CrossMark

Adv. Appl. Probab. **55,** 581–610 (2023)
*doi:*10.1017/apr.2022.43

THIN-ENDED CLUSTERS IN PERCOLATION IN $\mathbb{H}^d$

JAN CZAJKOWSKI ,* *Wrocław University of Science and Technology*

Abstract

Consider Bernoulli bond percolation on a graph nicely embedded in hyperbolic space $\mathbb{H}^d$ in such a way that it admits a transitive action by isometries of $\mathbb{H}^d$. Let p_a be the supremum of all percolation parameters such that no point at infinity of $\mathbb{H}^d$ lies in the boundary of the cluster of a fixed vertex with positive probability. Then for any parameter $p < p_a$, almost surely every percolation cluster is thin-ended, i.e. has only one-point boundaries of ends.

Keywords: Bernoulli percolation; hyperbolic space; cluster boundary; exponential decay

2020 Mathematics Subject Classification: Primary 82B43
Secondary 60K35; 20F55

1. Introduction

Consider an infinite, connected, locally finite graph G. Fix $p \in [0, 1]$ and for each edge, mark it as 'open' with probability p, independently for all edges of G. We mark as 'closed' the edges which are not declared open. The *state* of an edge is the information of whether it is open or closed. The (random) set ω of open edges of G forms, together with all the vertices of G, a random subgraph of G. We will often identify the latter with the set ω. This is a model of percolation, which we call *Bernoulli bond percolation* on G *with parameter* p. We will denote the corresponding probability measure by $\mathbb{P}_p$ to indicate the parameter p.

Among the main objects of interest in percolation theory are the connected components of the random subgraph ω, called *clusters*, and their 'sizes'. Here, 'size' can mean the number of vertices, the diameter, or some other property of a cluster which measures how 'large' it is. For example, one may ask whether there is a cluster containing infinitely many vertices in the random subgraph, with positive probability.

In this paper, all the graphs are connected, infinite, and locally finite.

Definition 1.1. We define the *critical probability* for the above percolation model on G as

$$p_c(G) := \inf\{p \in [0, 1] : \text{with positive probability } \omega \text{ has some infinite cluster}\}.$$

It follows from the Kolmogorov 0–1 law that if the graph G is connected and locally finite, then the probability $\mathbb{P}_p(\omega \text{ has some infinite cluster})$ always equals 0 or 1. Since this event is an *increasing* event (see Definition 5.3), its probability is an increasing function of p. Thus, it is 0 for $p < p_c(G)$ and 1 for $p > p_c(G)$.

Received 30 May 2020; revision received 1 July 2022.
* Postal address: Faculty of Pure and Applied Mathematics, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.
Email address: j.czajkowski@pwr.edu.pl

© The Author(s), 2023. Published by Cambridge University Press on behalf of Applied Probability Trust.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Definition 1.2. A graph G is *transitive* if its automorphism group acts transitively on the set of vertices of G. It is *quasi-transitive* if there are finitely many orbits of vertices of G under the action of its automorphism group.

If G is connected, locally finite, and quasi-transitive, then it is known from [18, Theorem 1] that the number of infinite clusters in ω is almost surely (a.s.) constant and equal to 0, 1, or ∞ (see also [16, Theorem 7.5]). Let us focus on the question of determining when this number is ∞.

Definition 1.3. The *unification probability* for the above percolation model on any graph G is the number

$$p_{\mathrm{u}}(G) := \inf\{p \in [0,1] : \text{a.s. there is a unique infinite cluster in } \omega\}.$$

Definition 1.4. A locally finite graph G is *non-amenable* if there is a constant $\Phi > 0$ such that for every non-empty finite set K of vertices of G, $|\mathrm{bd}\, K| \geq \Phi |K|$, where $\mathrm{bd}\, K$ is the set of edges of G with exactly one vertex in K (a kind of boundary of K). Otherwise, G is *amenable*.

It turns out that if a connected, transitive, locally finite graph G is amenable, then there is at most one infinite cluster a.s. for bond and site Bernoulli percolation; see [16, Theorem 7.6]. So if $p_{\mathrm{c}}(G) < p_{\mathrm{u}}(G)$, then the graph G must be non-amenable. It is an interesting question whether the converse is true.

Conjecture 1.1. ([4]) *If G is non-amenable and quasi-transitive, then $p_{\mathrm{c}}(G) < p_{\mathrm{u}}(G)$.*

There are several classes of non-amenable graphs for which the above conjecture has been established. Let us mention a few of them here. For an infinite regular tree it is actually folklore. It was shown for bond percolation on the Cartesian product of $\mathbb{Z}^d$ with an infinite regular tree of sufficiently high degree in [10]. Later, it was shown for site percolation on Cayley graphs of a wide class of Fuchsian groups in [15], and for site and bond percolation on transitive, non-amenable, planar graphs with one end in [5]. (A graph has *one end* if after throwing out any finite set of vertices, exactly one of its connected components is infinite.) These two results concern the hyperbolic plane $\mathbb{H}^2$. Similarly, the conjecture is obtained in [7] for many tiling graphs in $\mathbb{H}^3$. There is also a rather general result in [19] saying that any finitely generated non-amenable group has a Cayley graph G with $p_{\mathrm{c}}(G) < p_{\mathrm{u}}(G)$ for bond percolation. Recently a general result [12] was published which says that $p_{\mathrm{c}}(G) < p_{\mathrm{u}}(G)$ for any hyperbolic non-amenable quasi-transitive graph.

The interested reader may consult e.g. [9] and [16], which give quite wide introductions to percolation theory.

1.1. Boundaries of ends

In this paper, we consider percolation clusters on graphs 'naturally' embedded in $\mathbb{H}^d$ with $d \geq 2$. We define the *boundaries of ends* of a cluster in $\mathbb{H}^d$ as follows.

Definition 1.5. For any topological space X, by int_X and $\overline{\cdot}^X$ we mean the operations of taking the interior and the closure, respectively, in the space X.

Definition 1.6. Let X be a locally compact completely regular Hausdorff $\left(\mathrm{T}_{3\frac{1}{2}}\right)$ space.

- An *end* of a subset $C \subseteq X$ is a function e from the family of all compact subsets of X to the family of subsets of C that satisfies the following conditions:

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

- For any compact $K \subseteq X$, the set $e(K)$ is one of the connected components of $C \setminus K$.
- For $K \subseteq K' \subseteq X$—both compact—we have

$$e(K) \supseteq e(K').$$

One can think of an end of a given set $C \subseteq X$ as a 'direction' in C when escaping any compact subset of X.

Now let $\hat{X}$ be an arbitrary compactification of X. Then we make the following definitions:

- The *boundary* of $C \subseteq X$ (not to be confused with the usual topological notion of boundary) is given by

$$\partial C = \overline{C}^{\hat{X}} \setminus X.$$

- Finally, the *boundary of an end* e of $C \subseteq X$ is

$$\partial e = \bigcap_{\substack{K \subseteq X \\ K \text{ compact}}} \partial e(K).$$

We also put $\hat{C} = \overline{C}^{\hat{X}}$. Whenever we use the usual notion of boundary (taken in $\mathbb{H}^d$ by default), we denote it by bd to distinguish it from ∂ (and we do not call it the 'boundary').

We use these notions in the context of the hyperbolic space $\mathbb{H}^d$, where the underlying compactification is the compactification $\hat{\mathbb{H}}^d$ of $\mathbb{H}^d$ by its set of points at infinity (it is the same as the Gromov boundary of $\mathbb{H}^d$; see [2, Definition II.8.1] and [2, Section III.H.3]). The role of C above will be played by percolation clusters in $\mathbb{H}^d$.

Thus, $\partial\mathbb{H}^d = \hat{\mathbb{H}}^d \setminus \mathbb{H}^d$ is the set of points at infinity. If $\mathbb{H}^d$ is considered in its Poincaré disc (also called *ball*) model, $\partial\mathbb{H}^d$ is naturally identified with the boundary sphere of the Poincaré disc.

Remark 1.1. In this paper, whenever we consider a subset of $\mathbb{H}^d$ denoted by a longer formula e.g. of the form '$C(\dots)$', we use the notation $\hat{C}(\dots)$ for its closure in $\hat{\mathbb{H}}^d$, instead of $\widehat{C(\dots)}$, for aesthetic reasons.

