$I_1 \to \begin{cases} \mu([-\lambda, \beta-\epsilon]) & \text{if} \quad x < -\delta \\ 0 & \text{if} \quad x > -\delta \end{cases}$$

$$= \int_{-\lambda}^{\beta-\epsilon} P(N\tau(\alpha) - \delta > x)\,\mu(d\alpha)$$

if $x \neq -\delta$. Finally, we clearly have

$$\limsup_{j\to\infty} I_2 \leqslant \mu([\beta-\epsilon, \beta+\epsilon]).$$

Thus letting $j \to \infty$ and then $\epsilon \downarrow 0$, we obtain

$$I_1 + I_2 + I_3 \to \int_{-\lambda}^{\lambda} P(N\tau(\alpha) - \delta > x)\,\mu(d\alpha)$$

provided $x \neq -\delta$. Hence by (2·22) upon letting $\lambda \to \infty$ we obtain for $x \neq -\delta$ and x a continuity point of W

$$P(W > x) = \int_{-\infty}^{\infty} P(N\tau(\alpha) - \delta > x)\,\mu(d\alpha). \tag{3·18}$$

From this it immediately follows that (3·18) holds for all x, and so $W \overset{d}{=} N\tau(N_1) - \delta$ where $N_1 \overset{d}{=} N$ and N, N_1 are independent. Upon taking characteristic functions, we then obtain (3·15).

Now assume (3·16) and (3·17) hold for some δ and τ. We will show that τ necessarily satisfies (3·13), and that $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1} \to W$ where W is given by (3·15). First observe that by Theorem 3·2, $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1}$ is tight and so every subsequence of n_i contains a further subsequence n_j such that $({}^{(\tau_{n(j)})}S_{n_j} - \delta_{n_j})\,\gamma_{n_j}^{-1} \to W$ for some W. Thus, by what has already been proved, for some δ_1 and some σ satisfying (3·13)

$$E\exp(iuW) = \exp(-iu\delta_1) \int_{-\infty}^{\infty} \exp(-u^2\sigma^2(\alpha)/2)\,\mu(d\alpha).$$

Furthermore, we have also shown that for $({}^{(\tau_{n(j)})}S_{n_j} - \delta_{n_j})\,\gamma_{n_j}^{-1} \to W$, it is necessary that

$$\tau_{n_j}(\alpha)\,\gamma_{n_j}^{-1} \to \sigma(\alpha), \quad \delta_{n_j}\gamma_{n_j}^{-1} \to \delta_1.$$

Thus by (3·16) and (3·17), $\sigma = \tau$ and $\delta_1 = \delta$. Hence all subsequential limits of $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1}$ must be the same and equal to W. Since $({}^{(\tau_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma_{n_i}^{-1}$ is tight, this forces convergence to W along n_i. ∎

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A random variable W, with characteristic function of the form (3·15), is usually referred to as a variance mixture of normals. To complete the characterization of the subsequential limit laws, we will show that every random variable W of the form (3·15) for some τ satisfying (3·13), does arise as the limit of a normalized trimmed sum along some subsequence. This will be a consequence of the following result which proves, in Doeblin's terminology, the existence of universal laws. First let

$$\mathscr{C} = \{h : h \text{ is non-negative, non-decreasing and convex}\},$$

$$\mathscr{L} = \{W : E\exp(iuW) = \int_{-\infty}^{\infty} \exp(-u^2 h(\alpha)/2)\,\mu(d\alpha) \quad \text{for some } h \in \mathscr{C}\}.$$

Observe that if $h \in \mathscr{C}$ then $h^{\frac{1}{2}}$ satisfies (3·13).

THEOREM 3·8. *Assume that $r_n \to \infty$ and $r_n n^{-1} \to 0$; then there exists a symmetric random variable X and a normalizing sequence γ_n such that the class of subsequential limit laws of ${}^{(r_n)}S_n \gamma_n^{-1}$ is exactly $\mathscr{L}$.*

Proof. Let $-\infty < \alpha_1 < \ldots < \alpha_j < \infty$ and $0 < \lambda_1 < \ldots < \lambda_j < \infty$ be rational numbers. Set

$$h(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha \leqslant \alpha_1 \\ \sum_{p=1}^{i-1} \lambda_p(\alpha_{p+1} - \alpha_p) + \lambda_i(\alpha - \alpha_i) & \text{if} \quad \alpha_i \leqslant \alpha \leqslant \alpha_{i+1}, 1 \leqslant i \leqslant j-1 \\ \sum_{p=1}^{j-1} \lambda_p(\alpha_{p+1} - \alpha_p) + \lambda_j(\alpha - \alpha_j) & \text{if} \quad \alpha \geqslant \alpha_j. \end{cases} \tag{3·19}$$

Thus the graph of h is a polygonal line passing through $(\alpha_i, \sum_{p=1}^{i-1} \lambda_p(\alpha_{p+1} - \alpha_p))$ for $1 \leqslant i \leqslant j$, with slope 0 if $\alpha < \alpha_1$ and slope λ_j if $\alpha > \alpha_j$.

