Characterizing Common Cause Closed Probability Spaces*

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A probability space is common cause closed if it contains a Reichenbachian common cause of every correlation in it and common cause incomplete otherwise. It is shown that a probability space is common cause incomplete if and only if it contains more than one atom and that every space is common cause completable. The implications of these results for Reichenbach's Common Cause Principle are discussed, and it is argued that the principle is only falsifiable if conditions on the common cause are imposed that go beyond the requirements formulated by Reichenbach in the definition of common cause.

1. The Problem of Common Cause Closedness: Informal Definition and Article Overview. The aim of this article is to provide new results on common cause closedness and common cause completability of classical, Kolmogorovian probability measure spaces. The propositions presented here give a complete characterization of common cause closed classical probability spaces and show that classical probability measure spaces are common cause completable in general in the very strong sense of being embeddable into a common cause closed probability space. These results solve several open problems that have been formulated in earlier papers, especially in Hofer-Szabó, Rédei, and Szabó (1999, 2000) and Gyenis and Rédei (2004).

Informally, a classical probability space (X, S, p) (with the set X of

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elementary events; S, a Boolean algebra of some subsets of X; and $p{:}S \to [0, 1]$, a probability measure) is *common cause closed* (or common cause *complete*) if for every pair of events (a, b) that are correlated in p, there exists an event $c \in S$ that is a (Reichenbachian) common cause of the correlation. The latter means that c satisfies the conditions in definition 1. If S contains correlated events a, b for which there exists no common cause in S, then (X, S, p) is called common cause *incomplete*.

It is known that common cause incomplete probability spaces exist, and it also is known that common cause closedness, while nontrivial, is not an impossible feature of probability spaces: it was shown (Gyenis and Rédei 2004, proposition 2) that no probability space with a Boolean algebra of finite cardinality can be common cause closed and that purely nonatomic probability spaces are common cause closed (Gyenis and Rédei 2004, proposition 7; see definition 2 for the measure-theoretic meaning of nonatomicity). Since probability spaces with a Boolean algebra of finite cardinality are atomic, these results indicate that the (lack of) atomicity of (X, S, p) is directly relevant from the perspective of common cause closedness of (X, S, p); however, the precise nature of the role of atomicity remained unclear in Gyenis and Rédei (2004). In this article, we clarify this role by showing (theorem 1) that the probability space (X, S, p) is common cause closed if and only if there is at most one atom in it.

If a probability space (X, S, p) is common cause incomplete, then the question arises whether it can be common cause completed. Common cause completion of (X, S, p) with respect to a set $\{(a_i, b_i) : i \in I\}$ of pairs of correlated elements is met by finding an extension (X', S', p') of (X, S, p) (definition 5) such that (X', S', p') contains a common cause $c_i \in S'$ for every correlated pair (a_i, b_i), $i \in I$. It was shown in Hofer-Szabó et al. (1999) that every common cause incomplete probability space is common cause completable with respect to any *finite* set of correlations. It remained open in Hofer-Szabó et al. (1999), however, whether common cause completability holds with respect to an *infinite* set of correlations as well. We prove in this article that this is the case. In fact, we prove much more: it will be shown that every common cause incomplete probability space has an extension that is common cause closed.

Common cause (in)completeness of probability spaces is relevant from the perspective of what is called Reichenbach's Common Cause Principle: according to this principle, which goes back to Reichenbach's work (1956) in which the notion of common cause is defined explicitly for the first time, correlations are signs of causal relations. Put more precisely, the principle asserts that if two events are correlated, then either it is the case that the correlation is due to a direct casual link between the correlated events or, if a direct causal connection between the correlated entities is excluded for some reason, there exists a common cause that explains the

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correlation. The Common Cause Principle has been extensively discussed in the literature (see the references in sec. 5); in particular, it has been debated whether the principle holds in nature. Common cause (in)completeness and common cause completability have ramifications for how Reichenbach's Common Cause Principle can(not) be falsified, and in section 5 we will discuss how the presented technical results on common cause closedness and completability contribute to the further specification of the conditions under which such a falsification is feasible.

The structure of the article is as follows: in section 2, we give the basic definitions and fix the notation. Section 3 characterizes common cause closedness in terms of atomicity. Section 4 presents the general common cause completability result. This result is based on a construction that provides extensions of classical probability spaces. Since the existence of extensions of probability spaces is important for other problems in philosophy of science involving probability theory (e.g., in Bayesianism), this construction has some interest in its own right, but it is somewhat technical and lengthy. The technical details (including the proofs of the mathematical claims involved in the construction) are thus just sketched in this article, with the emphasis put on the features of the construction that are crucial from the perspective of the general common cause completability result.

