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m-full ideals II

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Introduction

In his paper [**10**] the author investigated the structure of m-full ideals by analysing their syzygies and, as one special case, showed how the Betti numbers of Borel stable ideals over polynomial rings can be computed. The same result, among other things, was also obtained by Eliahou and Kervaire[**1**] by a different method.

Let $\mathfrak{a}$ be a Borel stable ideal in a polynomial ring R and let V_i be the ideal generated by i generic linear forms and let t_{n-i-1} be the type of the ideal $\mathfrak{a}+V_i/V_i$ over R/V_i. Then the Betti numbers of $\mathfrak{a}$ are linear combinations of $t_0, \ldots, t_{n-1}$ with certain binomial numbers as coefficients, and conversely from the numbers $t_0, \ldots, t_{n-1}$ the Betti numbers can be recovered (see [**10**], corollary 9 and proposition 4).

The present paper grew out of the question of deciding what sequences $t_0, \ldots, t_{n-1}$ of integers can arise from a Borel stable ideal as above. Our goal of this paper is to prove Theorems 4·1 and 4·2 of Section 4, in which we show that if we make the restriction on the ideal that it be generated by monomials of a fixed degree then the sequence $t_0, \ldots, t_{n-1}$ is precisely the same as what has been known as the Hilbert series of a graded Artin algebra.

In an earlier paper [**11**] the author showed some properties of m-primary m-full ideals. In this present paper we need to generalize these results to ideals which are not necessarily m-primary. Most results of [**11**] can be generalized to general m-full ideals. The 0th local cohomology module $\bigcup \mathfrak{a}:\mathfrak{m}^i/\mathfrak{a}$ of $R/\mathfrak{a}$ plays an important role. These are the contents of Section 1 and have independent interest.

In Section 2 we summarize some combinatorial formulae derived from Macaulay's theorem which characterizes the Hilbert function of homogeneous algebras (Theorem 2·2).

In Section 3 we consider m-full ideals $\mathfrak{a}$ in regular local rings which satisfy the condition $\mathfrak{m}^{d+1} \cap \mathfrak{a} = \mathfrak{m}\mathfrak{a}$. The result (Theorem 3·1) will be applied, in Section 4, to Borel stable ideals generated by monomials all of degree d to obtain the theorems mentioned above.

1. *A characterization of* m*-full ideals*

We quickly review some fundamental definitions and notation that we use throughout.

Let $(R, \mathfrak{m}, k)$ be a local ring, where $\mathfrak{m}$ is generated by $x_1, \ldots, x_n$. Let $R^* = R(y_1, \ldots, y_n)$ be the polynomial ring $R[y_1, \ldots, y_n]$ localized at the prime ideal $\mathfrak{m}R[y_1, \ldots, y_n]$. Then we will call the element $z = y_1x_1 + \ldots + y_nx_n$ in R^* a general element of R and regard it as an element of R. An ideal $\mathfrak{a}$ of R is said to be m-full if $\mathfrak{m}\mathfrak{a}:z = \mathfrak{a}$, where

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z is a general element of R. (By the convention we have just made this means $\mathfrak{m}\mathfrak{a}R^*:zR^* = \mathfrak{a}R^*$, to be precise.) One sees easily that if k is infinite then $\mathfrak{a}$ is $\mathfrak{m}$-full if and only if $\mathfrak{m}\mathfrak{a}:y = \mathfrak{a}$ for some y in R. For details we refer the reader to [**10**] and [**11**]. We define $\tau(\mathfrak{a}) = \text{length}\,(\mathfrak{a}:\mathfrak{m}/\mathfrak{m})$. Suppose that R is a regular local ring of dimension n. Then it is equal to the nth free rank in the minimal free resolution of $R/\mathfrak{a}$. If $\tau(\mathfrak{a}) > 0$, we call it the type of $\mathfrak{a}$. By $\mu(\mathfrak{a})$ we denote the minimal number of generators of an ideal $\mathfrak{a}$, and l stands for length.

LEMMA 1·1 (Rees). *Let $(R, \mathfrak{m}, k)$ be a local ring. Assume $\mathfrak{m}\mathfrak{a}:z = \mathfrak{a}$ for an element $z \in \mathfrak{m}$. Then*

(i) $\mathfrak{a}:z = \mathfrak{a}:\mathfrak{m}$,

(ii) $\mathfrak{a}:\mathfrak{b}$ *is $\mathfrak{m}$-full for any ideal $\mathfrak{b} \subset R$, and*

(iii) $\mathfrak{a}:z^i = \mathfrak{a}:\mathfrak{m}^i$ *for any $i \geqslant 0$.*

Proof. (i) Obviously $\mathfrak{a}:z \supset \mathfrak{a}:\mathfrak{m}$. Let $x \in \mathfrak{a}:z$. Then $\mathfrak{m}zx \subset \mathfrak{m}\mathfrak{a}$. Hence we conclude that $\mathfrak{m}x \subset \mathfrak{m}\mathfrak{a}:z = \mathfrak{a}$. So $x \in \mathfrak{a}:\mathfrak{m}$.