Let $\mathscr{P}$ be the collection of all functions of the form (3·19) for some j and some $-\infty < \alpha_1 < \ldots < \alpha_j < \infty, 0 < \lambda_1 < \ldots < \lambda_j$ rational numbers. Note that $\mathscr{P}$ is countable. Furthermore $\mathscr{P} \subseteq \mathscr{C}$ and $\mathscr{P}$ is dense in $\mathscr{C}$ in the topology of uniform convergence on compact sets.

Let $h_1, h_2, \ldots$ be an enumeration of functions in $\mathscr{P}$ such that every function occurs infinitely often. Since a symmetric distribution is determined by its absolute value, it suffices to construct $G(x) = P(|X| > x)$ for all $x \geqslant 0$. This will be done inductively; at the kth stage we will define $G(x)$ on an interval $[z_{k-1}, z_k)$ where $z_k \to \infty$. At the same time we will define an increasing sequence of integers n_k and define the normalizer γ_n for $n_{k-1} < n \leqslant n_k$.

We begin by defining $G(x) = 1 - x/2$ for $0 \leqslant x < 1$ and setting $z_0 = 1$, $n_0 = 1$ and $\gamma_1 = 1$. Observe that G is right continuous and non-increasing on $[0, z_0)$, $G(0) = 1$ and $G(z_0 -) > 0$. Now assume G has been defined for $0 \leqslant x < z_{k-1}$ so that G is right continuous and non-increasing on $[0, z_{k-1})$, $G(0) = 1$ and $G(z_{k-1} -) > 0$. Also assume that $n_0 < \ldots < n_{k-1}$ and γ_n has been defined for $n \leqslant n_{k-1}$. We will show how to determine z_k and n_k, how to define G on $[z_{k-1}, z_k)$, and how to define γ_n for $n_{k-1} < n \leqslant n_k$. Assume that h_k is given by (3·19). Choose $n_k > n_{k-1}$ large enough that

$$u_{n_k}(\alpha_1) < G(z_{k-1} -), \tag{3·20}$$

$$\alpha_j < r_{n_k}^{\frac{1}{4}}, \tag{3·21}$$

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$$u_{n_k}(r_{n_k}^{\frac{1}{4}}) > 0. \tag{3·22}$$

First define
$$G(z_{k-1}) = u_{n_k}(\alpha_1). \tag{3·23}$$

Then $H(u_{n_k}(\alpha_1))$ is well defined. Let

$$\nu_k = r_{n_k}^{\frac{1}{2}}(n_k H(u_{n_k}(\alpha_1)))^{-1} \tag{3·24}$$

and set
$$\eta_k = k(\nu_k^{-1} \vee 2\lambda_1^{-1}). \tag{3·25}$$

Define
$$x_{k,1} = z_{k-1}; \quad x_{k,i} = (\eta_k \lambda_{i-1})^{\frac{1}{2}} x_{k,1}, \quad 2 \leqslant i \leqslant j+1, \tag{3·26}$$

$$z_k = x_{k,j+1} + 1. \tag{3·27}$$

Observe that $z_{k-1} = x_{k,1} < \ldots < x_{k,j+1} < x_{k,j+1} + 1 = z_k$ and $z_k z_{k-1}^{-1} > (\eta_k \lambda_j)^{\frac{1}{2}} \to \infty$ as $k \to \infty$. Now define

$$G(x) = \begin{cases} u_{n_k}(\alpha_i) & \text{if} \quad x_{k,i} \leqslant x < x_{k,i+1}, \quad 1 \leqslant i \leqslant j, \\ u_{n_k}(r_{n_k}^{\frac{1}{4}}) & \text{if} \quad x_{k,j+1} \leqslant x < z_k. \end{cases} \tag{3·28}$$

This completes the construction of G on $[z_{k-1}, z_k)$. The normalizer γ_n for $n_{k-1} < n \leqslant n_k$ is defined by
$$\gamma_n = x_{k,1} r_{n_k}^{\frac{1}{4}} \eta_k^{\frac{1}{2}}. \tag{3·29}$$

Observe that by construction, (3·20) and (3·21), G is non-increasing on $[0, z_k)$. By (3·22), $G(z_k -) > 0$ and clearly G is right continuous with $G(0) = 1$. Thus the induction procedure can continue. Let G be the function defined this way. Observe that $G(x) \to 0$ since G is non-increasing and $G(x_{k,j+1}) = u_{n_k}(r_{n_k}^{\frac{1}{4}}) < r_{n_k} n_k^{-1} \to 0$. Thus we can define a symmetric random variable X by $P(|X| > x) = G(x)$.