Let us define a percolation threshold p_e as the supremum of percolation parameters p such that a.s. for every infinite cluster, the boundary of each of its ends is a singleton. (We say that such a cluster has *one-point boundaries of ends* or is *thin-ended* for short.) The question is whether $p_c < p_e < p_u$, e.g., for some natural tiling graphs in $\mathbb{H}^d$ for $d \geq 3$. In such a case one will have an additional percolation threshold in the non-uniqueness phase. In this paper, we give a sufficient condition for p-Bernoulli bond percolation to have only thin-ended infinite clusters, for a large class of transitive graphs embedded in $\mathbb{H}^d$. That sufficient condition is '$p < p_a$', where p_a is a threshold defined in Definition 1.8. (In Section 1.3 we will make a few remarks about how p_a may be related to the other percolation thresholds.) The key part of the proof is an adaptation of the proof of Theorem (5.4) from [9], which in turn is based on [17].

In the next section, we formulate the assumptions on the graph and the main theorem.

1.2. The graph, the sufficient condition, and the main theorem

Definition 1.7. For any graph G, let $V(G)$ denote its set of vertices and $E(G)$ its set of edges. In this paper, we often think of an $\omega \subseteq E(G)$ as a sample, called a *percolation configuration*. Accordingly, the power set $2^{E(G)}$ is the sample space for modelling Bernoulli

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

bond percolation. The accompanying measure $\mathbb{P}_p$ on it is induced by the product measure $\prod_{e\in E(G)}(p\delta_1 + (1-p)\delta_0)$ on $\{0,1\}^{E(G)}$, so $\mathbb{P}_p(\omega \ni e) = p$ for any $e \in E(G)$. In this way one can define formally the configuration ω as a set-valued random variable. For any graph G embedded in an arbitrary metric space, we call this embedded graph *transitive under isometries* if some group of isometries of the space acts on G by graph automorphisms transitively on its set of vertices.

A graph embedding in a topological space is *locally finite* if every point has a neighbourhood intersecting only finitely many vertices and edges of the embedded graph.

By a *simple* graph we mean a graph without loops or multiple edges.

Assumption 1.1. *Throughout this paper we assume that G is an infinite, connected (simple) graph embedded in $\mathbb{H}^d$, such that*

- *its edges are geodesic segments;*
- *the embedding is locally finite; and*
- *it is transitive under isometries.*

Let us also pick a vertex o (for 'origin') of G and fix it once and for all.

Note that by these assumptions, $V(G)$ is countable, every vertex of G has finite degree, and G is a closed subset of $\mathbb{H}^d$.

Definition 1.8. For $v \in V(G)$, by $C(v)$ we mean the percolation cluster of v in G. Let $\mathcal{N}(G)$ (for 'null'), or $\mathcal{N}$ for short, be defined by

$$\mathcal{N}(G) = \left\{p \in [0,1] : \left(\forall x \in \partial\mathbb{H}^d\right)\left(\mathbb{P}_p(x \in \partial C(o)) = 0\right)\right\}, \tag{1.1}$$

and put

$$p_{\mathrm{a}} = p_{\mathrm{a}}(G) = \sup \mathcal{N}(G).$$

Here the subscript 'a' stands for 'accumulation point'.

Remark 1.2. In words, $\mathcal{N}$ is the set of parameters p of Bernoulli bond percolation on G such that no point of $\partial\mathbb{H}^d$ lies in the boundary of the cluster of o with positive probability. Note that $\mathcal{N}$ is an interval (the author does not know whether it is right-open or right-closed), because the events $\{x \in \partial C(o)\}$ for $x \in \partial\mathbb{H}^d$ are all increasing, so $\mathbb{P}_p(x \in \partial C(o))$ is a non-decreasing function of p (see [9, Theorem (2.1)]). That allows us to think of p_{a} as the point of a phase transition.

We now formulate the main theorem.

Theorem 1.1. *Let G satisfy Assumption 1.1. Then, for any $0 \le p < p_{\mathrm{a}}$, a.s. every cluster in p-Bernoulli bond percolation on G is thin-ended, i.e. has only one-point boundaries of ends.*

This theorem was already known in dimension $d = 2$, when the group of the isometries preserving the graph acts cocompactly on $\mathbb{H}^2$, i.e. for vertex-transitive graphs of tilings of $\mathbb{H}^2$ by bounded hyperbolic polygons. It is proved in that generality in [6] (based on [5]), but is also easily seen in a much earlier paper, [15], for Cayley graphs of a wide class of Fuchsian groups.

The key ingredient of the proof of Theorem 1.1 is Lemma 3.1, which is a corollary of Theorem 3.2. The latter is quite interesting in its own right. These results are presented (along with a proof of Lemma 3.1) in Section 3. The elaborate proof of Theorem 3.2, rewritten from

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

the proof of Theorem (5.4) in [9], is deferred to Section 5. The proof of the main theorem, Theorem 1.1, is presented in Section 4.

1.3. Remarks on the sufficient condition

In this section, we give some remarks on the threshold p_a and on the events $\{x \in \partial C(o)\}$ (used to define $\mathcal{N}$).

Definition 1.9. For $A, B \subseteq \mathbb{H}^d$, let $A \leftrightarrow B$ be the event that there is an open path in the percolation process (given by the context) intersecting both A and B. We also say that such a path *joins* A and B. If any of the sets is of the form $\{x\}$, we write x instead of $\{x\}$ in that formula and those phrases.

Remark 1.3. The threshold p_a is bounded as follows:

$$p_c \le p_a \le p_u.$$

The inequality $p_c \le p_a$ is obvious, and the inequality $p_a \le p_u$ can be shown as follows: if p is such that $\mathbb{P}_p$-a.s. there is a unique infinite cluster in G, then with some probability $a > 0$, o belongs to the infinite cluster, and by the Harris–FKG inequality, for any $v \in V(G)$,

$$\mathbb{P}_p(o \leftrightarrow v) \ge a^2.$$

Take $x \in \partial G$. Choose a decreasing (in the sense of set inclusion) sequence $(H_n)_n$ of half-spaces such that $\bigcap_{n=1}^{\infty} \operatorname{int}_{\partial \mathbb{H}^d} \partial H_n = \{x\}$. Since $x \in \partial G$, we have $V(G) \cap H_n \neq \emptyset$ for all n. Therefore

$$\begin{aligned}\mathbb{P}_p(x \in \partial C(o)) &= \mathbb{P}_p\left(\bigcap_{n\in\mathbb{N}} \{(\exists v \in V(G) \cap H_n)(o \leftrightarrow v)\}\right) \\ &= \lim_{n\to\infty} \mathbb{P}_p((\exists v \in V(G) \cap H_n)(o \leftrightarrow v)) \ge a^2.\end{aligned}$$

Hence, $p \notin \mathcal{N}$, so $p \ge p_a$, as desired.

The main theorem (Theorem 1.1) is interesting when $p_c < p_a$. The author does not know what the class of embedded graphs G (even among those arising from Coxeter reflection groups as in [7]) satisfying $p_c(G) < p_a(G)$ is. The author suspects that $p_a = p_u$ for such graphs as in [7] in the cocompact case (see Remark 1.5; in such a case, most often we would have $p_a > p_c$). In dimension $d = 2$ the equality $p_a = p_u$ follows easily from the following fact. If such a graph embedded in $\mathbb{H}^2$ has one end, then in the middle phase of Bernoulli percolation on it we have many bi-infinite curves disjoint from the percolation subgraph and dividing $\mathbb{H}^2$ into pieces (see e.g. Subsection 2.7 in [14]). This implies exponential decay of the probability of joining o to a distant hyperbolic half-plane, which gives $\mathbb{P}_p(x \in \partial C(o)) = 0$ for any $x \in \partial \mathbb{H}^d$.

On the other hand, there are examples where $p_a < p_u$ (see Example 1.1 below).

Remark 1.4. In the assumptions of Theorem 1.1, p_a can be replaced by

$$p_a' = \sup\left\{p \in [0, 1] : g_p(r) \xrightarrow[r\to\infty]{} 0\right\}$$

with $g_p(r)$ from Definition 3.1, because only the fact that for $p < p_a$, $g_p(r) \xrightarrow[r\to\infty]{} 0$ (Proposition 5.1) is used. Accordingly, $p_a' \ge p_a$. Nevertheless, the author does not know whether it is possible that $p_a' > p_a$.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

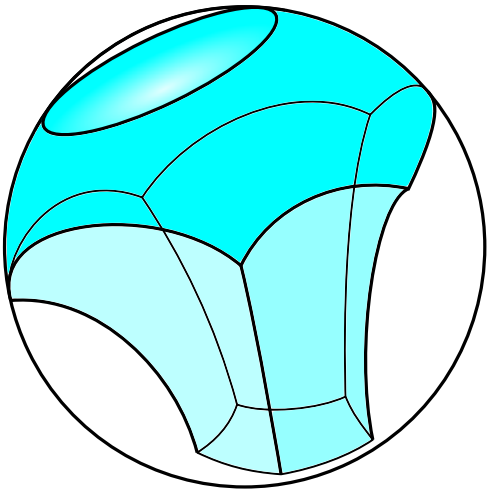


FIGURE 1. Polyhedron in Example 1.1 (shown in Poincaré disc model of $\mathbb{H}^3$).