(ii) We will prove $\mathfrak{m}(\mathfrak{a}:\mathfrak{b}):z = \mathfrak{a}:\mathfrak{b}$. One inclusion is obvious. Let $x \in \mathfrak{m}(\mathfrak{a}:\mathfrak{b}):z$. Then $zx \in \mathfrak{m}(\mathfrak{a}:\mathfrak{b}) \subset \mathfrak{m}\mathfrak{a}:\mathfrak{b}$. So $x \in (\mathfrak{m}\mathfrak{a}:\mathfrak{b}):z = (\mathfrak{m}\mathfrak{a}:z):\mathfrak{b} = \mathfrak{a}:\mathfrak{b}$, as wanted.

(iii) This is immediate from (i) and (ii) by induction on i.

LEMMA 1·2. *Let $(R, \mathfrak{m}, k)$ be a local ring, and $\mathfrak{a} \subset R$ any ideal. Let z be any non-unit element of R and let $^-:R \to R/zR$ be the natural surjection. Then $\mu(\mathfrak{a}) - \mu(\bar{\mathfrak{a}}) = l(\mathfrak{a}:z/\mathfrak{m}\mathfrak{a}:z)$.*

Proof. This follows immediately from the exact sequence

$$0 \to \mathfrak{a}:z/\mathfrak{m}\mathfrak{a}:z \xrightarrow{\cdot z} \mathfrak{a}/\mathfrak{m}\mathfrak{a} \to \mathfrak{a} + zR/\mathfrak{m}\mathfrak{a} + zR \to 0.$$

PROPOSITION 1·3. *Let $(R, \mathfrak{m}, k)$ be a local ring and $\mathfrak{a} \subset R$ an ideal. Then $\mathfrak{a}$ is $\mathfrak{m}$-full if and only if*

(1) $\mathfrak{a}:z = \mathfrak{a}:\mathfrak{m}$ *for a general element z of R, and*

(2) $\mu(\mathfrak{a}) - \mu(\bar{\mathfrak{a}}) = l(\mathfrak{a}:\mathfrak{m}/\mathfrak{a})$ *where '$^-$' denotes the reduction by z.*

Proof. Let z be a general element. Note that $\mathfrak{a} \subset \mathfrak{m}\mathfrak{a}:z$. Equality follows from (1), (2) and Lemma 1·2. Conversely assume that $\mathfrak{a} = \mathfrak{m}\mathfrak{a}:z$. (1) was proved in Lemma 1·1. (2) follows from Lemma 1·2.

PROPOSITION 1·4. *Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a}$ an ideal and z any element of $\mathfrak{m}$. Let $\bar{\mathfrak{a}}$ denote $\mathfrak{a} + zR/zR$. Furthermore let $\mathfrak{h} = \bigcup_i (\mathfrak{a}:\mathfrak{m}^i)$, $H = \mathfrak{h}/\mathfrak{a}$, and $\tilde{H} = \mathfrak{h}/\mathfrak{m}\mathfrak{a}$. Then*

(i) $\tau(\mathfrak{a}) \leqslant l(H/zH)$,

(ii) $\mu(\mathfrak{a}) \leqslant l(\tilde{H}/z\tilde{H})$, *and*

(iii) $\mu(\bar{\mathfrak{a}}) + l(H/zH) \leqslant l(\tilde{H}/z\tilde{H})$.

(iv) *If z is a general element, then $\mu(\bar{\mathfrak{a}}) + l(H/zH) = l(\tilde{H}/z\tilde{H})$.*

Proof. Consider the exact sequence

$$0 \to \frac{(\mathfrak{a}:z) \cap \mathfrak{h}}{\mathfrak{a}} \to H \xrightarrow{\cdot z} H \to H/zH \to 0.$$

Since $\mathfrak{a}:\mathfrak{m} \subset (\mathfrak{a}:z) \cap \mathfrak{h}$, we have

$$\tau(\mathfrak{a}) = l(\mathfrak{a}:\mathfrak{m}/\mathfrak{a}) \leqslant l(\mathfrak{a}:z \cap \mathfrak{h}/\mathfrak{a}) = l(H/zH).$$

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This proves (i). Consider the similar exact sequence as above for $\tilde{H}$. Then since $\mathfrak{a} \subset \mathfrak{m}\mathfrak{a} : z \cap \mathfrak{h}$, we have

$$\mu(\mathfrak{a}) = l(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) \leqslant l(\mathfrak{m}\mathfrak{a} : z \cap \mathfrak{h}/\mathfrak{m}\mathfrak{a}) = l(\tilde{H}/z\tilde{H}).$$

This proves (ii). To prove (iii) consider the inclusions

$$\mathfrak{m}\mathfrak{a} \subset \mathfrak{a} \subset (\mathfrak{m}\mathfrak{a} : z) \cap \mathfrak{h} \subset (\mathfrak{a} : z) \cap \mathfrak{h},$$

in which we know

$$l(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) = \mu(\mathfrak{a}), \quad l(\mathfrak{a} : z \cap \mathfrak{h}/\mathfrak{a}) = l(H/zH), \quad \text{and} \quad l(\mathfrak{m}\mathfrak{a} : z \cap \mathfrak{h}/\mathfrak{m}\mathfrak{a}) = l(\tilde{H}/z\tilde{H}).$$