2. Notation, Definitions. In what follows, (X, S, p) denotes a classical probability measure space (*probability space*, for short), where X is a set, S is a Boolean algebra of some subsets of X, and p is a σ-additive measure on S. Typically, and throughout this article, S is assumed to be a σ algebra. Whenever it is not important on which set X the σ algebra S is represented, we write (S, p) and call (S, p) a probability space.

Given a classical probability measure space, events $a, b \in S$ are called correlated in p if

$$\mathrm{Corr}_p(a, b) \doteq p(a \wedge b) - p(a)p(b) > 0. \tag{1}$$

Definition 1. Event $c \in S$ is said to be the *common cause of the correlation* (1) if the following (independent) conditions hold:

$$p(a \wedge b|c) = p(a|c)p(b|c), \tag{2}$$

$$p(a \wedge b|c^{\perp}) = p(a|c^{\perp})p(b|c^{\perp}), \tag{3}$$

$$p(a|c) > p(a|c^{\perp}), \tag{4}$$

$$p(b|c) > p(b|c^{\perp}), \tag{5}$$

where $p(x|y) = p(x \wedge y)/p(y)$ denotes the conditional probability of

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x on condition y, and it is assumed that none of the probabilities $p(x)$, $x = a, b, c, c^{\perp}$ are equal to zero.

Note that by taking c to be either a or b, conditions (2)–(5) are satisfied, so, formally, both a and b are common causes of the correlation (1) between a and b; intuitively, however, such a "common cause" is not a "proper" common cause. A common cause is *proper* if it differs from both a and b by more than a measure-zero set (see the remarks below about treating measure-zero sets). In what follows, a common cause will always mean a proper common cause.

In connection with every problem involving measure spaces, one has to deal with measure-zero sets. Since we have the Common Cause Principle in mind, especially its status in physics, and since from a physical point of view, measure-zero sets are frequently irrelevant, we want to exclude measure-zero sets. One way of doing this is to assume that p is *faithful* in the sense that $p(a) = 0$ ($a \in S$) entails $a = \emptyset$. Faithfulness of p can always be achieved by passing from (X, S, p) to the corresponding *probability algebra*: Denoting by $\mathcal{N}$ the σ ideal of S consisting of measure-zero elements of S, one can factorize S by $\mathcal{N}$ and extend p from S to the quotient Boolean algebra $\hat{S} \doteq S/\mathcal{N}$ on which the extended $\hat{p}$ is a faithful probability measure (see Halmos 1950, 166, for details).

Note that $a, b \in S$ are (positively) correlated if and only if their representatives $\hat{a}, \hat{b}$ in the corresponding probability algebra $\hat{S}$ are correlated; moreover, $c \in S$ is a common cause of the correlation between a and b if and only if $\hat{c}$ in $\hat{S}$ is a common cause of correlation between $\hat{a}$ and $\hat{b}$. This means that investigating a probability space and its corresponding probability algebra (with respect to common cause) are equivalent. This allows us to work with the probability algebra instead of the concrete probability space. A further advantage of working in the probability algebra rather than in the probability space is that the notions of an atom in the algebraic and the measure theoretic sense coincide in a probability algebra:

Definition 2. Let (S, p) be a probability space with $\leq$ as the nonstrict (and $<$ as the corresponding strict) partial order arising from the Boolean lattice operations in S.

1. Element $a \in S$ is said to be a p *atom* if $p(a) > 0$, and for all $b < a$ we have $p(b) = p(a)$ or $p(b) = 0$.
2. The space (S, p) is *purely atomic* if for every $a \in A$ with $p(a) \neq 0$ there exists a p atom $b \in A$ such that $b \leq a$.
3. The space is *purely nonatomic* if there are no p atoms in it; that is, for all $a \in A$ with $p(a) \neq 0$, there exists $b < a$ such that $0 < p(b) < p(a)$.