Example 1.1. Let Π be an 'unbounded polyhedron' with six faces in $\mathbb{H}^3$, five of which are cyclically perpendicular, with the sixth one disjoint from them (see Figure 1). Then the group Γ generated by the (hyperbolic) reflections in the faces of Π is isomorphic to the free product of $\mathbb{Z}_2$ and the Coxeter group $\Gamma_5 < \mathrm{Isom}(\mathbb{H}^2)$ generated by the reflections in the sides of a right-angled pentagon in $\mathbb{H}^2$. Let G and G_5 be the Cayley graphs of Γ and Γ_5, respectively. Then G has infinitely many ends, so from [16, Exercise 7.12(b)] $p_{\mathrm{u}}(G) = 1$. Next, if $p > p_{\mathrm{u}}(G_5)$, then with positive probability $\partial C(o)$ contains the whole circle $\partial(\Gamma_5 \cdot o)$. (This is implied by Theorem 4.1 and Lemma 4.3 from [5].) Hence, $p_{\mathrm{a}}(G) \le p_{\mathrm{u}}(G_5) < p_{\mathrm{u}}(G)$, as $p_{\mathrm{u}}(G_5) < 1$ by [1, Theorem 10] (because G_5 is one-ended). Moreover, the conclusion of the main theorem (Theorem 1.1) fails for any $p > p_{\mathrm{u}}(G_5)$.

Remark 1.5. This remark is hoped to somewhat explain the suspicion that for the Cayley graph of a cocompact Coxeter reflection group in $\mathbb{H}^d$, we have $p_{\mathrm{a}} = p_{\mathrm{u}}$ (Remark 1.3). It was based on another suspicion: that for $p < p_{\mathrm{u}}$ in the same setting,

$$\mathbb{P}_p\text{-a.s.}|\partial C(o)| = 0, \tag{1.2}$$

where $|\cdot|$ is the Lebesgue measure on $\partial\mathbb{H}^d = \mathbb{S}^{d-1}$.

During the preparation of this article, the author was able to verify (1.2) for some $p > p_{\mathrm{c}}$ in the case when the group in Assumption 1.1 is a reflection group of a bounded right-angled polyhedron in $\mathbb{H}^3$ with at least 16 faces, acting faithfully on the vertex set of G. It seems that this can easily be generalized to many cocompact groups in higher dimensions. This is based on recent work by Sidoravicius, Wang, and Xiang concerning the boundary of the trace of a branching random walk on a hyperbolic group; see [20, Theorem 1.2, Remark 1.3(iii)].

If one proves (1.2), then the probability vanishing in (1.1) follows for $|\cdot|$-almost every point $x \in \partial\mathbb{H}^d$. In addition, because the induced action of such a cocompact group on $\partial\mathbb{H}^d$ has only dense orbits (see e.g. [13, Proposition 4.2]), one might suspect that in such a situation as above, $\mathbb{P}_p(x \in \partial C(o)) = 0$ holds for all $x \in \partial\mathbb{H}^d$.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

2. Definitions: percolation on a subset of $\mathbb{H}^d$

Here we are going to introduce some notions and notation used in Theorem 3.2 and Lemma 3.1 and in the proof of the main theorem.

Definition 2.1. Let us adopt the convention that the natural numbers include 0. We denote the set of all positive natural numbers by $\mathbb{N}_+$.

Definition 2.2. For the rest of this paper, consider $\mathbb{H}^d$ in its fixed half-space Poincaré model (being the upper half-space $\mathbb{R}^{d-1} \times (0, \infty)$), in which the point o (the distinguished vertex of G) is represented by $(0, \dots, 0, 1)$. (This point will play the role of the origin of both $\mathbb{H}^d$ and G.) The half-space model of $\mathbb{H}^d$ and its relation to the Poincaré ball model are explained in [2, Chapter I.6, p. 90]. Note that the inversion of $\mathbb{R}^d$ mapping the Poincaré ball model $\mathbb{B}^d$ to our fixed half-space model sends one point of the sphere bd $\mathbb{B}^d$ to infinity. In the context of the half-space model, we treat that 'infinity' as an abstract point (outside $\mathbb{R}^d$) compactifying $\mathbb{R}^d$. We call it the *point at infinity* and denote it by ∞.

Let $\widehat{\mathbb{H}}^d$ be the closure of $\mathbb{H}^d$ in $\mathbb{R}^d$, and let $\eth\mathbb{H}^d = \widehat{\mathbb{H}}^d \setminus \mathbb{H}^d$ (so here $\widehat{\mathbb{H}}^d = \mathbb{R}^{d-1} \times [0, \infty)$ and $\eth\mathbb{H}^d = \mathbb{R}^{d-1} \times \{0\}$). We identify $\widehat{\mathbb{H}}^d$ with $\hat{\mathbb{H}}^d \setminus \{\infty\}$ and $\eth\mathbb{H}^d$ with $\partial\mathbb{H}^d \setminus \{\infty\}$ in a natural way. Also, for any closed $A \subseteq \mathbb{H}^d$, let $\widehat{A} = \overline{A}^{\widehat{\mathbb{H}}^d}$ and $\eth A = \widehat{A} \setminus A$. (Here, for complex notation for a subset of $\mathbb{H}^d$ (of the form e.g. $A_x^y(z)$), we use the same notational convention for $\widehat{\cdot}$ as for $\hat{\cdot}$—see Remark 1.1.)

Although sometimes we use the linear and Euclidean structure of $\mathbb{R}^d$ in $\mathbb{H}^d$, the default geometry on $\mathbb{H}^d$ is the hyperbolic one, unless indicated otherwise. On the other hand, by the *Euclidean metric of the disc model* we mean the metric on $\hat{\mathbb{H}}^d$ induced by the embedding of $\hat{\mathbb{H}}^d$ in $\mathbb{R}^d$ (as a unit disc) arising from the Poincaré disc model of $\mathbb{H}^d$. Nevertheless, we are going to treat that metric as a metric on the set $\widehat{\mathbb{H}}^d \cup \{\infty\} = \hat{\mathbb{H}}^d$, never really considering $\mathbb{H}^d$ in the disc model.

Definition 2.3. For $k > 0$ and $x \in \mathbb{R}^{d-1} \times \{0\}$, by $y \mapsto k \cdot y$ and $y \mapsto y + x$ (or $k\cdot$, $\cdot + x$, respectively, for short) we mean always just a scaling and a translation of $\mathbb{R}^d$, respectively, often as isometries of $\mathbb{H}^d$. (Note that restricted to $\mathbb{H}^d$ they are indeed hyperbolic isometries.)

Definition 2.4. Let $\mathrm{Isom}(\mathbb{H}^d)$ denote the isometry group of $\mathbb{H}^d$.

For any $h \in (0, 1]$ and $R \in O(d)$ (the orthogonal linear group of $\mathbb{R}^d$) the pair (h, R) uniquely determines an isometry of $\mathbb{H}^d$, denoted by $\Phi^{(h,R)}$, such that $\Phi^{(h,R)}(o) = (0, \dots, 0, h)$ and $D\Phi^{(h,R)}(o) = hR$ (as an ordinary derivative of a function $\mathbb{R}^{d-1} \times (0, \infty) \to \mathbb{R}^d$).

Let $G^{(h,R)}$ denote $\Phi^{(h,R)}[G]$. Similarly, for any $\Phi \in \mathrm{Isom}(\mathbb{H}^d)$ let $G^\Phi = \Phi[G]$. Furthermore, in the same fashion, let $o^{(h,R)} = \Phi^{(h,R)}(o)$ (which is $h \cdot o$) and $o^\Phi = \Phi(o)$.

Definition 2.5. For any $p \in [0, 1]$, whenever we consider p-Bernoulli bond percolation on G^Φ for $\Phi \in \mathrm{Isom}(\mathbb{H}^d)$, we just take $\Phi[\omega]$, where ω denotes the random configuration in p-Bernoulli bond percolation on G.

Remark 2.1. One can say that this is a way of coupling the Bernoulli bond percolation processes on G^Φ for $\Phi \in \mathrm{Isom}(\mathbb{H}^d)$.

Formally, the notion of 'p-Bernoulli bond percolation on G^Φ' is not well-defined, because for different isometries Φ_1, Φ_2 of $\mathbb{H}^d$ such that $G^{\Phi_1} = G^{\Phi_2}$, the processes $\Phi_1[\omega]$ and $\Phi_2[\omega]$ are still different. Thus, we are going to use the convention that the isometry Φ used to determine the process $\Phi[\omega]$ is the same as used in the notation G^Φ determining the underlying graph.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Definition 2.6. Let $L^h = \mathbb{R}^{d-1} \times (0, h] \subseteq \mathbb{H}^d$ and put $L = L^1$. (In other words, L^h is the complement of some open horoball in $\mathbb{H}^d$, which viewed in the Poincaré disc model $\mathbb{B}^d$ is tangent to $\partial\mathbb{B}^d$ at the point corresponding to ∞.)