We will now show that for $\alpha \leqslant r_{n_k}^{\frac{1}{4}}$

$$0 \leqslant \tau_{n_k}^2(\alpha) \gamma_{n_k}^{-2} - m_k n_k^{-1} h_k(\alpha) \leqslant k^{-1}. \tag{3·30}$$

First observe that

$$a_{n_k}(\alpha) = \begin{cases} x_{k,1} & \text{if} \quad \alpha = \alpha_1 \\ x_{k,i+1} & \text{if} \quad \alpha_i < \alpha \leqslant \alpha_{i+1}, \quad 1 \leqslant i \leqslant j-1 \\ x_{k,j+1} & \text{if} \quad \alpha_j < \alpha \leqslant r_{n_k}^{\frac{1}{4}}. \end{cases}$$

Let $1 \leqslant i \leqslant j-1$ and $\alpha_i < \alpha \leqslant \alpha_{i+1}$; then

$$\begin{aligned} (\tau_{n_k}^2(\alpha) - \tau_{n_k}^2(\alpha_i)) \gamma_{n_k}^{-2} &= m_k E[\tilde{G}(U)^2; u_n(\alpha) \leqslant U < u_n(\alpha_i)] \gamma_{n_k}^{-2} \\ &= m_k x_{k,i+1}^2 r_{n_k}^{\frac{1}{2}} n_k^{-1} (\alpha - \alpha_i) \gamma_{n_k}^{-2} \\ &= m_k n_k^{-1} \lambda_i (\alpha - \alpha_i). \end{aligned} \tag{3·31}$$

Similarly for $\alpha_j < \alpha \leqslant r_{n_k}^{\frac{1}{4}}$

$$(\tau_{n_k}^2(\alpha) - \tau_{n_k}^2(\alpha_j)) \gamma_{n_k}^{-2} = m_k n_k^{-1} \lambda_j (\alpha - \alpha_j). \tag{3·32}$$

Thus for $\alpha_1 < \alpha \leqslant r_{n_k}^{\frac{1}{4}}$

$$(\tau_{n_k}^2(\alpha) - \tau_{n_k}^2(\alpha_1)) \gamma_{n_k}^{-2} = m_k n_k^{-1} h_k(\alpha). \tag{3·33}$$

For $\alpha \leqslant \alpha_1$ we have

$$\begin{aligned} \tau_{n_k}^2(\alpha) \gamma_{n_k}^{-2} &\leqslant \tau_{n_k}^2(\alpha_1) \gamma_{n_k}^{-2} \\ &= m_k a_{n_k}^2(\alpha_1) H(u_{n_k}(\alpha_1)) \gamma_{n_k}^{-2} \\ &= m_k H(u_{n_k}(\alpha_1)) (r_{n_k}^{\frac{1}{2}} \eta_k)^{-1} \\ &\leqslant m_k n_k^{-1} k^{-1} \end{aligned} \tag{3·34}$$

by (3·24) and (3·25). Since $m_k n_k^{-1} \leqslant 1$, (3·33) and (3·34) together imply (3·30).

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To complete the proof, let $h \in \mathscr{C}$ and set $\tau = h^{\frac{1}{2}}$. Let $n_{k(i)}$ be a subsequence of n_k such that $h_{k(i)}(\alpha) \to h(\alpha)$ for all α. Then by (3·30)

$$0 \leqslant \tau^2_{n_{k(i)}}(\alpha)\,\gamma^{-2}_{n_{k(i)}} - m_{k(i)}\,n^{-1}_{k(i)}\,h_{k(i)}(\alpha) \leqslant k(i)^{-1} \tag{3·35}$$

for $\alpha \leqslant r^{\frac{1}{4}}_{n_{k(i)}}$. Thus $\tau_{n_{k(i)}}(\alpha)\,\gamma^{-1}_{n_{k(i)}} \to \tau(\alpha)$ and so by Theorem 3·7 along $n_{k(i)}$

$$^{(r_n)}S_n\,\gamma_n^{-1} \to W, \tag{3·36}$$

where

$$E\exp(iuW) = \int_{-\infty}^{\infty} \exp(-u^2\tau^2(\alpha)/2)\,\mu(d\alpha). \tag{3·37}$$

Thus the class of subsequential limit laws of $^{(r_n)}S_n\,\gamma_n^{-1}$ contains $\mathscr{L}$ and by Theorem 3·7 it is necessarily contained in $\mathscr{L}$. ∎

Remarks. 1. As an immediate consequence of Theorems 3·7 and 3·8 we obtain the characterization of the subsequential limit laws given in the introduction.

2. By a slight modification of the construction, the universal law constructed in Theorem 3·8 can be made continuous.

3. It is natural to ask whether the class of limit laws along the entire sequence is smaller than the class of subsequential limit laws. The answer to this question depends on how it is phrased. For example, there exist sequences r_n satisfying (1·1) such that for any $W \in \mathscr{L}$ there exists a symmetric distribution X such that $^{(r_n)}S_n\,\gamma_n^{-1} \to W$ for some γ_n. On the other hand, given a sequence r_n satisfying (1·1), it may be the case that the class of limit laws of $^{(r_n)}S_n\,\gamma_n^{-1}$ along the entire sequence is smaller than $\mathscr{L}$. Thus to obtain a theory analogous to that of the stable laws, it is necessary to place some restriction on r_n. A natural choice is that r_n be non-decreasing. It would be interesting to determine the class of limit laws along the entire sequence in this case.