Thus

$$l(\mathfrak{a} : z \cap \mathfrak{h}/\mathfrak{m}\mathfrak{a} : z \cap \mathfrak{h}) = \mu(\mathfrak{a}) + l(H/zH) - l(\tilde{H}/z\tilde{H}).$$

The left-hand side of this equality is bounded by $l(\mathfrak{a} : z/\mathfrak{m}\mathfrak{a} : z)$, which is equal to $\mu(\mathfrak{a}) - \mu(\bar{\mathfrak{a}})$ by Lemma 1·2. Thus we get $l(\tilde{H}/z\tilde{H}) \geqslant l(H/zH) + \mu(\bar{\mathfrak{a}})$. Note that this inequality becomes an equality if $\mathfrak{a} : z \subset \mathfrak{h}$. So to prove (iv) it suffices to show that if z is a general element then $\mathfrak{a} : z \subset \mathfrak{h}$. Let $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{n-1} \cap \mathfrak{q}_n$ be a shortest primary decomposition of $\mathfrak{a}$. If no $\mathfrak{m}$-primary component appears then this means that $\mathfrak{a} : z = \mathfrak{a}$ since z is a general element. So the assertion is trivial. Otherwise one of the components, say $\mathfrak{q}_n$, is an $\mathfrak{m}$-primary ideal. Then we must have $\mathfrak{a} : z = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{n-1} \cap (\mathfrak{q}_n : z)$. Choose i so that $\mathfrak{m}^i(\mathfrak{q}_n : z) \subset \mathfrak{q}_n$. Then $\mathfrak{m}^i(\mathfrak{a} : z) \subset \mathfrak{a}$, as desired. ∎

THEOREM 1·5. *Let $(R, \mathfrak{m})$ be a local ring. Let $\mathfrak{a}$ be an ideal and z a general element of R. Let H and $\tilde{H}$ be the same as in Proposition* 1·4. *Then the following conditions are equivalent*:

(1) $\mathfrak{a}$ *is* $\mathfrak{m}$-*full*;
(2) $\tau(\mathfrak{a}) = l(H/zH)$ *and* $\mu(\mathfrak{a}) = \tau(\mathfrak{a}) + \mu(\bar{\mathfrak{a}})$;
(3) $\mu(\mathfrak{a}) = l(H/zH) + \mu(\bar{\mathfrak{a}})$;
(4) $\mu(\mathfrak{a}) = l(\tilde{H}/z\tilde{H})$.

Proof. Let $\mathfrak{h}$ be as in Proposition 1·4. In the proof of Proposition 1·4 we proved that $\mathfrak{a} : z \subset \mathfrak{h}$ for any ideal $\mathfrak{a}$, and that it implies that $l(\mathfrak{m}\mathfrak{a} : z/\mathfrak{m}\mathfrak{a}) = l(\tilde{H}/z\tilde{H})$ and $l(\mathfrak{a} : z/\mathfrak{a}) = l(H/zH)$. First we want to prove the implication (1) ⇒ (2). Assume (1). Then $\mathfrak{m}\mathfrak{a} : z = \mathfrak{a}$ and $\mathfrak{a} : z = \mathfrak{a} : \mathfrak{m}$ by Lemma 1·1. Hence we have $\mu(\mathfrak{a}) = l(\tilde{H}/z\tilde{H})$ and $\tau(\mathfrak{a}) = l(H/zH)$. Now we may use Proposition 1·4(iv) to conclude (2). (2) ⇒ (3) is trivial. (3) ⇒ (4) is also trivial if we use Proposition 1·4(iv). Notice that $\mu(\mathfrak{a}) = l(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) \leqslant l(\mathfrak{m}\mathfrak{a} : z/\mathfrak{m}\mathfrak{a}) = l(\tilde{H}/z\tilde{H})$. Thus (4) implies (1). Now the proof is complete.

Remark. If $\mathfrak{a}$ is $\mathfrak{m}$-primary then $H = R/\mathfrak{a}$ and $\tilde{H} = R/\mathfrak{m}\mathfrak{a}$. Hence Theorem 1·5 is a generalization of [**11**], theorem 2.

2. *Some combinatorial formulae*

We want to review Macaulay's theorem which characterizes the Hilbert functions of homogeneous rings and some related results.

Given two integers g, $i > 0$, it is possible to write

$$g = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}, \quad n_i > n_{i-1} > \ldots > n_j \geqslant j \geqslant 1.$$

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Such an expression is unique. After [5] we will call this the i-binomial expansion of g. We define $g^{\langle i\rangle}$, $g^{(i)}$ as follows.

$$g^{\langle i\rangle} = \binom{n_i+1}{i+1}+\binom{n_{i-1}+1}{i}+\ldots+\binom{n_j+1}{j+1},$$

$$g^{(i)} = \binom{n_i+1}{i}+\binom{n_{i-1}+1}{i-1}+\ldots+\binom{n_j+1}{j}.$$

Furthermore we make the following convention:

$$0^{\langle i\rangle} = 0^{(i)} = 0 \quad \text{for } i > 0.$$

Remark. The notation $h^{\langle i\rangle}$ is used in Stanley [8], but $g^{(i)}$ should not be confused with $g^{(i)}$ in [8]. To characterize the f-vectors of simplicial complexes one needs the latter definition, but we do not use this in this paper.