Definition 2.7. Consider any closed set $A \subseteq \mathbb{H}^d$ intersecting each geodesic line only in finitely many intervals and half-lines of that line (every set from the algebra of sets generated by convex sets satisfies this condition, e.g. $A = L^h$). Then by $G^\Phi \cap A$ we mean an embedded graph in A with the set of vertices consisting of $V\big(G^\Phi\big) \cap A$ and the points of intersection of the edges of G^Φ with $\operatorname{bd} A$, and with the edges being all the non-degenerate connected components of intersections of edges of G^Φ with A. The percolation process on $G^\Phi \cap A$ considered in this paper is, by default, the process $\Phi[\omega] \cap A$. The same convention as in Remark 2.1 is used for these processes.

Remark 2.2. To prove the main theorem, we use the process $\Phi^{(h,R)}[\omega] \cap L^H$ for $p \in [0, 1]$ and for different H. In some sense, it is p-Bernoulli bond percolation on $G^{(h,R)} \cap L^H$: on one hand, this process is defined in terms of the independent random states of the edges of $G^{(h,R)}$, but on the other hand, sometimes different edges of the graph $G^{(h,R)} \cap L^H$ are obtained from the same edge of $G^{(h,R)}$, so their states are stochastically dependent. Nevertheless, we are going to use some facts about Bernoulli percolation for the percolation process on $G^{(h,R)} \cap L^H$. In such a situation, we consider the edges of $G^{(h,R)}$ intersecting L^H instead of their fragments obtained in the intersection with L^H.

3. Exponential decay of the cluster radius distribution

We are going to treat the percolation process $\Phi^{(h,R)}[\omega] \cap L^H$ roughly as a Bernoulli percolation process on the standard lattice $\mathbb{Z}^{d-1}$ (given a graph structure by joining every pair of vertices from $\mathbb{Z}^{d-1}$ with distance 1 by an edge). This is motivated by the fact that $\mathbb{Z}^{d-1}$ with the graph metric is quasi-isometric to $\partial\mathbb{H}^d$ or L^H with the Euclidean metric. (Two metric spaces are *quasi-isometric* if, loosely speaking, there are mappings in both directions between them which are bi-Lipschitz up to an additive constant. For a strict definition, see [2, Definition I.8.14]; cf. also Exercise 8.16(1) there.)

In the setting of $\mathbb{Z}^{d-1}$, we have a theorem on exponential decay of the cluster radius distribution, below the critical threshold of percolation.

Theorem 3.1. ([9, Theorem (5.4)].) *For any $p < p_c\big(\mathbb{Z}^d\big)$ there exists $\psi(p) > 0$ such that in p-Bernoulli bond percolation on $\mathbb{Z}^d$*

$$\mathbb{P}_p(\textit{the origin } (0, \ldots, 0) \textit{ is connected to the sphere of radius } n) < e^{-\psi(p)n}$$

for all n, where the spheres are considered in the graph metric on $\mathbb{Z}^d$.

The idea (of a slightly more general theorem) comes from [17], where a sketch of the proof is given, and a detailed proof of the above statement is presented in [9].

We adapt the idea of this theorem to the percolation process on $G^{(h,R)} \cap L$ in Theorem 3.2 and Lemma 3.1, appropriately rewriting the proof in [9], which is going to be the key part of the proof of the main theorem. In order to consider such a counterpart of the above theorem, we define a kind of tail of all the distributions of the cluster radius in $G^{(h,R)} \cap L$ for $(h, R) \in (0, 1] \times O(d)$, as follows.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Definition 3.1. Let π be the Euclidean orthogonal projection from $\mathbb{H}^d$ onto $\eth\mathbb{H}^d$, and for any $x, y \in \mathbb{H}^d$, let

$$d_{\eth}(x, y) = \|\pi(x) - \pi(y)\|_\infty,$$

where $\|\cdot\|_\infty$ is the maximum (i.e. l^∞) norm on $\eth\mathbb{H}^d = \mathbb{R}^{d-1} \times \{0\}$. Then, for $r > 0$ and $x \in \mathbb{H}^d$, let

$$B_r(x) = \left\{y \in \mathbb{H}^d : d_{\eth}(x, y) \le r\right\} \quad \text{and} \quad S_r(x) = \operatorname{bd} B_r(x),$$

and for $h > 0$, put

$$B_r^h(x) = B_r(x) \cap L^h.$$

If $x = o$ (or, more generally, if $\pi(x) = \pi(o)$), then we omit '(x)'. Finally, for $p \in [0, 1]$ and $r > 0$, let

$$g_p(r) = \sup_{(h,R)\in(0,1]\times O(d)} \mathbb{P}_p\big(o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L\big).$$

This can be thought of as the distribution function of the 'size' (projection radius) of the cluster at the origin in the percolation process restricted to L, 'made invariant' under all isometries $\Phi^{(h,R)}$.

Remark 3.1. In the Euclidean geometry, $B_r(x)$ and $B_r^h(x)$ are just cuboids of dimensions $r \times \ldots \times r \times \infty$ (unbounded in the direction of the dth axis) and $r \times \ldots \times r \times h$, respectively (up to removal of the face lying in $\eth\mathbb{H}^d$).

The condition '$p < p_c\big(\mathbb{Z}^d\big)$' in Theorem 3.1 will be replaced by '$p < p_{\mathrm{a}}$', which is natural because of the remark below. Before stating the remark, we introduce some notation concerning the percolation clusters.

Definition 3.2. For $\Phi \in \mathrm{Isom}\big(\mathbb{H}^d\big)$ and $v \in V\big(G^\Phi\big)$ and a set $A \subseteq \mathbb{H}^d$ from the algebra generated by the convex sets, let $C^\Phi(v)$ and $C_A^\Phi(v)$ be the clusters of v in G^Φ and $G^\Phi \cap A$, respectively, in the percolation configuration. Similarly, for $(h, R) \in (0, 1] \times O(d)$ and $\Phi = \Phi^{(h,R)}$, we use the notation $C^{(h,R)}(v)$ and $C_A^{(h,R)}(v)$, respectively.

If $v = \Phi(o)$, we omit '(v)' for brevity, and if $\Phi = \mathrm{Id}$, we omit 'Φ'.

Remark 3.2. If $p \in \mathcal{N}$, then for any $\Phi \in \mathrm{Isom}\big(\mathbb{H}^d\big)$, the cluster C^Φ is $\mathbb{P}_p$-a.s. bounded in the Euclidean metric. The reason is as follows. Take any $p \in \mathcal{N}$ and $\Phi \in \mathrm{Isom}\big(\mathbb{H}^d\big)$. Then, for any $x \in \partial\mathbb{H}^d$, we have $x \notin \hat{C}(o)$ $\mathbb{P}_p$-a.s. as well as $x \notin \hat{C}^\Phi$ $\mathbb{P}_p$-a.s. If we choose $x = \infty$ (for our half-space model of $\mathbb{H}^d$), then $\hat{C}^\Phi$ is $\mathbb{P}_p$-a.s. a compact set in $\widehat{\mathbb{H}}^d$, so C^Φ is bounded in the Euclidean metric.

Now we formulate the aforementioned counterpart of Theorem 3.1. Its proof (based on that of [9, Theorem (5.4)]) is deferred to Section 5.

Theorem 3.2. (Exponential decay of $g_p(\cdot)$) *Let a graph G embedded in $\mathbb{H}^d$ satisfy the conditions in Assumption 1.1. Then, for any $p < p_{\mathrm{a}}$, there exists $\psi = \psi(p) > 0$ such that for any $r > 0$,*

$$g_p(r) \le e^{-\psi r}.$$

The next lemma is a stronger version of the above theorem, where we take the union of all the clusters meeting some $B_{r_0}^1$ instead of the cluster of $o^{(h,R)}$ in $G^{(h,R)} \cap L$. In other words, here

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

the role played by $o^{(h,R)}$ in Theorem 3.2 is taken over by its thickened version $B^1_{r_0} \cap V\big(G^\Phi\big)$ for any $\Phi \in \mathrm{Isom}\big(\mathbb{H}^d\big)$. That leads to the following notation.