Given a random variable W which can arise as a subsequential limit, and the corresponding δ and τ, it is a simple matter to rephrase Theorem 3·7 so as to give necessary and sufficient conditions on the underlying distribution X, for the existence of γ_n, δ_n such that $(^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma^{-1}_{n_i} \to W$. If W is degenerate, at δ say, then for any distribution one can always choose γ_n so that $^{(r_n)}S_n\,\gamma_n^{-1} \xrightarrow{P} 0$ and then choose $\delta_n = -\delta\gamma_n$. Thus we will assume that W is non-degenerate. In particular, this means that τ is not identically zero.

THEOREM 3·9. *Let W be a non-degenerate random variable of the form* (3·15) *for some δ and τ. Then there exist γ_n, δ_n such that $(^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma^{-1}_{n_i} \to W$ if and only if, for some (all) α_0 such that $\tau(\alpha_0) > 0$,*

$$\tau_{n_i}(\alpha)\,\tau^{-1}_{n_i}(\alpha_0) \to \tau(\alpha)\,\tau^{-1}(\alpha_0) \quad \textit{for all} \quad \alpha. \tag{3·38}$$

In this case one may take $\gamma_{n_i} = \tau_{n_i}(\alpha_0)\,\tau^{-1}(\alpha_0)$ and $\delta_{n_i} = \delta\gamma_{n_i}$.

Proof. Assume (3·38) holds for some α_0 and let $\gamma_{n_i} = \tau_{n_i}(\alpha_0)\,\tau^{-1}(\alpha_0)$ and $\delta_{n_i} = \delta\gamma_{n_i}$. Then (3·16) and (3·17) hold, and hence $(^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma^{-1}_{n_i} \to W$.

Now assume $(^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\,\gamma^{-1}_{n_i} \to W$ and α_0 is such that $\tau(\alpha_0) > 0$. Then by (3·16)

$$\tau_{n_i}(\alpha)\,\tau^{-1}_{n_i}(\alpha_0) = \tau_{n_i}(\alpha)\,\gamma^{-1}_{n_i}\,\tau^{-1}_{n_i}(\alpha_0)\,\gamma_{n_i} \to \tau(\alpha)\,\tau^{-1}(\alpha_0). \quad ∎ \tag{3·39}$$

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The preceding theorem now leads to necessary and sufficient conditions for asymptotic normality.

THEOREM 3·10. *There exist* γ_n, δ_n *such that* $({}^{(r_{n(i)})}S_{n_i} - \delta_{n_i})\gamma_{n_i}^{-1} \to N$ *if and only if*

$$\lim_{i\to\infty} \tau_{n_i}(\alpha)\tau_{n_i}^{-1}(0) = 1 \quad \textit{for all} \quad \alpha. \tag{3·40}$$

In this case one may take $\gamma_{n_i} = \tau_{n_i}(0)$ *and* $\delta_{n_i} = 0$.

Proof. Set $W = N$, then by uniqueness $\delta = 0$ and $\tau(\alpha) \equiv 1$. The result now follows immediately from Theorem 3·9. ∎

A similar remark to the one made after Theorem 3·5 applies here, namely that (3·40) is just a condition on truncated second moments. For a general distribution (3·40) is equivalent to

$$\lim_{i\to\infty} \frac{E[\tilde{G}(U)^2 : U \geqslant u_{n_i}(\alpha)]}{E[\tilde{G}(U)^2 : U \geqslant u_{n_i}(0)]} = 1 \quad \text{for all} \quad \alpha;$$

which in the case of continuous distributions reduces to

$$\lim_{i\to\infty} \frac{EX^2 1(|X| \leqslant a_{n_i}(\alpha))}{EX^2 1(|X| \leqslant a_{ni}(0))} = 1 \quad \text{for all} \quad \alpha.$$

Condition (3·40) can be reformulated in several different ways. We give one below which will be used to check that distributions in the Feller class satisfy Theorem 3·10.

LEMMA 3·11. *Condition* (3·40) *is equivalent to*

$$\lim_{i\to\infty} n_i H(u_{n_i}(\beta)) r_{n_i}^{-\frac{1}{2}} = \infty \quad \textit{for all} \quad \beta. \tag{3·41}$$

Proof. Assume (3·40); then $\tau_{n_i}(\alpha)\tau_{n_i}^{-1}(\beta) \to 1$ for all $\alpha > \beta$. Thus (3·41) follows from (2·16). Now assume (3·41) and let $\alpha > \beta$. Then $\tau_{n_i}(\alpha) \geqslant \tau_{n_i}(\beta)$, while by (2·17)

$$\begin{aligned} 1 &\leqslant \tau_{n_i}^2(\beta)\tau_{n_i}^{-2}(\alpha) + (\alpha-\beta) r_{n_i}^{\frac{1}{2}} (n_i H(u_{n_i}(\alpha)))^{-1} \\ &\sim \tau_{n_i}^2(\beta)\tau_{n_i}^{-2}(\alpha). \end{aligned}$$

Thus $\tau_{n_i}(\alpha)\tau_{n_i}^{-1}(\beta) \to 1$, from which (3·40) follows. ∎

As we mentioned in the introduction, Pruitt [11] has shown that under (1·1), if F is continuous and in the Feller class, then the trimmed sum can be normalized to converge to the normal. It is a simple consequence of (3·41) that this holds for discontinuous distributions also.