LEMMA 2·1. *Assume that $d \geqslant 1$ and $n \geqslant 2$ are integers and h and g are positive integers such that*

$$h+g = \binom{n+d-1}{d}.$$

Then

$$h^{\langle d\rangle}+g^{(n-1)} = \binom{n+d}{d+1}.$$

Since no good references seem available we give a brief discussion of this lemma. Let $V(n,d)$ denote the set of monomials is n variables $x_1,\ldots,x_n$ of degree d. We put the lexicographic order on $V(n,d)$, which is defined as follows:

$$x^A = x_1^{a_1}\ldots x_n^{a_n} > x^B = x_1^{b_1}\ldots x_n^{b_n} \Leftrightarrow \begin{cases}\text{the first non zero component}\\ \text{of } A-B \text{ is positive.}\end{cases}$$

Now consider two monomials $x^A > x^B$, where x^A is the gth monomial (from the first) and x^B is the hth monomial from the last in $V(n,d)$. Then $g+h = \#\, V(n,d)$ if and only if x^A and x^B are adjacent. Suppose that this is the case. Then notice that $x^A.x_n$ and $x^B.x_k$ are adjacent in $V(n,d+1)$, where k is the maximal integer such that $b_k \neq 0$. If we take for granted the Sublemmas 2·1 (a) and 2·1 (b) below it follows immediately that $g^{(n-1)}+h^{\langle d\rangle} = \#\, V(n,d+1)$, as we want to prove. The proofs of the sublemmas are similar, so we give a proof only for Sublemma 2·1 (a).

SUBLEMMA 2·1 (a). *If x^A is the gth monomial in $V(n,d)$, then $x^A.x_n$ is the $g^{(n-1)}$th monomial in $V(n,d+1)$.*

SUBLEMMA 2·1 (b). *If x^B is the hth monomial from the last in $V(n,d)$, with $B = (b_1,\ldots,b_k,0,\ldots,0)$, $b_k \neq 0$, then $x^B.x_k$ is the $h^{\langle d\rangle}$th monomial in $V(n,d+1)$ from the last.*

Proof of Sublemma 2·1 (a). If either $d = 1$ or $n = 2$, the assertion is easily verified. We assume $d \geqslant 2$ and $n \geqslant 3$. Let $W(n-1,d)$ be the set of monomials of degree d in $x_2,\ldots,x_n$. Then $V(n,d)$ contains $W(n-1,d)$, and we have the disjoint union $V(n,d) = x_1.V(n,d-1) \cup W(n-1,d)$. Moreover the map $V(n,d-1) \to V(n,d)$ defined by multiplication by x_1 takes consecutive terms to consecutive terms and the first

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element to the first element. So if x^A is divisible by x_1 then we can use the induction on d applied to $(1/x_1).x^A$. Suppose $x^A \in V(n,d)$ is the gth element and let

$$g = \binom{m_{n-1}}{n-1} + \ldots + \binom{m_j}{j}$$

be the $(n-1)$-binomial expansion. One sees easily that x^A does not involve x_1 if and only if

$$g > \binom{n+d-2}{n-1}.$$

Assume that this is the case. Then one sees furthermore that $m_{n-1} = n+d-2$ and $j < n-1$ unless $x^A = x_n^d$. If $x^A = x_n^d$ then $m_{n-1} = n+d-1$ and $j = n-1$, and the assertion of Sublemma 2·1 (a) is obvious. So we assume $x^A \neq x_n^d$. As a monomial in $W(n-1,d)$, x^A occurs at the fth place, where $f = g - \# x_1.V(n,d-1)$. Therefore by deleting the first term from the binomial expansion of g, we get the $(n-2)$-binomial expansion of f. Now the proof is finished by induction on n. ∎

The following is generally attributed to F. S. Macaulay [**6**], but a number of authors have simplified the proof. (Cf. Stanley [**8**] for a historical note.)

THEOREM 2·2. *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k in n variables. We view it as a graded ring with $\deg x_i = 1$ for each i. Let $R = \oplus R_d$ be the decomposition into the homogeneous parts. Let $I = \oplus I_d$ be a graded ideal of R, and $R/I = \oplus R_d/I_d$ the quotient ring. Now let $h_d = \dim (R/I)_d$, and $g_d = \dim I_d$. Then*

(i) $h_{d+1} \leqslant h_d^{\langle d \rangle}$, *for* $d \geqslant 1$, *and*

(ii) $g_{d+1} \geqslant g_d^{\{n-1\}}$, *for* $d \geqslant 1$ *and* $n \geqslant 2$.

Remark. For $n \geqslant 2$, (i) and (ii) are equivalent by Lemma 2·1. For the proof of (i) see Greene and Kleitman [**4**] and Stanley [**9**]. (So far as ideals generated by monomials are concerned there is a proof in [**4**]. By [**9**], theorem 2·1 the general case reduces to the monomial case.) Also see Green [**3**] for a new proof and related results. Originally Macaulay [**6**] proved (ii) essentially. More consequences of Macaulay's theorem are found in Elias *et al.* [**2**].