Definition 3.3. For any $C \subseteq \mathbb{H}^d$, we define its *projection radius* by

$$r_{\eth}(C) = \sup_{x \in C} d_{\eth}(o, x).$$

Lemma 3.1. *Let a graph G embedded in $\mathbb{H}^d$ satisfy the conditions in Assumption 1.1. Fix an arbitrary $r_0 > 0$ and let $\mathbf{o}_\Phi = B^1_{r_0} \cap V\big(G^\Phi\big)$ for $\Phi \in \mathrm{Isom}\big(\mathbb{H}^d\big)$. Then, for any p such that the conclusion of Theorem 3.2 holds (in particular, for $p < p_{\mathrm{a}}$), there exist $\alpha = \alpha(p, r_0)$, $\varphi = \varphi(p, r_0) > 0$ such that for any $r > 0$ and $\Phi \in \mathrm{Isom}\big(\mathbb{H}^d\big)$,*

$$\mathbb{P}_p\big(\mathbf{o}_\Phi \leftrightarrow S_r \text{ in } G^\Phi \cap L\big) \le \alpha e^{-\varphi r}, \tag{3.1}$$

which is to say that for any $r > 0$

$$\sup_{\Phi \in \mathrm{Isom}\left(\mathbb{H}^d\right)} \mathbb{P}_p\left(r_{\eth}\left(\bigcup_{v \in \mathbf{o}_\Phi} C^\Phi_L(v)\right) \ge r\right) \le \alpha e^{-\varphi r}. \tag{3.2}$$

Remark 3.3. The version of the conclusion of the lemma with the inequality (3.2) is equivalent to the version with the inequality (3.1), though the probabilities involved are not necessarily equal. Still,

$$\mathbb{P}_p\big(\mathbf{o}_\Phi \leftrightarrow S_r \text{ in } G^\Phi \cap L\big) \le \mathbb{P}_p\left(r_{\eth}\left(\bigcup_{v \in \mathbf{o}_\Phi} C^\Phi_L(v)\right) \ge r\right) \le \mathbb{P}_p\big(\mathbf{o}_\Phi \leftrightarrow S_{r-\varepsilon} \text{ in } G^\Phi \cap L\big)$$

for any $r > \varepsilon > 0$, so if the continuous function $\alpha e^{-\varphi r}$ bounds one of these probabilities, then it also bounds the other.

Before we prove the lemma, let us outline the proof. The first step is the following observation.

Definition 3.4. Put $\mathbf{o} = \mathbf{o}_\Phi$. For $x \in \mathbb{H}^d \subseteq \mathbb{R}^d$, let $h(x)$ denote the dth coordinate of x (or the Euclidean distance from x to $\eth\mathbb{H}^d$), which we call the *height* of x.

Observation 3.1. There exists $H \ge 1$ such that a.s. if $\mathbf{o} \leftrightarrow S_r$ in $G^\Phi \cap L$, then there exists $v_{\mathrm{h}} \in V\big(G^\Phi\big) \cap B^1_r$ such that $v_{\mathrm{h}} \leftrightarrow S_{\frac{r-r_0}{2}}(v_{\mathrm{h}})$ in $G^\Phi \cap L^{Hh(v_{\mathrm{h}})}$ (not only in $G^\Phi \cap L$).

Then we derive the inequality

$$\mathbb{P}_p\big(\mathbf{o} \leftrightarrow S_r \text{ in } G^\Phi \cap L\big) \le \sum_{v \in V\left(G^\Phi\right) \cap B^1_r} g_p\left(\frac{r - r_0}{2Hh(v)}\right),$$

and we estimate the right-hand side using Theorem 3.2, obtaining (3.1).

We now prove the observation, then turn to proving the lemma.

Proof of the observation. Assume that $\mathbf{o} \leftrightarrow S_r$ in $G^\Phi \cap L$ (note that this event may have probability 0, e.g. when $\mathbf{o} = \emptyset$). Consider all open paths in $G^\Phi \cap L$ joining $\mathbf{o}$ to S_r, and consider all the vertices of G^Φ visited by those paths before they reach S_r starting from $\mathbf{o}$. There is a non-zero finite number of vertices of maximal height among them, because the embedding of

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

G^Φ is locally finite. Choose one of these vertices and call it v_h. We prove that v_h satisfies the condition in the observation. Take an open path P in $G^\Phi \cap L$ joining $\mathbf{o}$ to S_r passing through v_h which ends once it reaches S_r. In fact, all the vertices of P (except the last one) lie in $B_r^{h(v_\mathrm{h})}$.

Hyperbolic lengths of edges in G^Φ are bounded from above (by the transitivity of G^Φ under isometries). That implies that for any edge of G^Φ the ratio between the heights of any two of its points is also bounded from above by some constant $H \geq 1$ (this is going to be the H in the observation). The reasons for that are the following two basic properties of the half-space model of $\mathbb{H}^d$:

- The heights of points of any fixed hyperbolic ball (of finite radius) are bounded from above and from below by some positive constants.
- Any hyperbolic ball can be mapped onto any other hyperbolic ball of the same radius by a translation by a vector from $\mathbb{R}^{d-1} \times \{0\}$ composed with a linear scaling of $\mathbb{R}^d$.

That implies that the path $P \subseteq L^{Hh(v_\mathrm{h})}$.

Now, because P contains some points $x \in B_{r_0}^1$ and $y \in S_r$, and because, by the triangle inequality, $d_\partial(x, y) \geq r - r_0$, it follows that $d_\partial(v_\mathrm{h}, x)$ or $d_\partial(v_\mathrm{h}, y)$ is at least $\frac{r-r_0}{2}$ (again by the triangle inequality). Hence, P intersects $S_{\frac{r-r_0}{2}}(v_\mathrm{h})$, which finishes the proof. □

Proof of Lemma 3.1. Note that, thanks to Remark 3.3, it is sufficient to prove inequality (3.1),

$$\mathbb{P}_p(\mathbf{o}_\Phi \leftrightarrow S_r \text{ in } G^\Phi \cap L) \leq \alpha e^{-\varphi r},$$

for any $\Phi \in \mathrm{Isom}(\mathbb{H}^d)$ and $r > 0$, and for some α and φ independent of r and Φ.

Let $r > r_0$ and $\Phi \in \mathrm{Isom}(\mathbb{H}^d)$. Using Observation 3.1 with an appropriate H, we estimate

$$\begin{aligned} &\mathbb{P}_p\big(\mathbf{o} \leftrightarrow S_r \text{ in } G^\Phi \cap L\big) \qquad (*)\\ &\quad\leq \sum_{v \in V\left(G^\Phi\right) \cap B_r^1} \mathbb{P}_p\big(v \leftrightarrow S_{\frac{r-r_0}{2}}(v) \text{ in } G^\Phi \cap L^{Hh(v)}\big)\\ &\quad= \sum_{v \in V\left(G^\Phi\right) \cap B_r^1} \mathbb{P}_p\Big(\tfrac{1}{H} \cdot o \leftrightarrow S_{\frac{r-r_0}{2Hh(v)}}\Big(\tfrac{1}{H} \cdot o\Big) \text{ in } \tfrac{1}{Hh(v)}\big(G^\Phi - \pi(v)\big) \cap L^1\Big), \end{aligned}$$

by mapping the situation via the (hyperbolic) isometry $\frac{1}{Hh(v)}(\cdot - \pi(v))$ for each v. Note that because for $v \in V\left(G^\Phi\right) \cap B_r^1$, $\frac{1}{H} \cdot o$ indeed is a vertex of $\frac{1}{Hh(v)}\big(G^\Phi - \pi(v)\big)$, by the transitivity of G under isometries, we can replace the isometry $\frac{1}{Hh(v)}(\cdot - \pi(v))$ with an isometry giving the same image of G and mapping o to $\frac{1}{H} \cdot o$, hence of the form $\Phi^{(1/H,R)}$. That, combined with the assumption on p (the conclusion of Theorem 3.2), gives

$$(*) \leq \sum_{v \in V\left(G^\Phi\right) \cap B_r^1} g_p\left(\frac{r - r_0}{2Hh(v)}\right) \leq \sum_{v \in V\left(G^\Phi\right) \cap B_r^1} e^{-\psi \frac{r-r_0}{2Hh(v)}}, \tag{3.3}$$

where ψ is as in Theorem 3.2.