COROLLARY 3·12. *Assume that* X *is in the Feller class and* (1·1) *holds; then there exists* γ_n *such that* ${}^{(r_n)}S_n \gamma_n^{-1} \to N$.

Proof. A necessary and sufficient condition for X to be in the Feller class, as given by Feller [4], is

$$\liminf_{x\to\infty} \frac{K(x)}{G(x)} > 0. \tag{3·42}$$

Now by (2·3) and (2·5) for $0 < u < G(0)$

$$\frac{H(u)}{u} \geqslant \frac{K(\tilde{G}(u)-)}{G(\tilde{G}(u)-)}.$$

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Thus if $\bar{G}(u) \to \infty$ as $u \downarrow 0$, then by (3·42)

$$\liminf_{u \downarrow 0} \frac{H(u)}{u} > 0. \tag{3·43}$$

If $\bar{G}(u)$ remains bounded as $u \downarrow 0$, then by (2·4) $H(u)\, u^{-1} \to \infty$. Thus in either case (3·43) holds. Hence for any β, if n is sufficiently large

$$\begin{aligned} nH(u_n(\beta))\, r_n^{-\frac{1}{2}} &\geqslant cnu_n(\beta)\, r_n^{-\frac{1}{2}} \\ &\sim cr_n\, r_n^{-\frac{1}{2}} \\ &\to \infty. \end{aligned}$$

Thus ${}^{(r_n)}S_n\, \tau_n^{-1}(0) \to N$ by Theorem 3·10 and Lemma 3·11. ∎

We conclude this section by pointing out the modifications needed to deal with the case $r_n\, n^{-1} \nrightarrow 0$. The restriction we must impose on r_n to ensure that we do not trim too many terms is $\limsup r_n\, n^{-1} < G(0)$. Then essentially the only difference is that $u_n(\alpha)$ instead of being given by (2·9), is defined by

$$u_n(\alpha) = ((r_n - \alpha(r_n(1 - r_n\, n^{-1}))^{\frac{1}{2}})\, n^{-1} \vee 0) \wedge 1.$$

This change is needed in order that $\mu_n \to \mu$ in Lemma 2·4. The proofs then go through almost exactly as before and the results remain valid as stated except for: Theorem 3·7 where (3·16) should be replaced with $\tau_{n_i}(\alpha)\, v_{n_i}^{-\frac{1}{2}}(\alpha)\, \gamma_{n_i}^{-1} \to \tau(\alpha)$; Theorem 3·8 which fails if $\liminf r_n\, n^{-1} > 0$, see Theorem 3·13 below; Theorem 3·9 where one should take $\gamma_{n_i} = \tau_{n_i}(\alpha_0)\, v_{n_i}^{-\frac{1}{2}}(\alpha_0)\, \tau^{-1}(\alpha_0)$; Theorem 3·10 where one should take $\gamma_{n_i} = \tau_{n_i}(0)\, v_{n_i}^{-\frac{1}{2}}(0)$. This leads to

THEOREM 3·13. *For any symmetric distribution X, if*

$$0 < \liminf_{n \to \infty} r_n\, n^{-1} \leqslant \limsup_{n \to \infty} r_n\, n^{-1} < G(0), \tag{3·44}$$

then ${}^{(r_n)}S_n\, \gamma_n^{-1} \to N$ where $\gamma_n = \tau_n(0)\, v_n^{-\frac{1}{2}}(0)$.

Proof. By the above remarks it suffices to show that (3·40), or equivalently (3·41), is a consequence of (3·44). But by (3·44), there exist u_0, u_1 such that for any β if n is sufficiently large

$$0 < u_0 < u_n(\beta) < u_1 < G(0).$$

Hence for large n, by (2·4),

$$\begin{aligned} H(u_n(\beta)) &\geqslant (\bar{G}(u_1)\, \bar{G}(u_n(\beta))^{-1})^2\, H(u_1) \\ &\geqslant (\bar{G}(u_1)\, \bar{G}(u_0)^{-1})^2\, H(u_1) \\ &> 0. \end{aligned}$$

Thus (3·41) holds and the result follows. ∎

Theorem 3·13 should be contrasted with a result of Stigler [**12**] which considers the trimmed sum formed by removing the upper $[\lambda_1 n]$ and lower $[\lambda_2 n]$ order statistics of $X_1, \ldots, X_n$. In this case asymptotic normality may fail. This contrast has also been noted independently, for continuous distributions, by Maller [**9**] who considers trimming a fixed proportion of the sample in $\mathbb{R}^d$.