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Note that p atoms and algebraic atoms are different notions in general (see examples below). However, in the probability algebra, these notions coincide: if a is a p atom and $b < a$ with $p(b) = p(a)$, then a and b are equal modulo measure zero. Hence, in the corresponding probability algebra they are the same, so it follows that in a probability algebra every p atom is also an algebraic atom. (The converse is obvious.) Moreover, a is a p atom in (S, p) if and only if $\hat{a}$ is a $\hat{p}$ atom in $(S/\mathcal{N}, \hat{p})$. This motivates the following definition:

> **Definition 3**. For a probability space (S, p), the number of p atoms in (S, p) is the cardinality of the set of $\hat{p}$ atoms in the corresponding probability algebra $(S/\mathcal{N}, \hat{p})$ (which is equal to the cardinality of the set of algebraic atoms in $\hat{S}$).

We also note that every measure space decomposes into purely nonatomic and purely atomic parts, as do probability measure spaces: for any probability measure space (X, S, p), there exist mutually singular probability measures p_0 and p_1 such that p is a convex combination of p_0 and p_1, and the spaces (X, S, p_0) and (X, S, p_1) are purely nonatomic and purely atomic, respectively. Two measures p_0 and p_1 are mutually singular by definition if there exists $Z \subset X$ such that for all $a \in S$ we have $(Z \cap a) \in S$, $(Z^{\perp} \cap a) \in S$, and $p_0(Z \cap a) = p_1(Z^{\perp} \cap a) = 0$ (see Halmos 1950, 126).

Examples are as follows:

1. Every finite probability space is purely atomic.
2. Let $([0, 1], \mathfrak{L}, \lambda)$ denote the space of Lebesgue-measurable subsets of the interval $[0, 1]$ with the Lebesgue measure λ. This probability space is purely nonatomic; note, however, that the one-element sets $\{x\}$ $(x \in [0, 1])$ are algebraic atoms in $\mathfrak{L}$, all having measure zero.
3. If $r \in (0, 1)$ and $x \in [0, 1]$ are fixed numbers, and p' is the probability space on $\mathfrak{L}$ defined by

$$p' \doteq (1 - r)\lambda + r\delta_{\{x\}},$$

where $\delta_{\{x\}}$ is the Dirac measure concentrated on $\{x\}$, then the probability space $([0, 1], \mathfrak{L}, p')$ has exactly one atom, which is $\{x\}$. Thus, it is not purely nonatomic and not purely atomic.

From now on, we assume that p in the probability space (S, p) is faithful (equivalently, that (S, p) is a probability algebra).

3. Characterization of Common Cause Closed Spaces.

> **Definition 4**. The probability space (X, S, p) is *common cause incomplete* if there exist events $a, b \in S$ such that a, b are correlated in p but S does not contain a common cause of the correlation. The space

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(X, S, p) is *common cause closed* (*complete*) if for every pair of correlated events $a, b \in S$ there exists a common cause $c \in S$.

There exist trivially common cause closed spaces: the ones that do not contain any correlated events (e.g., if p is the probability measure concentrated at one single point in X, then the probability space is trivially common cause closed). Since we are interested in nontrivial common cause closedness, in what follows, "common cause closed" will mean "nontrivially common cause closed."

Both common cause incomplete and common cause closed probability spaces exist (Gyenis and Rédei 2004); however, a general characterization of common cause (in)complete spaces was not given in Gyenis and Rédei (2004). Our aim in this section is to give such a characterization in terms of the atomic structure of the probability spaces. The first result is as follows:

Proposition 1. Let (S, p) be a probability algebra with two distinct atoms. Then (S, p) is not common cause closed.

In the proof, we make use of the following lemma:

Lemma 1 (Gyenis and Rédei 2004, lemma 1). Let (S, p) be arbitrary and assume that $a, b \in S$ are (positively) correlated. Then c cannot be a proper common cause of the correlation between a and b if any of the following conditions hold: (i) $c \leq a \wedge b^{\perp}$, (ii) $a \wedge b = c \wedge b$ and $a \leq c$, (iii) $c = a \wedge b$, (iv) $c = a \vee b$, or (v) $a \leq b \leq c$.

Proof (of proposition 1). Let $a, b \in S$ be two atoms. We may assume that $p(a \vee b) \neq 1$ because otherwise we are dealing with essentially the case of the Boolean algebra generated by two atoms, which is not common cause closed. Thus, a and $a \vee b$ are correlated because

$$p(a \wedge (a \vee b)) = p(a) > p(a)p(a \vee b)$$

and $p(a \vee b) < 1$. We claim that this correlated pair has no common cause in S. By i of lemma 1 (applying to the pair a and $a \vee b$), if c is a common cause then $c \not\leq a^{\perp}$. Hence, $c > a$ ($c < a$ is impossible because a is an atom). Consequently, c is of form $a \vee d$ for some $d \in S$. Since b is an atom, there are two cases, depending on whether $d \wedge b = 0$ or $d > b$.