PROPOSITION 2·3. *Fix an integer $i \geqslant 1$. Suppose that $g_0, g_1, \ldots, g_d$ is a sequence of positive integers such that $g_j \geqslant g_{j-1}^{\{i\}}$, for $j = 1, 2, \ldots, d$. Then $g_0 + g_1 + \ldots + g_d \leqslant g_d^{\langle i \rangle}$.*

For the proof we prepare two easy lemmas.

LEMMA 2·4. *For positive integers a, a' and i, we have* (i) $a^{\langle i \rangle} + a' \leqslant (a+a')^{\langle i \rangle}$, (ii) $a \leqslant a^{\langle i \rangle}$, *and* (iii) $a^{\langle i \rangle} + a^{\{i\}} = a^{\{i\}\langle i \rangle}$.

Proof. (i) It suffices to prove the case $a' = 1$. Write the i-binomial expansion

$$a = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}$$

and similarly for $a+1$. In these expressions if the leading terms are the same the proof is finished by induction on i. So assume that they are different. One sees easily that that occurs only if

$$a = \binom{n_i}{i} + \binom{n_i - 1}{i-1} + \ldots + \binom{n_i - i + 1}{1} \quad \text{and} \quad a+1 = \binom{n_i + 1}{i}.$$

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Now the desired inequality is immediate. (ii) follows easily from (i) by induction on a with $a' = 1$, and (iii) is easy.

LEMMA 2·5. *Let a, b and n be positive integers. Suppose that $a^{\{i\}} \leqslant b$. Then $a^{\langle i\rangle} + b \leqslant b^{\langle i\rangle}$.*

Proof. Let b' be the integer such that $b = a^{\{i\}} + b'$. Use Lemma 2·4 (i) and (iii) to see that

$$a^{\langle i\rangle} + b = a^{\langle i\rangle} + a^{\{i\}} + b' = a^{\{i\}\langle i\rangle} + b' \leqslant (a^{\{i\}} + b')^{\langle i\rangle} = b^{\langle i\rangle}.$$

Proof of Proposition 2·3. If $d = 0$ we can use Lemma 2·4 (ii). By induction on d we may assume that $a_0 + \ldots + a_{d-1} \leqslant a_{d-1}^{\langle i\rangle}$. Now the proof is complete by Lemma 2·5.

Remark. In Proposition 2·3 the equality occurs for example when

$$a_0 = \binom{i}{i} \quad \text{and} \quad a_j = a_{j-1}^{\{i\}} \quad \text{for } j \geqslant 1.$$

3. *The length of the module* $\bigcup \mathfrak{a} : \mathfrak{m}^i/\mathfrak{a}$

Let $(R, \mathfrak{m}, k)$ be a local ring such that all powers of $\mathfrak{m}$ are $\mathfrak{m}$-full. (Regular local rings are such an example. See Proposition 3·3 below.) Let $\mathfrak{a}$ be an $\mathfrak{m}$-full ideal of R and let d be the smallest integer such that $\mathfrak{a} \subset \mathfrak{m}^d$. Let $G = \mathrm{Gr}(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \ldots$ be the associated graded ring with respect to $\mathfrak{m}$ and let $A = G/\mathfrak{m}^{d+1}G$. Assume that $\mathfrak{m}\mathfrak{a} = \mathfrak{m}^{d+1} \cap \mathfrak{a}$. Then the diagram

$$\begin{array}{ccc} \mathfrak{a} & \subset & \mathfrak{m}^d \\ \cup & & \cup \\ \mathfrak{m}\mathfrak{a} & \subset & \mathfrak{m}^{d+1} \end{array}$$

induces an injection $\mathfrak{a}/\mathfrak{m}\mathfrak{a} \subset \mathfrak{m}^d/\mathfrak{m}^{d+1}$. Let z be a general element of R and apply $(-) : z$ to the above diagram. Then since $\mathfrak{a} : z = \mathfrak{a} : \mathfrak{m}$ by Lemma 1·1 and $\mathfrak{m}^i : z = \mathfrak{m}^{i-1}$ for each i by assumption, we get the following diagram:

$$\begin{array}{ccc} \mathfrak{a} : \mathfrak{m} & \subset & \mathfrak{m}^{d-1} \\ \cup & & \cup \\ \mathfrak{a} & \subset & \mathfrak{m}^d. \end{array}$$

Again this induces an injection $\mathfrak{a} : \mathfrak{m}/\mathfrak{a} \subset \mathfrak{m}^{d-1}/\mathfrak{m}^d$, since $\mathfrak{m}^d \cap \mathfrak{a} : \mathfrak{m} = \mathfrak{a}$. The last equality can be proved by applying $(-) : z$ to $\mathfrak{m}^{d+1} \cap \mathfrak{a} = \mathfrak{m}\mathfrak{a}$. By Lemma 1·1 we may iterate the process to obtain the inclusions

$$\mathfrak{a} : \mathfrak{m}^i/\mathfrak{a} : \mathfrak{m}^{i-1} \subset \mathfrak{m}^{d-i}/\mathfrak{m}^{d-i+1}$$

for all $i = 0, 1, 2, \ldots, d$. (We make the convention that $\mathfrak{a} : \mathfrak{m}^{-1} = \mathfrak{m}\mathfrak{a}$.) Put $I = \sum_{i=0}^{d} \mathfrak{a} : \mathfrak{m}^{d-i}/\mathfrak{a} : \mathfrak{m}^{d-i-1}$. Then this is a vector subspace of A, but as one sees easily it is actually an ideal of A.