4. *Examples*

As we have just seen, every distribution in the Feller class satisfies (3·40) with $n_i = i$. However the class of distributions to which Theorem 3·10 applies is much

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larger. Indeed it follows from Theorem 3·13 that for any symmetric distribution there exists a sequence $r_n \to \infty$ such that $r_n n^{-1} \to 0$ and ${}^{(r_n)}S_n \gamma_n^{-1} \to N$ for some γ_n. Of course, in general, the fatter the tail of the underlying distribution the more rapidly r_n must increase. We will now give an example of a distribution which is not in the Feller class and investigate the asymptotic distribution of the trimmed sum.

Example 4·1. Let X be symmetric with density $p(x) = (2|x|\,(\log|x|)^2)^{-1}$ for $|x| \geqslant e$. Then $G(x) = (\log x)^{-1}$ for $x \geqslant e$ and so $a_n(\alpha) = \exp(u_n^{-1}(\alpha))$. Next observe that

$$EX^2 1(|X| \leqslant x) = \int_e^x y(\log y)^{-2}\,dy \sim x^2/2(\log x)^2.$$

Since the distribution of X is continuous,

$$\begin{aligned}\tau_n^2(\alpha) &= mEX^2 1(|X| \leqslant a_n(\alpha)) \\ &\sim 2^{-1} n a_n^2(\alpha)\,(\log a_n(\alpha))^{-2} \\ &= 2^{-1} n \exp(2u_n^{-1}(\alpha))\,u_n^2(\alpha).\end{aligned}$$

A simple computation now shows that for any α, β

$$\begin{aligned}\tau_n(\alpha)\,\tau_n^{-1}(\beta) &\sim \exp(u_n^{-1}(\alpha) - u_n^{-1}(\beta)) \\ &= \exp((\alpha-\beta)\,n r_n^{\frac{1}{2}}((r_n - \alpha r_n^{\frac{1}{2}})\,(r_n - \beta r_n^{\frac{1}{2}}))^{-1}).\end{aligned}$$

Thus if $n r_n^{-\frac{3}{2}} \to \lambda, 0 \leqslant \lambda \leqslant \infty$ then

$$\tau_n(\alpha)\,\tau_n^{-1}(\beta) \to \exp(\lambda(\alpha-\beta)). \tag{4·1}$$

If we now apply Theorem 3·9 we see that if $\lambda < \infty$, then

$${}^{(r_n)}S_n\,\tau_n^{-1}(0) \to W, \tag{4·2}$$

where W is given by (3·15) with $\delta = 0$ and $\tau(\alpha) = \exp(\lambda\alpha)$. In particular, the limit is normal when $\lambda = 0$, i.e. when $n r_n^{-\frac{3}{2}} \to 0$, and non-normal when $0 < \lambda < \infty$. If $\lambda = \infty$ in (4·1), then (3·10) fails, so ${}^{(r_n)}S_n$ can not even be normalized to be stochastically compact.

We will now give an example to show that for any sequence $r_n \to \infty$ such that $r_n n^{-1} \to 0$, there exists a random variable X which is 'almost' symmetric, more precisely which is in the domain of normal attraction of a symmetric stable law, and for which $({}^{(r_n)}S_n - \delta_n)\gamma_n^{-1}$ is not asymptotically normal for any choice of γ_n and δ_n. This might be somewhat surprising for two reasons. First, the distribution of X is very close to the limiting symmetric stable distribution which, being in the Feller class, gives rise to trimmed sums which are asymptotically normal. Second, Csörgő, Horvath and Mason [3] have shown that the C.L.T. for $S_n(r_n, r_n)$ holds for all distributions (not necessarily symmetric ones) in the domain of attraction of a stable law.

Example 4·2. Fix $0 < p < 2$ and let c_i, d_i be any two sequences tending to infinity such that $c_1 = 1$ and

$$c_i < d_i < c_{i+1}. \tag{4·3}$$

Set $u_i = \sum_{j=i}^{\infty}(c_j^{-p} - d_j^{-p})$ and assume that

$$u_i = o(c_i^{-p}). \tag{4·4}$$

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Observe that this forces $c_i \sim d_i$. Now let X be a random variable with density

$$f(x) = \begin{cases} px^{-1-p} & c_i \leqslant x \leqslant d_i \qquad i = 1, 2, \ldots \\ 2^{-1}p|x|^{-1-p} & d_i < |x| < c_{i+1} \qquad i = 1, 2, \ldots \\ 0 & -d_i \leqslant x \leqslant -c_i \quad \text{or} \quad |x| < 1 \qquad i = 1, 2, \ldots \end{cases} \tag{4·5}$$

We claim that X is in the domain of normal attraction of a symmetric stable law of index p. By [5], p. 581, this is equivalent to showing

$$\lim_{x \to \infty} P(X > x)/P(X < -x) = 1, \tag{4·6}$$

$$\lim_{x \to \infty} x^p G(x) = C \quad \text{for some} \quad C > 0. \tag{4·7}$$

A computation shows that for $x \geqslant 1$

$$2P(X > x) = \begin{cases} 2x^{-p} - d_i^{-p} + u_{i+1} & c_i \leqslant x \leqslant d_i \\ x^{-p} + u_{i+1} & d_i \leqslant x \leqslant c_{i+1}, \end{cases}$$

$$2P(X < -x) = \begin{cases} d_i^{-p} - u_{i+1} & c_i \leqslant x \leqslant d_i \\ x^{-p} - u_{i+1} & d_i \leqslant x \leqslant c_{i+1}. \end{cases}$$