Case 1. Suppose $d \wedge b = 0$. Then $a \wedge (a \vee b) = c \wedge (a \vee b)$, which contradicts ii of lemma 1.
Case 2. Suppose $d > b$. Then $a \leq a \vee b \leq c$, which contradicts v of lemma 1.

QED

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Our next aim is to prove that any probability algebra having at most one atom is common cause closed (proposition 2). In order to do this, we make use of the following lemmas and facts. First, observe that if $x, y \in [0, 1]$, then the inequality

$$x - xy + y \leq 1 \tag{6}$$

holds. One can see this by rearranging terms as $(1 - x)(1 - y) \geq 0$.

Lemma 2 (Gyenis and Rédei 2004, lemma 4). Let (S, p) be a probability algebra and $a, b \in S$ be two correlated events such that $a \not\leq b$ and $b \not\leq a$. Then $c \in S$ is a common cause of the correlation between a and b if one of the following two conditions holds.

1. $c \leq (a \wedge b)$ and

$$p(c) = \frac{p(a \wedge b) - p(a)p(b)}{1 + p(a \wedge b) - (p(a) + p(b))}. \tag{7}$$

2. $(a \vee b) \leq c$ and

$$p(c) = \frac{p(a)p(b)}{p(a \wedge b)}. \tag{8}$$

Recall that every purely nonatomic probability algebra (space) possesses the *denseness property* (see, e.g., Halmos 1950); that is, if (S, p) is purely nonatomic, then what follows is true for all $\gamma \in (0, 1)$ and $a \in S$:

$$\text{if } 0 \leq \gamma \leq p(a), \text{ then there exists } b \leq a \text{ such that } p(b) = \gamma. \tag{9}$$

Replacing a and b with their complements, we get the following equivalent (dual) form of the denseness property:

$$\text{if } p(a) \leq \gamma \leq 1, \text{ then there exists } b \geq a \text{ such that } p(b) = \gamma. \tag{10}$$

If the space is not purely nonatomic, then denseness holds in a limited sense:

Lemma 3. Let (S, p) be a probability algebra with exactly one atom, and let

$$p = \alpha p_1 + (1 - \alpha)p_2 \tag{11}$$

be the decomposition of p, with p_1 and p_2 being mutually singular probability measures that are purely nonatomic and purely atomic, respectively. Then the following denseness properties hold:

For $x \in S$ and $\gamma \in (0, 1)$, if either a or b below is fulfilled, then there exists $y \leq x$ with $p(y) = \gamma$.

a) $p_1(x) \neq 0$, $p_2(x) = 0$, and $0 \leq \gamma \leq p(x)$.
b) $p_1(x) \neq 0$, $p_2(x) \neq 0$, and $1 - \alpha \leq \gamma \leq p(x)$.

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Dually, for $x \in S$ and $\gamma \in (0, 1)$, if one of a' or b' below is fulfilled, then there exists $y \geq x$ with $p(y) = \gamma$.

a') $p_1(x) \neq 1$, $p_2(x) = 1$, and $p(x) \leq \gamma \leq 1$.
b') $p_1(x) \neq 1$, $p_2(x) \neq 1$, and $p(x) \leq \gamma \leq \alpha$.

Proof. By replacing x, y, and γ with $x^\perp$, $y^\perp$, and $1 - \gamma$, it is easy to see that a' and b' are equivalent to a and b. Therefore, it is enough to prove a and b.

Suppose $p_1(x) \neq 0$, $p_2(x) = 0$, and $0 \leq \gamma < p(x)$. Then, by definition, $0 \leq \gamma < \alpha p_1(x) + (1 - \alpha)p_2(x) = \alpha p_1(x)$. Hence $0 \leq \gamma/\alpha < p_1(x)$, and by the pure nonatomicity of p_1 and the denseness property (9), we can conclude that there is some $y < x$ such that $p_1(y) = \gamma/\alpha$. Then, $p(y) = \alpha p_1(y) + (1 - \alpha)p_2(y) = \gamma$, as desired.