Because $B_r^1 = [-r, r]^{d-1} \times (0, 1]$, one can cover it by $\left\lceil \frac{r}{r_0} \right\rceil^{d-1}$ translations of $B_{r_0}^1$ by vectors from $\mathbb{R}^{d-1} \times \{0\}$. So, let

$$\left\{ B_{r_0}^1(x_i) : i = 1, \ldots, \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \right\}$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

be such a covering. Moreover, each $B^1_{r_0}(x_i)$ can be tessellated by infinitely many isometric (in the hyperbolic sense) copies of $K = B^1_{r_0} \setminus L^{\frac{1}{2}}$—more precisely, by a translation of K, 2^{d-1} translations of $\frac{1}{2}K$, $\left(2^{d-1}\right)^2$ translations of $\frac{1}{2^2}K$, etc., all along $\mathbb{R}^{d-1} \times \{0\}$. Let $U = \sup_{\varphi \in \mathrm{Isom}\left(\mathbb{H}^d\right)} \#(V\left(G^\Phi\right) \cap \varphi[K])$ (we have $U < \infty$ by Assumption 1.1). Then, splitting the sum from (3.3) according to those tessellations,

$$\begin{aligned}
(*) &\leq \sum_{i=1}^{\left\lceil \frac{r}{r_0} \right\rceil^{d-1}} \sum_{v \in V\left(G^\Phi\right) \cap B^1_{r_0}(x_i)} e^{-\psi \frac{r-r_0}{2Hh(v)}} \\
&\leq \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \sum_{k=0}^{\infty} \left(2^{d-1}\right)^k U \sup_{h \in \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right]} e^{-\psi \frac{r-r_0}{2Hh}} \\
&\leq U \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \sum_{k=0}^{\infty} \left(2^{d-1}\right)^k e^{-\frac{\psi}{H} 2^{k-1}(r-r_0)} \qquad (3.4) \\
&= U \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \sum_{k=0}^{\infty} e^{\ln 2 \cdot k(d-1) - \frac{\psi}{H} 2^{k-1}(r-r_0)}.
\end{aligned}$$

Now we are going to show that the above bound is finite and tends to 0 at an exponential rate as $r \to \infty$. First, we claim that there exists $k_0 \in \mathbb{N}$ such that

$$(\forall k \geq k_0)(\forall r \geq 2r_0) \left(\ln 2 \cdot k(d-1) - 2^{k-1} \frac{\psi}{H}(r - r_0) \leq -kr \right). \qquad (3.5)$$

Indeed, for sufficiently large k we have $2^{k-1}\frac{\psi}{H} - k > 0$, so for $r \geq 2r_0$,

$$\left(2^{k-1}\frac{\psi}{H} - k\right) r \geq \left(2^{k-1}\frac{\psi}{H} - k\right) \cdot 2r_0$$

and

$$2^{k-1}\frac{\psi}{H}(r - r_0) - kr \geq 2^{k-1}\frac{\psi}{H} r_0 - 2kr_0 \geq k(d-1)\ln 2$$

for sufficiently large k. So, let k_0 satisfy (3.5). Then, for $r \geq 2r_0$,

$$\begin{aligned}
(*) &\leq U \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \left(\sum_{k=0}^{k_0-1} \left(2^{d-1}\right)^k e^{-2^{k-1}\frac{\psi}{H}(r-r_0)} + \sum_{k=k_0}^{\infty} e^{-kr} \right) \\
&\leq U \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \left(k_0 \left(2^{d-1}\right)^{k_0-1} e^{-\frac{\psi}{2H}(r-r_0)} + e^{-k_0 r} \underbrace{\frac{1}{1-e^{-r}}}_{\leq \frac{1}{1-e^{-2r_0}}} \right) \\
&\leq U \left\lceil \frac{r}{r_0} \right\rceil^{d-1} \left(D e^{-Er} \right)
\end{aligned}$$

for some constants $D, E > 0$. If we choose $r_1 \geq 2r_0$ such that

$$(\forall r \geq r_1) \left(\left\lceil \frac{r}{r_0} \right\rceil^{d-1} \leq e^{\frac{Er}{2}} \right)$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

(which is possible), then

$$(*) \leq UDe^{-\frac{Er}{2}} \quad \text{for } r \geq r_1.$$

For $r < r_1$ we have simply $(*) \leq 1$, so picking $\varphi = -\frac{E}{2}$ and sufficiently large α, we complete the proof of the lemma. □

4. Scaling—proof of the main theorem

Now we complete the proof of the main theorem.

Theorem 4.1. (Recalled from Theorem 1.1) *Let G satisfy Assumption 1.1. Then, for any $0 \leq p < p_a$, a.s. every cluster in p-Bernoulli bond percolation on G is thin-ended, i.e. has only one-point boundaries of ends.*

Proof of Theorem 1.1. Fix $p \in [0, p_a)$ and suppose towards a contradiction that with some positive probability there is some cluster with some end with a non-one-point boundary. Note that by Remark 3.2 and by the transitivity of G under isometries, for any $v \in V(G)$ a.s. $C(v)$ is bounded in the Euclidean metric, so a.s. all the percolation clusters in G are bounded in the Euclidean metric. Then, for some $\delta > 0$ and $r > 0$, there exists with probability $a > 0$ a cluster, bounded in the Euclidean metric, with the boundary of some end having Euclidean diameter greater than or equal to δ and intersecting the open disc $\text{int}_{\partial\mathbb{H}^d}\partial B_r$. Let C and e be such a cluster and its end, respectively. For $A \subseteq \mathbb{H}^d$, let the *projection diameter* of A be the Euclidean diameter of $\pi(A)$. Then for $h > 0$,

- the set $\overline{C \backslash L^h}$ is compact;
- $e(\overline{C \backslash L^h})$ is a cluster in the percolation configuration on $G \cap L^h$;
- $e(\overline{C \backslash L^h})$ has projection diameter at least δ and intersects $B_r \cap V(G)$.

All of the above implies that for any $k \in \mathbb{N}$,

$$\mathbb{P}_p\Big(\exists\, C_{L^{1/2^k}}(v) \text{ of proj. diam. } \geq \delta, \text{ intersecting } B_r \cap V(G)\Big) \geq a,$$

so, by scaling by 2^k in $\mathbb{R}^d$ (which is a hyperbolic isometry), we obtain

$$\mathbb{P}_p\Big(\exists\, C_L^{2^k \cdot}(v) \text{ of proj. diam. } \geq 2^k\delta \text{ intersecting } B_{2^k r} \cap V\big(G^{2^k \cdot}\big)\Big) \geq a$$

(where we take the cluster in the intersection of L and $G^{2^k \cdot}$, the image of G under the scaling). The set $B_{2^k r} \cap L$ is a union of $\big(2^k\big)^{d-1}$ isometric copies of $B_r \cap L$, so the left-hand side of the above inequality is bounded from above by

$$\big(2^k\big)^{d-1} \sup_{\Phi \in \text{Isom}\left(\mathbb{H}^d\right)} \mathbb{P}_p\Big(\exists\, C_L^{\Phi}(v) \text{ of proj. diam. } \geq 2^k\delta \text{ intersecting } B_r \cap V\big(G^{\Phi}\big)\Big)$$

$$\leq \big(2^k\big)^{d-1} \sup_{\Phi \in \text{Isom}\left(\mathbb{H}^d\right)} \mathbb{P}_p\left(r_{\partial}\left(\bigcup_{v \in B_r^1 \cap V\left(G^{\Phi}\right)} C_L^{\Phi}(v)\right) \geq \frac{2^k\delta}{2}\right)$$

(because the projection radius of a cluster is at least half its projection diameter). Therefore, by Lemma 3.1, for any $k \in \mathbb{N}$,

$$a \leq \big(2^k\big)^{d-1} \alpha e^{-\varphi\delta 2^{k-1}},$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

where $\alpha, \varphi > 0$ are constants (as well as δ, a, and r). But the right-hand side of this inequality tends to 0 as $k \to \infty$, which yields a contradiction. □

5. Proof of the exponential decay

In this section, we prove Theorem 3.2.

Theorem 5.1. (Recalled from Theorem 3.2) *Let a graph G embedded in $\mathbb{H}^d$ satisfy the conditions in Assumption 1.1. Then, for any $p < p_a$, there exists $\psi = \psi(p) > 0$ such that for any $r > 0$,*

$$g_p(r) \le e^{-\psi r}.$$

Before giving the proof, we present a rough outline of it with some preliminaries. As mentioned earlier, the proof is an adaptation of the proof of Theorem (5.4) in [9], based on the work [17]. Its structure and most of its notation are also borrowed from [9], so it is quite easy to compare both proofs. (The differences are technical; they are summarized in Remark 5.2.)

We consider the following events depending only on a finite fragment of the percolation configuration; cf. Remark 5.1.

Definition 5.1. Fix an arbitrary $(h, R) \in (0, 1] \times O(d)$. Let $p \in [0, 1]$, $r > 0$, and $\delta \in (0, h]$, and define $L_\delta = \mathbb{R}^{d-1} \times [\delta, 1] \subseteq \mathbb{H}^d$ (not to be confused with L^δ). Let the event

$$A^\delta(r) = \left\{o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L_\delta\right\}$$

and let

$$f_p^\delta(r) = \mathbb{P}_p\big(A^\delta(r)\big).$$

We will use Russo's formula for the events $A^\delta(r)$. Before we formulate it, we provide a couple of definitions needed there.