Thus for all $x \geqslant 1$

$$G(x) = x^{-p}, \tag{4·8}$$

which proves (4·7). If $c_i \leqslant x \leqslant d_i$

$$1 \leqslant \frac{P(X > x)}{P(X < -x)} = \frac{2x^{-p} - d_i^{-p} + u_{i+1}}{d_i^{-p} - u_{i+1}} \leqslant \frac{c_i^{-p} + u_i}{d_i^{-p} - u_{i+1}} \to 1$$

by (4·4), since $c_i \sim d_i$. Similarly for $d_i \leqslant x \leqslant c_{i+1}$

$$1 \leqslant \frac{P(X > x)}{P(X < -x)} = \frac{x^{-p} + u_{i+1}}{x^{-p} - u_{i+1}} = \frac{1 + u_{i+1}x^p}{1 - u_{i+1}x^p} \to 1$$

again by (4·4). This proves (4·6).

Now assume $r_n \to \infty$ and $r_n n^{-1} \to 0$. Choose a subsequence n_i such that

$$\sum_{j=i}^{\infty} r_{n_j}^3 n_j^{-1} = o(r_{n_i} n_i^{-1}) \tag{4·9}$$

$$u_{n_i}(r_{n_i}^1) > u_{n_{i+1}}(-r_{n_{i+1}}^1). \tag{4·10}$$

We will take

$$c_i = a_{n_i}(0), \quad d_i = a_{n_i}(r_{n_i}^1). \tag{4·11}$$

Observe that by (4·8)

$$a_{n_i}(\alpha) = (n_i(r_{n_i} - \alpha r_{n_i}^2)^{-1})^{1/p}, \tag{4·12}$$

and so

$$c_i = (n_i r_{n_i}^{-1})^{1/p}, \quad d_i = (n_i(r_{n_i} - r_{n_i}^3)^{-1})^{1/p}. \tag{4·13}$$

In particular, (4·3) holds by (4·10) and (4·4) holds by (4·9). We will now show that

$$(^{(r_{n(i)})}S_{n_i} - EV_{m_i}(0))\tau_{n_i}^{-1}(0) \to N + ((2-p)p^{-1})^{\frac{1}{2}}(N_1 \vee 0)$$

where $N \overset{d}{=} N_1$ and N, N_1 are independent. We begin by computing some relevant quantities. First

$$\begin{aligned} \tau_{n_i}^2(\alpha) &= m_i \int_1^{a_{n(i)}(\alpha)} py^{1-p}\,dy \\ &= m_i p(2-p)^{-1}(a_{n_i}^{2-p}(\alpha) - 1). \end{aligned} \tag{4·14}$$

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Thus by (4·12),
$$\tau_{n_i}(\alpha)\tau_{n_i}^{-1}(0)\to 1 \quad \text{u.c.} \tag{4·15}$$

Next, for $\alpha > 0$,

$$\begin{aligned} EV_{m_i}(\alpha)-EV_{m_i}(0) &= m_i(EX1(|X|\leqslant a_{n_i}(\alpha))\,v_{n_i}^{-1}(\alpha)-EX1(|X|\leqslant a_{n_i}(0))\,v_{n_i}^{-1}(0)) \\ &= m_i\,EX1(a_{n_i}(0)<|X|\leqslant a_{n_i}(\alpha))(v_{n_i}^{-1}(\alpha)-v_{n_i}^{-1}(0))\,EX1(|X|\leqslant a_{n_i}(0)) \\ &\quad + m_i(v_{n_i}^{-1}(\alpha) \\ &= \mathrm{I}+\mathrm{II}. \end{aligned}$$

Now if $0<\alpha<r_{n_i}^{\frac{1}{4}}$ then $c_i = a_{n_i}(0) < a_{n_i}(\alpha) < d_i$, thus

$$\begin{aligned} \mathrm{I} &= m_i\,v_{n_i}^{-1}(\alpha)\int_{a_{n(i)}(0)}^{a_{n(i)}(\alpha)} py^{-p}\,dy \\ &\sim m_i\,v_{n_i}^{-1}(\alpha)\,pa_{n_i}^{-p}(0)\,(a_{n_i}(\alpha)-a_{n_i}(0)) \quad \text{u.c.} \\ &\sim \alpha r_{n_i}^{\frac{1}{2}}(n_i\,r_{n_i}^{-1})^{1/p} \quad \text{u.c.} \end{aligned}$$

by (4·12). For II we have

$$\begin{aligned} |\mathrm{II}| &= m_i(v_{n_i}^{-1}(0)-v_{n_i}^{-1}(\alpha))\,EX1(|X|\leqslant a_{n_i}(0)) \\ &\sim \alpha r_{n_i}^{\frac{1}{2}}\,EX1(|X|\leqslant a_{n_i}(0)) \quad \text{u.c.} \\ &\leqslant \alpha r_{n_i}^{\frac{1}{2}}(EX^2 1(|X|\leqslant a_{n_i}(0)))^{\frac{1}{2}} \\ &= \alpha(r_{n_i}\,m_i^{-1})^{\frac{1}{2}}\tau_{n_i}(0) \\ &= o(\tau_{n_i}(0)) \quad \text{u.c.} \end{aligned}$$