To prove b, suppose $p_1(x) \neq 0$, $p_2(x) \neq 0$, and $1 - \alpha \leq \gamma < p(x)$. Then,

$$0 \leq \frac{\gamma - (1 - \alpha)}{\alpha} < p_1(x).$$

Again by (9), there exists $y_0 < x$ such that $p_1(y) = [\gamma - (1 - \alpha)]/\alpha$. If we denote the unique atom of (S, p) by q (thus, $p_1(q) = 0$ and $p_2(q) = 1$), then $y \doteq y_0 \vee q$, as desired. QED

Lemma 4. Let (S, p) be a probability algebra, and suppose $a, b \in S$ are correlated events. Then, the following inequalities hold:

$$p(a \vee b) \neq 1, \tag{12}$$

$$p(a \wedge b) \neq 0, \tag{13}$$

$$p(a \vee b) \leq \frac{p(a)p(b)}{p(a \wedge b)}, \tag{14}$$

$$p(a \wedge b) \geq \frac{p(a \wedge b) - p(a)p(b)}{1 + p(a \wedge b) - (p(a) + p(b))}. \tag{15}$$

Proof. Equation (13) follows directly from the definition of correlation. To see (14), note that the following inequality holds:

$$p(a \vee b) \leq p(a) + p(b) \leq p(a)p(b) \leq \frac{p(a)p(b)}{p(a \wedge b)},$$

where $p(a) + p(b) \leq p(a)p(b)$ follows from (6). To see (12), suppose, seeking a contradiction, that $p(a \vee b) = 1$. Then, by the general additivity rule,

$$p(a) + p(b) = p(a \wedge b) + p(a \vee b), \tag{16}$$

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and by the correlation inequality, we get $p(a) + p(b) - 1 > p(a)p(b)$. But this is impossible, again by (6). Finally, (15) can be obtained after rearranging terms and using (12), (14), and (6). QED

Proposition 2. Assume that the probability algebra (S, p) has at most one atom. Then (S, p) is common cause closed.

Proof. Let $p = \alpha p_1 + (1 - \alpha)p_2$ ($\alpha \in [0, 1]$) be the decomposition of p into mutually singular probability measures p_1 and p_2, which are purely nonatomic and purely atomic, respectively. Since there is exactly one atom in S (denote it by q), for all $b \in S$, we have $p_2(b) \neq 0$ if and only if $q \leq b$, and then $p_2(b) = p_2(q) = 1$.

Now let $a, b \in S$ be two correlated events (clearly $p(a), p(b) \neq 0$). We need to find a common cause $c \in S$ for this pair of events. We proceed by distinguishing cases. There are basically four cases determined by the relation between a and b, and some cases can be dealt with quickly: $a \wedge b = 0$ is impossible by (13), and by symmetry we do not have to differentiate between $a \leq b$ and $b \leq a$.

Case 1. Suppose $a \not\leq b$ and $b \not\leq a$. There are two subcases:

i) If $p_2(a \vee b) = 1$, then $p_1(a \vee b) \neq 1$ since otherwise $p(a \vee b)$ would be equal to 1, which contradicts (12). Let

$$\gamma = \frac{p(a)p(b)}{p(a \wedge b)}.$$

Then by (14), we get

$$p(a \vee b) \leq \gamma \leq 1,$$

and by a' of lemma 3, it follows that there exists $c \geq (a \vee b)$ with $p(c) = \gamma$. By 2 of lemma 2, c is a common cause.

ii) If $p_2(a \vee b) = 0$, then $p_2(a \wedge b) = 0$ as well, and hence $p_1(a \wedge b) \neq 0$ since otherwise we would get $p(a \wedge b) = 0$, which is impossible by (13). Let

$$\gamma = \frac{p(a \wedge b) - p(a)p(b)}{1 + p(a \wedge b) - (p(a) + p(b))}.$$

Then by (15), we get

$$0 \leq \gamma \leq p(a \wedge b),$$

and by a of lemma 3, it follows that there exists $c < (a \wedge b)$ with $p(c) = \gamma$. By 1 of lemma 2, c is a common cause.

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Case 2. Suppose $a \leq b$ and $p_1(b) = 0$. Then q obviously satisfies equations (2)–(5); hence, it is a common cause.
Case 3. Assume $a < b$ and $p_1(b) \neq 0$. Then any c satisfying $a \leq c \leq b$ is a common cause.

QED

Theorem 1. The probability algebra (S, p) is common cause closed if and only if there is at most one atom in it.