THEOREM 3·1. *Suppose that $(R, \mathfrak{m}, k)$ is a regular local ring of dimension $n \geqslant 2$, and $\mathfrak{a}$ is an $\mathfrak{m}$-full ideal such that $\mathfrak{m}^{d+1} \cap \mathfrak{a} = \mathfrak{m}\mathfrak{a}$. Let $H = \bigcup \mathfrak{a} : \mathfrak{m}^i/\mathfrak{a}$ and $\tilde{H} = \bigcup \mathfrak{a} : \mathfrak{m}^i/\mathfrak{m}\mathfrak{a}$. Then*

$$l(H) \leqslant \tau(\mathfrak{a})^{\langle n-1\rangle} \quad \textit{and} \quad l(\tilde{H}) \leqslant \mu(\mathfrak{a})^{\langle n-1\rangle}.$$

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Proof. First we note that the condition $\mathfrak{m}^{d+1} \cap \mathfrak{a} = \mathfrak{m}\mathfrak{a}$ implies $\mathfrak{a} \subset \mathfrak{m}^d$. In fact $\mathfrak{a} = \mathfrak{m}\mathfrak{a}:z = (\mathfrak{m}^{d+1} \cap \mathfrak{a}):z \subset \mathfrak{m}^{d+1}:z = \mathfrak{m}^d$. Put

$$G = \mathrm{Gr}_{\mathfrak{m}}(R), \quad A = G/\mathfrak{m}^{d+1}G, \quad I = \sum_{i=0}^{d} \mathfrak{a}:\mathfrak{m}^{d-i}/\mathfrak{a}:\mathfrak{m}^{d-i-1}.$$

Then G is a polynomial ring in n variables, A is a truncated polynomial ring and I is an ideal of A. Let

$$a_i = \dim(\mathfrak{a}:\mathfrak{m}^{d-i}/\mathfrak{a}:\mathfrak{m}^{d-i-1}).$$

Then because I is an ideal of A, we have $a_i \geqslant a_{i-1}^{\langle n-1\rangle}$ for all i by Theorem 2·2. Notice that $a_{d-1} = \dim(\mathfrak{a}:\mathfrak{m}/\mathfrak{a}) = \tau(\mathfrak{a})$ and $a_d = \dim(\mathfrak{a}/\mathfrak{m}\mathfrak{a}) = \mu(\mathfrak{a})$. Since we have $l(H) = a_0 + \ldots + a_{d-1}$, and $l(\tilde{H}) = a_0 + \ldots + a_{d-1} + a_d$, we get the assertions by Proposition 2·3.

PROPOSITION 3·2. *Let $(R, \mathfrak{m})$ be a regular local ring with $\dim R = n \geqslant 3$, and let z be a general element of R. Put $\bar{R} = R/zR$ and $\bar{\mathfrak{m}} = \mathfrak{m}/zR$. Let $\mathfrak{a} \subset R$ be an ideal such that both $\mathfrak{a}$ and $\bar{\mathfrak{a}} = \mathfrak{a} + zR/zR$ are $\mathfrak{m}$-full. Furthermore assume that $\bar{\mathfrak{m}}^{d+1} \cap \bar{\mathfrak{a}} = \bar{\mathfrak{m}}\bar{\mathfrak{a}}$ with some integer d. Then we have* (i) $\tau(\mathfrak{a}) \leqslant \tau(\bar{\mathfrak{a}})^{\langle n-2\rangle}$, (ii) $\mu(\mathfrak{a}) \leqslant \mu(\bar{\mathfrak{a}})^{\langle n-2\rangle}$.

Proof. Let $H = \bigcup \mathfrak{a}:\mathfrak{m}^i/\mathfrak{a}$. Consider the exact sequence

$$0 \to H \to R/\mathfrak{a} \to R/H \to 0.$$

Note that z is a non-zero divisor on R/H unless $H = R$. In any case we have the natural injection $H/zH \subset \bar{R}/\bar{\mathfrak{a}}$. Put $\bar{H} = \bigcup \bar{\mathfrak{a}}:\bar{\mathfrak{m}}^i/\bar{\mathfrak{a}}$. Then it is easy to see that $H/zH \subset \bar{H}$ as ideals in $\bar{R}/\bar{\mathfrak{a}}$. By Theorem 3·1 and by Theorem 1·5(2) we see that $\tau(\mathfrak{a}) = l(H/zH) \leqslant l(\bar{H}) \leqslant \tau(\bar{\mathfrak{a}})^{\langle n-2\rangle}$. The second assertion is proved similarly. ∎

The following proposition was essentially proved by Sally. (See [7], proposition 3·2 and remarks following it on pp. 7–8.) We include it here for completeness.