A straightforward computation using (4·12) and (4·14) then shows that for $\alpha \geqslant 0$

$$(EV_{m_i}(\alpha)-EV_{m_i}(0))\,\tau_{n_i}^{-1}(0)\to((2-p)\,p^{-1})^{\frac{1}{2}}\alpha \quad \text{u.c.} \tag{4·16}$$

On the other hand, if $-r_{n_i}^{\frac{1}{4}}<\alpha<0$, then by (4·10) $d_{i-1} < a_{n_i}(\alpha) < a_{n_i}(0) = c_i$ and so

$$\begin{aligned} EV_{m_i}(\alpha)-EV_{m_i}(0) &= -m_i\,EX1(a_{n_i}(\alpha)<|X|\leqslant a_{n_i}(0))\,v_{n_i}^{-1}(\alpha) \\ &\quad + m_i(v_{n_i}^{-1}(\alpha)-v_{n_i}^{-1}(0))\,EX1(|X|\leqslant a_{n_i}(0)) \\ &= m_i(v_{n_i}^{-1}(\alpha)-v_{n_i}^{-1}(0))\,EX1(|X|\leqslant a_{n_i}(0)) \\ &= o(\tau_{n_i}(0)) \quad \text{u.c.} \end{aligned}$$

as above. Thus for $\alpha \leqslant 0$

$$(EV_{m_i}(\alpha)-EV_{m_i}(0))\,\tau_{n_i}^{-1}(0)\to 0 \quad \text{u.c.} \tag{4·17}$$

Next, a direct computation using (4·5) and (4·14) shows that

$$m_i(E|X|\,1(|X|\leqslant a_{n_i}(\alpha)))^2 = o(\tau_{n_i}^2(\alpha)) \quad \text{u.c.}$$

Thus
$$\operatorname{Var} V_{m_i}(\alpha)\,\tau_{n_i}^{-2}(\alpha)\to 1 \quad \text{u.c.} \tag{4·18}$$

Finally, since $|X(u_{n_i}(\alpha))-EX(u_{n_i}(\alpha))| \leqslant 2a_{n_i}(\alpha)$, we have

$$\begin{aligned} \frac{E|X(u_{n_i}(\alpha))-EX(u_{n_i}(\alpha))|^3}{m_i^{\frac{3}{2}}(\operatorname{Var} X(u_{n_i}(\alpha)))^{\frac{3}{2}}} &\leqslant \frac{2a_{n_i}(\alpha)}{m_i^{\frac{1}{2}}(\operatorname{Var} X(u_{n_i}(\alpha)))^{\frac{1}{2}}} \\ &\to 0 \quad \text{u.c.} \end{aligned}$$

by (4·12), (4·14) and (4·18). Thus by the Berry–Esseen Theorem

$$\sup_y |P((V_{m_i}(\alpha)-EV_{m_i}(\alpha))\,(\operatorname{Var} V_{m_i}(\alpha))^{-\frac{1}{2}} > y)-P(N>y)|\to 0 \quad \text{u.c.} \tag{4·19}$$

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Thus by (4·15)–(4·19) we have

$$\begin{aligned}\int_{-\lambda}^{\lambda} & P((V_{m_i}(\alpha)-EV_{m_i}(0))\,\tau_{n_i}^{-1}(0) > x)\,\mu_{n_i}(d\alpha)\\ &= \int_{-\lambda}^{\lambda} P((V_{m_i}(\alpha)-EV_{m_i}(\alpha)+EV_{m_i}(\alpha)-EV_{m_i}(0))\,\tau_{n_i}^{-1}(0) > x)\,\mu_{n_i}(d\alpha)\\ &\to \int_{-\lambda}^{\lambda} P(N+((2-p)\,p^{-1})^{\frac{1}{2}}(\alpha \vee 0) > x)\,\mu(d\alpha). \qquad (4\cdot20)\end{aligned}$$

Now letting $\lambda \to \infty$ and using (2·22) we have

$$({}^{(r_{n(i)})}S_{n_i}-EV_{m_i}(0))\,\tau_{n_i}^{-1}(0) \to N+((2-p)\,p^{-1})^{\frac{1}{2}}(N_1 \vee 0)$$

as claimed.

The above example shows that when the underlying distribution is not symmetric, subsequential limits other than variance mixtures of normals are possible. The construction can be modified to give other examples. For instance, for any $-\infty \leqslant \alpha \leqslant \beta \leqslant \infty$, both $N+((2-p)\,p^{-1})^{\frac{1}{2}}((\alpha \vee N_1) \wedge \beta)$ and $N+((2-p)\,p^{-1})^{\frac{1}{2}}|(\alpha \vee N_1) \wedge \beta|$ can arise as subsequential limits.

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