Proof. The proof is straightforward, by combining propositions 1 and 2. QED

4. Common Cause Closure. Given a common cause incomplete probability space, it is natural to ask whether it can be common cause completed in the sense that it can be embedded into a larger probability space that is common cause closed. To make this question precise, we need the following definitions of extension and embedding:

Definition 5. The probability space (X', S', p') is called an *extension* of (X, S, p) if there exists a Boolean algebra σ-embedding h of S into S' such that

$$p(x) = p'(h(x)) \qquad \text{for all } x \in S. \tag{17}$$

If (X', S', p') is an extension of (X, S, p), then we say that (X, S, p) can be *embedded* into (X', S', p').

This definition, and in particular condition (17), implies that if (X', S', p') is an extension of (X, S, p) (with respect to the embedding h), then every single correlation $\mathrm{Corr}_p(a, b)$ in (X, S, p) is carried over intact by h into the correlation $\mathrm{Corr}_{p'}(h(a), h(b))$ in (X', S', p') because

$$p'(h(a) \cap h(b)) = p'(h(a \cap b))$$

$$= p(a \cap b) > p(a)p(b) = p'(h(a))p'(h(b)).$$

Hence, it does make sense to ask whether a correlation in (X, S, p) has a Reichenbachian common cause in the extension (X', S', p').

The main result in this section is as follows:

Theorem 2. Every probability space has a purely nonatomic common cause closed extension. In particular, every probability space is common cause completable with respect to any set of correlated events.

To prove theorem 2, all we need to show is that for any probability space (X, S, p) there exists an extension (X', S', p'), which is purely nonatomic

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since, by theorem 1, such an extension is common cause closed. The main idea and the steps for constructing such an extension are as follows.

Given two probability spaces $\mathfrak{A} = (X, \mathcal{A}, \mu)$ and $\mathfrak{B} = (Y, \mathcal{B}, \nu)$, we construct a space $\mathfrak{A} \circledast \mathfrak{B}$ in such a way that (i) $\mathfrak{A} \circledast \mathfrak{B}$ is an extension of both $\mathfrak{A}$ and $\mathfrak{B}$ and (ii) $\mathfrak{A} \circledast \mathfrak{B}$ inherits the purely nonatomic character of $\mathfrak{B}$. Taking the purely nonatomic space $([0, 1], \mathfrak{L}, \lambda)$, the product $(X, \mathcal{A}, \mu) \circledast ([0, 1], \mathfrak{L}, \lambda)$ is the required extension.

The construction of $\mathfrak{A} \circledast \mathfrak{B}$ is done in the following four steps.

Step 1. The space $\mathcal{D}$ of *simple functions* from Y to $\mathcal{A}$ (defined below) is a sub-Boolean algebra of $\mathcal{A}^Y$:

$$\mathcal{D} \doteq \{\bigvee_{i=0}^{n-1} \chi_{B_i}^{a_i} : Y = \bigsqcup_{i<n} B_i, B_i \in \mathcal{B}, a_i \in \mathcal{A}, n \in \mathbb{N}\},$$

where for $a \in \mathcal{A}$ and $B \subseteq Y$, the function χ_B^a is defined as $\chi_B^a(y) \doteq a$ if $y \in B$ and $0_{\mathcal{A}}$ otherwise. This $\mathcal{D}$ is not necessarily σ complete.

Step 2. The function $\rho(f) \doteq \int_Y \mu \circ f d\nu$ defined on $\mathcal{D}$ is a probability measure (it is σ additive on $\mathcal{D}$), and the space $(\mathcal{D}, \rho)$ is an extension of both $(\mathcal{A}, \mu)$ and $(\mathcal{B}, \nu)$ as the two embeddings $h_{\mathcal{A}}(a) \doteq \chi_Y^a$ and $h_{\mathcal{B}}(B) \doteq \chi_B^1$ show. Further, the purely nonatomic character of $(\mathcal{B}, \nu)$ is inherited by $(\mathcal{D}, \rho)$.

Step 3. Define a σ algebra Σ as

$$\Sigma \doteq \{E \subseteq \bigcup_{y \in Y} (X \times \{y\}) : \{x : (x, y) \in E\} \in \mathcal{A} \text{ for all } y \in Y\}$$

and let $\iota : \mathcal{A}^Y \to \Sigma$ be $\iota(f) \doteq \bigcup_{y \in Y} (f(y) \times \{y\})$. Then, ι is a σ isomorphism, and the space $\mathfrak{D} = (\bigcup_{y \in Y} (X \times \{y\}), \{\iota(f) : f \in \mathcal{D}\}, \rho \circ \iota^{-1})$ is a set-represented, isomorphic copy of $(\mathcal{D}, \rho)$, which is thus an extension of both $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$.