PROPOSITION 3·3. *Let $(R, \mathfrak{m})$ be a local ring and let*

$$\mathrm{Gr}_{\mathfrak{m}}(R) = \mathrm{Gr}(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \ldots$$

be the associated graded ring. Then

(i) $\mathrm{depth}\,\mathrm{Gr}(R) > 0$ *if and only if all powers of* $\mathfrak{m}$ *are* $\mathfrak{m}$*-full, and*

(ii) $\mathrm{depth}\,R > 0$ *if and only if all high powers of* $\mathfrak{m}$ *are* $\mathfrak{m}$*-full.*

Proof. (i) is straightforward.

(ii) Let $\bar{z} \in \mathfrak{m}/\mathfrak{m}^2$ be the image of a general element of R in G. Then it is also a general element of G. Since $[0:\bar{z}]_G \subset \bigcup_i [0:\bar{\mathfrak{m}}^i]_G$, as we saw in the proof of Proposition 1·4, $[0:\bar{z}]_G$ has finite length. Thus $\bar{z}$ is a non-zero divisor of the module $\bigoplus_{i \geqslant c} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ for some c. This means that $\mathfrak{m}^c \cap (\mathfrak{m}^{i+1}:z) = \mathfrak{m}^i$ for all $i \geqslant c$. On the other hand, by the Artin-Rees Lemma, there is an integer j such that $\mathfrak{m}^i:z \subset \mathfrak{m}^{i-j} + (0:z)$ for all $i \geqslant j$. Thus if $\mathrm{depth}\,R > 0$ then $0:z = 0$ and $\mathfrak{m}^{i+1}:z = \mathfrak{m}^i$ for all $i \geqslant j+c$. Conversely assume that $\mathfrak{m}^{i+1}:z = \mathfrak{m}^i$ for $i \gg 0$. Then, by Krull's Intersection Theorem,

$$0:z = \left(\bigcap_i \mathfrak{m}^i\right):z = \bigcap_i (\mathfrak{m}^i:z) = \bigcap_i \mathfrak{m}^{i-1} = 0.$$

4. *The Betti numbers of certain Borel stable ideals*

Let $R = k[z_1, \ldots, z_n]$ be a polynomial ring over a field k of characteristic 0. An ideal $\mathfrak{a}$ is called Borel stable if it is generated by monomials and if for any monomial $f = z_1^{a_1} \ldots z_n^{a_n}$ in $\mathfrak{a}$ and for any pair of indices $i < j$ with $a_j > 0$ we have $z_i f/z_j \in \mathfrak{a}$. It is

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easy to see that a Borel stable ideal is $\mathfrak{m}$-full. In fact if $\mathfrak{a}$ is Borel stable we have $\mathfrak{m}\mathfrak{a}:z_n = \mathfrak{a}$. Of course $\mathfrak{m}$ is now the irrelevant maximal ideal, and we have moved to the graded analogue of the theory. Now set

$$R_i = R/(z_{i+1}, z_{i+2}, \ldots, z_n)R, \quad (i = 0, \ldots, n-1),$$
$$R_n = R,$$
$$t_i = \mu(\mathfrak{a}R_{i+1}) - \mu(\mathfrak{a}R_i) \quad (i = 0, \ldots, n-1).$$

(We have $\mu(\mathfrak{a}R_0) = 0$ because $R_0 = k$ and $\mathfrak{a}R_0 = 0$. So $t_0 = \mu(\mathfrak{a}R_1) = 1$.) Notice that if $\mathfrak{a}$ is Borel stable then $\mathfrak{a}R_i$ is $\mathfrak{m}$-full for all i, as was proved in [**10**], example 3. In [**10**] such an ideal is called completely $\mathfrak{m}$-full. For such an ideal we will call the sequence $t_0, \ldots, t_{n-1}$ the t-sequence of $\mathfrak{a}$. Notice that $t_i = \tau(\mathfrak{a}R_{i+1})$ by Proposition 1·3. ('t' stands for type.)

THEOREM 4·1. *Suppose that $\mathfrak{a} \subset k[z_1, \ldots, z_n]$ is a Borel stable ideal generated by monomials of the same degree d and the t-sequence of $\mathfrak{a}$ is $t_0, \ldots, t_{n-1}$. Then $t_0 = 1$, and $t_{i+1} \leqslant t_i^{\langle i \rangle}$ for $i = 1, \ldots, n-2$. Hence it is the Hilbert series of some homogeneous Artin algebra.*

Proof. It is easy to see that $\mathfrak{m}^{d+1} \cap \mathfrak{a} = \mathfrak{m}\mathfrak{a}$. If $n \leqslant 2$ there is nothing to prove. Assume $n \geqslant 3$. Then the first assertion follows from Proposition 3·2. The second assertion is Macaulay's theorem.