Definition 5.2. For an event A in the percolation on any graph G, call an edge *pivotal* for a given configuration if and only if changing the state of that edge (and preserving the states of the other edges) causes A to change its state as well (from occurring to not occurring, or vice versa). Then let $N(A)$ be the (random) number of all edges that are pivotal for A.

Definition 5.3. We say that an event A (being a set of configurations) is *increasing* if and only if for any configurations $\omega \subseteq \omega'$, if $\omega \in A$, then $\omega' \in A$.

Theorem 5.2. (Russo's formula) *Consider Bernoulli bond percolation on any graph G, and let A be an increasing event defined in terms of the states of only finitely many edges of G. Then*

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \mathbb{E}_p(N(A)).$$

This formula is proved as Theorem (2.25) in [9] for G being the classical lattice $\mathbb{Z}^d$, but the proof applies for any graph G.

We will use Russo's formula to derive a functional inequality for $f_p^\delta(r)$ involving $\mathbb{E}_p\big(N\big(A^\delta(r)\big)|A^\delta(r)\big)$ (see (5.3)). Then we will estimate $\mathbb{E}_p\big(N\big(A^\delta(r)\big)|A^\delta(r)\big)$ from below (Lemma 5.2), looking at the cluster in $G^{(h,R)} \cap L_\delta$ joining $o^{(h,R)}$ to S_r as a 'chain of sausages' separated from each other by the pivotal edges for $A^\delta(r)$. We will compare that 'chain of sausages' to a renewal process with inter-renewal times distributed roughly as the projection radius of the cluster at the origin. For this, we would like to use random variables with left-continuous distribution function $1 - g_p$. (By the *left-continuous distribution*

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

function of a probability distribution (measure) μ on $\mathbb{R}$, we mean the function $\mathbb{R} \ni x \mapsto \mu((-\infty, x))$.) Because g_p does not need to be left-continuous, we replace it, when needed, by its left-continuous version $\tilde{g}_p$, defined as follows.

Definition 5.4. Put $\tilde{g}_p(r) = \lim_{\varrho \to r^-} g_p(\varrho)$ for $r > 0$.

In this way we will derive the following functional inequality:

$$f_\alpha^\delta(r) \le f_\beta^\delta(r) \exp\left(-(\beta - \alpha)\left(\frac{r}{a + \int_0^r \tilde{g}_\beta(m)\,\mathrm{d}m} - 1\right)\right) \tag{5.1}$$

for any $0 \le \alpha < \beta \le 1$, $r > 0$ and for $\delta \in (0, h)$, where a is a positive constant depending only on G. Then we will pass to some limits and to the supremum over (h, R), obtaining a functional inequality for $\tilde{g}_\cdot(\cdot)$: for any α, β s.t. $0 \le \alpha < \beta \le 1$ and for $r > 0$,

$$\tilde{g}_\alpha(r) \le \tilde{g}_\beta(r) \exp\left(-(\beta - \alpha)\left(\frac{r}{a + \int_0^r \tilde{g}_\beta(m)\,\mathrm{d}m} - 1\right)\right). \tag{5.2}$$

Note that this implies Theorem 3.2, provided that the integral in the denominator is a bounded function of r.

Then we arrive at a mild asymptotic estimate, $\tilde{g}_p(r) \le \delta(p)/\sqrt{r}$ (Lemma 5.3), whose proof uses the above functional inequality. That asymptotic estimate is then sharpened to that desired in Theorem 3.2, by repeatedly using the inequality (5.2).

Proof of the theorem. Let $p > 0$, $(h, R) \in (0, 1] \times O(d)$, $r > 0$, and $\delta \in (0, h]$ be fixed.

Note that if there is no path joining $o^{(h,R)}$ to S_r in $G^{(h,R)} \cap L_\delta$ at all, then for any $p \in [0, 1]$, $f_p^\delta(r) = 0$ and the inequality (5.1) is obvious. The same happens when $\alpha = 0$. Because in the proof of that inequality we need $f_p^\delta(r) > 0$ and $\alpha > 0$, we now make the following assumption (without loss of generality).

Assumption 5.1. *We assume that there is a path joining* $o^{(h,R)}$ *to* S_r *in* $G^{(h,R)} \cap L_\delta$ *and that* $\alpha > 0$. (*Then for* $p > 0$, $f_p^\delta(r) > 0$.)

The events $A^\delta(r)$ depend on the states of only finitely many edges of $G^{(h,R)}$ (namely, those intersecting $L_\delta \cap B_r$), so we are able to use Russo's formula for them, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}p} f_p^\delta(r) = \mathbb{E}_p\big(N\big(A^\delta(r)\big)\big).$$

Now, $A^\delta(r)$ is increasing, and for $e \in E\big(G^{(h,R)}\big)$, the event $\big\{e \text{ pivotal for } A^\delta(r)\big\}$ is independent of the state of e (which is easily seen; it is the rule for any event), so

$$\begin{aligned}\mathbb{P}_p\big(A^\delta(r) \wedge e \text{ is pivotal for } A^\delta(r)\big) &= \mathbb{P}_p\big(e \text{ is open and pivotal for } A^\delta(r)\big) \\ &= p\mathbb{P}_p\big(e \text{ is pivotal for } A^\delta(r)\big);\end{aligned}$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}p}f_p^\delta(r) &= \sum_{e\in E\left(G^{(h,R)}\right)} \mathbb{P}_p\left(e \text{ is pivotal for } A^\delta(r)\right)\\ &= \frac{1}{p}\sum_{e\in E\left(G^{(h,R)}\right)} \mathbb{P}_p\left(A^\delta(r)\wedge e \text{ is pivotal for } A^\delta(r)\right)\\ &= \frac{f_p^\delta(r)}{p}\sum_{e\in E\left(G^{(h,R)}\right)} \mathbb{P}_p\left(e \text{ is pivotal for } A^\delta(r)|A^\delta(r)\right)\\ &= \frac{f_p^\delta(r)}{p}\mathbb{E}_p\left(N\left(A^\delta(r)\right)|A^\delta(r)\right),\end{aligned}$$

which can be written as

$$\frac{\mathrm{d}}{\mathrm{d}p}\ln\left(f_p^\delta(r)\right) = \frac{1}{p}\mathbb{E}_p\left(N\left(A^\delta(r)\right)|A^\delta(r)\right).$$

For any $0<\alpha<\beta\le 1$, integrating over $[\alpha,\beta]$ and exponentiating the above equality gives

$$\frac{f_\alpha^\delta(r)}{f_\beta^\delta(r)} = \exp\left(-\int_\alpha^\beta \frac{1}{p}\mathbb{E}_p\left(N\left(A^\delta(r)\right)|A^\delta(r)\right)\mathrm{d}p\right),$$

which implies

$$f_\alpha^\delta(r) \le f_\beta^\delta(r)\exp\left(-\int_\alpha^\beta \mathbb{E}_p\left(N\left(A^\delta(r)\right)|A^\delta(r)\right)\mathrm{d}p\right). \tag{5.3}$$

At this point, our aim is to bound $\mathbb{E}_p\left(N\left(A^\delta(r)\right)|A^\delta(r)\right)$ from below. Fix any $r>0$ and $\delta\in(0,h]$ and assume for now that $A^\delta(r)$ occurs. Let us look at the structure of the cluster of $o^{(h,R)}$ in $G^{(h,R)}\cap L_\delta$ in the context of the pivotal edges for $A^\delta(r)$ (following [9] and [17]). If $e\in E\left(G^{(h,R)}\right)$ is pivotal for $A^\delta(r)$, then if we change the percolation configuration by closing e, we cause the cluster of $C_{L_\delta}^{(h,R)}\left(o^{(h,R)}\right)$ to be disjoint from S_r. So, in our situation, all the pivotal edges lie on any open path in $G^{(h,R)}\cap L_\delta$ joining $o^{(h,R)}$ to S_r, and they are visited by the path in the same order and direction (regardless of the choice of the path).

Definition 5.5. Assume that $A^\delta(r)$ occurs. Let $N=N\left(A^\delta(r)\right)$, and let $e_1,\ldots,e_N$ be the above ordering of the pivotal edges. Denote by x_i, y_i the end vertices of e_i, x_i being the one closer to $o^{(h,R)}$ along a path as above. Also, let $y_0=o^{(h,R)}$. By convention, whenever we mention e_i, we assume that $i\le N\left(A^\delta(r)\right)$.

Note that because there is no edge separating y_{i-1} from x_i in the open cluster in $G^{(h,R)}\cap L_\delta$ for $i=1,\ldots,N$, by Menger's theorem (see e.g. [8, Theorem 3.3.1, Corollary 3.3.5(ii)]), there exist two edge-disjoint open paths in that cluster joining y_{i-1} to x_i. (One can say, following the discoverer of this proof idea, that that open cluster resembles a chain of sausages.)