Step 4. Since $(\mathcal{D}, \rho)$ inherits the purely nonatomic structure of $(Y, \mathcal{B}, \nu)$, so does $\mathfrak{D}$. Then applying Charatéodory's extension theorem to $\mathfrak{D}$ (see, e.g., Halmos 1950, theorem A, sec. 13), we obtain a classical probability space $\mathfrak{A} \circledast \mathfrak{B}$ that extends $\mathfrak{D}$. Since the purely nonatomic character of $\mathfrak{D}$ is preserved through this extension, the resulting space is the desired extension.

We note that this construction (with some obvious modifications) seems to be applicable to certain general probability spaces as well, for example, for σ-continuous orthomodular but not distributive lattices. This is relevant from the perspective of common cause completability of quantum probability spaces, but this latter problem is not discussed in this article.

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5. The Common Cause Principle in the Light of Common Cause Completability. While the definition of common cause was given first by Reichenbach (1956), he did not formulate the Common Cause Principle explicitly. This was done later, mainly by Salmon (1978, 1980, 1984), and in the current research on causal modeling, the principle appears in the form of the Causal Markov condition (Glymour, Spirtes, and Scheines 1993; Pearl 2000). The Common Cause Principle has been extensively discussed and debated in the literature, especially by Suppes (1970), Salmon (1978, 1980, 1984), Van Fraassen (1982a, 1982b), Sober (1984, 1988, 2001, 2008), Cartwright (1987), Butterfield (1989, 2007), Spohn (1991), Arntzenius (1993), Chang and Cartwright (1993), Uffink (1999), Placek (2000a, 2000b, 2000c), Hoover (2003), and Henson (2005). There seems to be consensus that the principle is not universally valid. There are two types of arguments in favor of the claim that the principle does not hold in general:

1. Informal examples of simple, classical correlations between events that are, intuitively, causally independent and for which, again, intuitively, there should not exist a common cause. Sober's (2001) example of correlation between bread prices in England and Venetian sea levels is a typical example in this category (see also Sober 2008, sec. 3.8).
2. Arguments showing that the Einstein-Podolsky-Rosen (EPR) correlations cannot be explained by common causes.

The common cause completability result in the previous section (theorem 2) shows, however, that the Common Cause Principle can always be defended by referring to "hidden common causes": a falsifying attempt based on displaying a correlation in a common cause incomplete probability space can be refuted by saying that the common cause incomplete probability space in question is just "too meager" and there exists a larger space that contains a common cause of the correlation in question. This strategy of saving the principle was available already on the basis of an earlier result stating that every common cause incomplete probability space is "locally" (i.e., with respect to a given finite set of correlations) common cause completable (Hofer-Szabó et al. 1999); however, this defense had two weaknesses.

First, one could object that the extended probability space was not necessarily common cause closed, and hence it may violate the principle for correlations that are not in the given fixed set. This counterargument against the natural hidden common cause defense of the principle against falsification is now neutralized by theorem 2 because any common cause incomplete probability space can be embedded into a common cause closed one. One also can reject the above objection against the defense

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of the principle, by arguing that from the perspective of the Common Cause Principle, what matters is only whether any particular correlation or any specific set of correlations (the ones we are interested in) has a satisfactory explanation in terms of common causes and that therefore a defense based on local common cause completability is sufficient: the status of the principle does not hinge on whether common cause closed probability spaces can be achieved. On the basis of this reasoning, theorem 2 is an overkill. But an overkill is still a kill, and the claim that any probability space containing a correlation without a common cause in it can be extended into a common cause closed one does represent a defense of the principle; this defense is just logically much stronger than the protection based on earlier local extendability results.

The other weakness is that, since earlier common cause completability results were restricted to finite sets of correlations, those results did not allow one to conclude in full generality that any specific set of correlations we might be interested in can possibly have a common cause explanation—the strong common cause extendability result (theorem 2) entitles us to make such a general claim, no matter what the cardinality of the set of correlations is for which we wish to find common causes. This means that theorem 2 adds extra strength to the kind of defense of the principle that was already available on the basis of previous local extendability results. This extra strength also can, in principle, turn out to be an overkill if one can prove that given any probability space with any set of correlations in it, there exists an extension of the space that is not necessarily common cause closed but contains a common cause of every correlation in the given set. No result of this type is known at present, however—the only known way of extending a probability space with respect to a set of correlations of arbitrary cardinality is the one ensured by theorem 2.