Remark. For a given t-sequence of a Borel stable ideal in $R = k[z_1, \ldots, z_n]$, define the polynomial in λ by $T(\lambda) = \sum_{i=0}^{n-1} t_i \lambda^i$. Let $b_0, \ldots, b_{n-1}$ be the integers which satisfy $T(\lambda+1) = \sum_{i=0}^{n-1} b_i \lambda^i$. Then it was proved in [**10**] that $b_0, \ldots, b_{n-1}$ are the Betti numbers of the ideal $\mathfrak{a}$, i.e., $b_i = \dim \operatorname{Tor}_i^R(k, \mathfrak{a})$. On the other hand if we define the numbers $f_{-1}, f_0, \ldots, f_{n-2}$ by $\check{T}(\lambda+1) = \sum_{i=0}^{n-1} f_{n-2+i} \lambda^i$, where $\check{T}(\lambda) = \sum_{i=0}^{n-1} t_{n-1+i} \lambda^i$, then it is known that the sequence $f_{-1}, f_0, \ldots, f_{n-2}$ is the f-vector of some Cohen-Macaulay simplicial complex. See Stanley [**8**].

Next we want to prove the converse of the above Theorem. We fix some notation first. Let $k[i]_d$ be the set of all monomials of degree d in $k[z_0, \ldots, z_i]$. We fix the lexicographic order in the set $k[i]_d$ so the first few terms proceed as follows:

$$z_0^d, z_0^{d-1} z_1, z_0^{d-1} z_2, \ldots, z_0^{d-1} z_i z_0^{d-2} z_1^2, \ldots .$$

Let $t_0, t_1, \ldots, t_n$ and d be integers and let I_i be the subset of $k[i]_{d-1}$ consisting of the first t_i terms. Put $I = \bigcup_{i=0}^{n} I_i . z_i$.

THEOREM 4·2. *With the same notation as above assume that the integers satisfy $t_0 = 1$, $t_{i+1} \leqslant t_i^{\langle i \rangle}$ for $i = 1, \ldots, n-1$, and $d \geqslant t_1$. Let $R = k[z_0, \ldots, z_n]$ and let $\mathfrak{a}$ be the ideal of R generated by I which is defined above. Then $\mathfrak{a}$ is a Borel stable ideal and its t-sequence is $t_0, t_1, \ldots, t_n$.*

The proof will be given at the end of the next lemma.

Suppose that s is a positive integer. Write

$$s = \binom{s_i}{i} + \binom{s_{i-1}}{i-1} + \ldots + \binom{s_j}{j}, \quad s_i > s_{i-1} > \ldots > s_j \geqslant j \geqslant 1.$$

Then we define $s_{\langle i \rangle}$ by

$$s_{\langle i \rangle} = \binom{s_{i-1}}{i-1} + \binom{s_{i-1}-2}{i-2} + \ldots + \binom{s_j - 1}{j-1}.$$

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Obviously $(s^{\langle i-1\rangle})_{\langle i\rangle} = s$. With this notation we have

LEMMA 4·3. *With the same notation as above let* $f \in k[n-1]_{d-1}$. *Then we may also consider* $f \in k[n]_{d-1}$. *Suppose that* f *is the* s*th monomial of* $k[n-1]_{d-1}$, *and is the* t*th monomial of* $k[n]_{d-1}$. *Then* s *and* t *are related by*

$$s = t_{\langle n\rangle}.$$

Remark. It should be noted that the same notation $s_{\langle i\rangle}$ is used in [2] and [3], but the definition here is slightly different.

Proof of Lemma 4·3. We use the same argument as in the proof of Sublemma 2·1 (a). First we note that the assertion is true if either $n = 1$ or $d = 2$. Assume $n \geqslant 2$ and $d \geqslant 3$. If f is divisible by z_0, then use induction on d applied to f/z_0.

If f does not involve z_0, this means

$$t > \# k[n]_{d-2} = \binom{n+d-2}{n}.$$

Then use induction on n applied to

$$t - \binom{n+d-2}{n}.$$

Proof of Theorem 4·2. It suffices to prove that $\mathfrak{a}$ is Borel stable. We have to show that if $f \in \mathfrak{a}$ and $z_j | f$, then $(z_i/z_j)f \in \mathfrak{a}$ for $i < j$. We have to check this only for generators of $\mathfrak{a}$. If f does not involve z_n then we can use induction on n. Hence we assume that $f \in I_n . z_n$. If $(z_i/z_j)f$ is divisible by z_n, which occurs if and only if $z_n^2 | f$ or $j < n$, then we have $(1/z_n)(z_i/z_j)f < (1/z_n)f$ in $k[n]_{d-1}$. Hence by definition we have $(z_i/z_j)f \in \mathfrak{a}$. Thus it suffices to treat the case when $j = n$ and z_n divides f only once. Moreover, arguing that $(z_i/z_{n-1})(z_{n-1}/z_n)f = (z_i/z_n)f$, if necessary, we may assume $i = n-1$. Now what we have to show is that if $(1/z_n)f$ is the sth element in I_{n-1} then $s \leqslant t_{n-1}$. By Lemma 4·3, it follows that $s = (t_n)_{\langle n\rangle} \leqslant (t_{n-1}^{\langle n-1\rangle})_{\langle n\rangle} = t_{n-1}$. ∎

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