We wish to emphasize, however, that we do not interpret the strong common cause completability result (theorem 2) as proof that the Common Cause Principle holds. In our view, the theorem's main message is that in order to falsify the principle rigorously, it is imperative that one requires some conditions that the common causes of correlations have to satisfy in addition to the ones in Reichenbach's definition of common cause—without such additional conditions, the principle is not falsifiable.

Those additional conditions can, in principle, be of two sorts:

> **Probabilistic**. Formulated in terms of probabilities of the correlated events a, b and their common causes c and possibly involving probabilities of additional events x beyond a, b and c.
> **Algebraic**. Formulated in terms of the Boolean algebra operations between a, b, c and possibly involving additional events x beyond a, b and c.

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The additional conditions fix a certain probabilistic or algebraic type of the common cause, and it is a highly nontrivial problem whether a common cause incomplete probability space is common cause completable in such a way that the extension of the space contains common causes of a certain fixed type of the correlations in the common cause incomplete space. A technically more explicit analysis of the (lack of) common cause of the correlation between bread prices in Britain and Venetian sea levels would thus require the specification of the particular type of common cause that is claimed to be absent in this case. Whether a common cause closed space containing the required type of common case exists remains an open question since theorem 2 only shows that there exists a particular type of common cause in the common cause closed space, which might or might not be the type suiting the example. (One also can specify and analyze Sober's example in terms of the theory of time series, as Hoover does [2003]. Hoover's conclusion is that the example is not a counterexample to the principle.) It is to be expected that a purely nonatomic probability space, into which any probability space can be embedded, will contain various types of common causes since such a space is extremely rich in events (cf. the denseness property), but one cannot expect even such probability spaces to be unconditionally common cause closed, that is, common cause closed no matter what type of common causes one wishes to have. One can show, for instance, that one cannot in general require a common cause of a correlation to be a common cause of other correlations as well ("common causes are not *common* common causes"; see Hofer-Szabó, Rédei, and Szabó 2002, 623) because the probabilistic constraints entailed by the assumption that common causes are common common causes are not satisfiable in general. Hence, common cause extensions of the common common cause type do not always exist.

This is also the case with the EPR correlations: here, one has more than one (typically three or four) correlations, and in general they do not have a common common cause. But even if one does not require that the different correlations appearing in an EPR correlation experiment have a common common cause, one can impose additional probabilistic conditions on the common cause models of these correlations: in the EPR situation, one has information about the spatiotemporal (hence, causal) relation of the events involved, and one can use this information to motivate further probabilistic constraints on the correlated events and on the events in the hypothetical common causes. There are two sorts of extra requirements: "locality" and "no conspiracy."

"Locality" is the expression in terms of probabilistic independence conditions of the fact that the outcome of the measurement in one wing of the apparatus in the experimental setup measuring EPR correlations cannot be causally influenced by the choice of the measurement direction in

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the opposite wing of the measurement setup since the acts of measurements are causally independent (they are spacelike separated). The so-called no-conspiracy condition is intended to express that there is no direct causal dependence between the measurement choices and the hypothetical common causes. Whether the locality and no-conspiracy conditions are intuitively acceptable can be debated, but it can be shown that local and nonconspiratorial common cause explanations of the EPR correlations are not possible (see Szabó 2000; Grasshoff, Portmann, and Wüthrich 2005; Butterfield 2007; Portmann and Wüthrich 2007; Hofer-Szabó 2008, 2011, for proofs and further discussion). The significance of the strong common cause completability result in this article (theorem 2) from the perspective of the EPR correlations is that it is a "no-go" theorem on no-go theorems: it shows that the further conditions of locality and no conspiracy are absolutely indispensable in the proofs of no-go propositions stating the impossibility of common cause explanations of EPR correlations. Without these additional requirements, even any countably infinite subset of all the correlations inherent in an entangled quantum state are in principle explainable by referring to hidden causes.

Note, finally, that one can raise the problem of common cause completability and common cause closedness of nonclassical probability spaces (Hofer-Szabó et al. 1999; Rédei and Summers 2002, 2007), where the Boolean algebra is replaced by a more general orthocomplemented lattice, and the classical probability measure, by a general countably additive bounded measure. The product construction described in this article for classical probability spaces seems to be extendible to general probability spaces, and common cause closedness of nonclassical probability spaces might be established along the lines presented here.

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