Now, for $i=1,\ldots,N$, let $\varrho_i=d_{\mathfrak{F}}(y_{i-1},x_i)$ (this way of defining ϱ_i, which one can view as the 'projection length' of the ith 'sausage', is an adaptation of the definition of ρ_i in [9]). We use the convention that for $i\in\mathbb{N}$ such that $i>N\left(A^\delta(r)\right)$ (i.e. e_i, ϱ_i are undefined), $\varrho_i=+\infty$ (being greater than any real number).

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

The next lemma is used to compare $(\varrho_1, \dots, \varrho_N)$ to some renewal process with inter-renewal times of roughly the same distribution as the projection radius of $C_L^{(h,R)}\big(o^{(h,R)}\big)$. Its proof is deferred to Subsection 5.1.

Definition 5.6. Let a denote the supremum of the projection distance (in the sense of $d_{\mathfrak{H}}$) between the endpoints of an edge, taken over all $(h, R) \in (0, 1] \times O(d)$ and all the edges of $G^{(h,R)}$ intersecting L.

Lemma 5.1. (Cf. [9, Lemma (5.12)]) *Let $k \in \mathbb{N}_+$ and let $r_1, \dots, r_k \geq 0$ be such that $\sum_{i=1}^k r_i \leq r - (k-1)a$. Then for $0 < p < 1$,*

$$\mathbb{P}_p\left(\varrho_k < r_k,\ \bigwedge_{i<k} \varrho_i = r_i \middle| A^\delta(r)\right) \geq (1 - g_p(r_k))\mathbb{P}_p\left(\bigwedge_{i<k} \varrho_i = r_i \middle| A^\delta(r)\right).$$

Now we want to do some probabilistic reasoning using random variables with the left-continuous distribution function $1 - \tilde{g}_p$. The function $1 - \tilde{g}_p$ is non-decreasing (because for $(h, R) \in (0, 1] \times O(d)$, $\mathbb{P}_p\big(o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L\big)$ is non-increasing with respect to r, so g_p and $\tilde{g}_p$ are non-increasing as well), left-continuous, with values in $[0, 1]$, and such that $1 - \tilde{g}_p(0) = 0$, so it is the left-continuous distribution function of a random variable with values in $[0, \infty]$. So let $M_1, M_2, \dots$ be an infinite sequence of independent random variables all distributed according to $1 - \tilde{g}_p$ and all independent of the whole percolation process. Because their distribution depends on p, we will also denote them by $M_1^{(p)}, M_2^{(p)}, \dots$. (Here, an abuse of notation is going to happen, as we are still writing $\mathbb{P}_p$ for the whole probability measure used also for defining the variables $M_1, M_2, \dots$.)

We can now state the following corollary of Lemma 5.1.

Corollary 5.1. *For any $r > 0$, positive integer k, and $0 < p < 1$,*

$$\mathbb{P}_p\big(\varrho_1 + \cdots + \varrho_k < r - (k-1)a | A^\delta(r)\big) \geq \mathbb{P}_p(M_1 + \cdots + M_k < r - (k-1)a).$$

This corollary is proved in Subsection 5.2 and is used to prove the following lemma in Subsection 5.3.

Lemma 5.2. (Cf. [9, Lemma (5.17)]) *For $0 < p < 1$, $r > 0$,*

$$\mathbb{E}_p\big(N\big(A^\delta(r)\big) | A^\delta(r)\big) \geq \frac{r}{a + \int_0^r \tilde{g}_p(m)\,\mathrm{d}m} - 1.$$

Now, combining Lemma 5.2 above with the inequality (5.3) for $0 < \alpha < \beta \leq 1$, we have

$$\begin{aligned} f_\alpha^\delta(r) &\leq f_\beta^\delta(r) \exp\left(-\int_\alpha^\beta \mathbb{E}_p\big(N\big(A^\delta(r)\big) | A^\delta(r)\big)\,\mathrm{d}p\right) \\ &\leq f_\beta^\delta(r) \exp\left(-\int_\alpha^\beta \left(\frac{r}{a + \int_0^r \tilde{g}_p(m)\,\mathrm{d}m} - 1\right)\,\mathrm{d}p\right) \\ &\leq f_\beta^\delta(r) \exp\left(-(\beta - \alpha)\left(\frac{r}{a + \int_0^r \tilde{g}_\beta(m)\,\mathrm{d}m} - 1\right)\right) \end{aligned} \tag{5.4}$$

(because $\tilde{g}_p \leq \tilde{g}_\beta$ for $p \leq \beta$), which completes the proof of the inequality (5.1). (Let us now drop Assumption 5.1.)

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Now, note that for any $r > 0$ and $p \in [0, 1]$, the event $A^\delta(r)$ increases as δ decreases. Thus, taking the limit with $\delta \to 0$, we have

$$\lim_{\delta \to 0^+} f_p^\delta(r) = \mathbb{P}_p\left(\bigcup_{\delta > 0} A^\delta(r) \right) = \mathbb{P}_p\left(o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L \right).$$

So for any $r > 0$ and $0 \le \alpha < \beta \le 1$, using this for the inequality (5.1) gives

$$\mathbb{P}_\alpha\left(o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L \right) \le \mathbb{P}_\beta\left(o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L \right) \exp\left(-(\beta - \alpha) \left(\frac{r}{a + \int_0^r \tilde{g}_\beta(m)\, \mathrm{d}m} - 1 \right) \right).$$

Further, we take the supremum over $(h, R) \in (0, 1] \times O(d)$, obtaining

$$g_\alpha(r) \le g_\beta(r) \exp\left(-(\beta - \alpha) \left(\frac{r}{a + \int_0^r \tilde{g}_\beta(m)\, \mathrm{d}m} - 1 \right) \right).$$

Finally, taking the limits with r from the left, we get the functional inequality (5.2) involving only $\tilde{g}_\cdot(\cdot)$:

$$\tilde{g}_\alpha(r) \le \tilde{g}_\beta(r) \exp\left(-(\beta - \alpha) \left(\frac{r}{a + \int_0^r \tilde{g}_\beta(m)\, \mathrm{d}m} - 1 \right) \right). \tag{5.5}$$

(Note that the exponent remains unchanged throughout, from (5.4) until now.)

Recall that once we have

$$\int_0^\infty \tilde{g}_\beta(m)\, \mathrm{d}m = \mathbb{E}\left(M_1^{(\beta)} \right) < \infty,$$

we then obtain Theorem 3.2 for $\tilde{g}_\alpha(r)$, for $\alpha < \beta$. This bound is going to be established by showing the rapid decay of $\tilde{g}_p$, using (5.5) repeatedly. The next lemma is the first step of this procedure.

Lemma 5.3. (Cf. [9, Lemma (5.24)]) *For any $p < p_{\mathrm{a}}$, there exists $\delta(p)$ such that*

$$\tilde{g}_p(r) \le \delta(p) \cdot \frac{1}{\sqrt{r}} \quad \textit{for } r > 0.$$

We defer the proof of the above lemma to Subsection 5.4.

It is relatively easy to obtain Theorem 3.2 (which is being proved) from Lemma 5.3. First, we deduce that for $r > 0$ and $p < p_{\mathrm{a}}$,

$$\int_0^r \tilde{g}_p(m)\, \mathrm{d}m \le 2\delta(p)\sqrt{r},$$

so if $r \ge a^2$, then

$$a + \int_0^r \tilde{g}_p(m)\, \mathrm{d}m \le (2\delta(p) + 1)\sqrt{r}.$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Then, using (5.5), for $0 \le \alpha < \beta < p_{\mathrm{a}}$ we have

$$\begin{aligned}\int_{a^2}^{\infty} \tilde{g}_\alpha(r)\,\mathrm{d}r &\le \int_{a^2}^{\infty} \exp\left(-(\beta-\alpha)\left(\frac{r}{a+\int_0^r \tilde{g}_\beta(m)\,\mathrm{d}m}-1\right)\right)\mathrm{d}r\\ &\le e\int_{a^2}^{\infty}\exp\left(-\underbrace{\frac{\beta-\alpha}{2\delta(\beta)+1}}_{=C>0}\sqrt{r}\right)\mathrm{d}r\\ &= e\int_a^{\infty} e^{-Cx}\cdot 2x\,\mathrm{d}x,\end{aligned}$$

so

$$\mathbb{E}\left(M_1^{(\alpha)}\right) = \int_0^{\infty}\tilde{g}_\alpha(r)\,\mathrm{d}r \le a^2 + e\int_a^{\infty} e^{-Cx}\cdot 2x\,\mathrm{d}x < \infty,$$

as desired. Finally, we use the finiteness of $\mathbb{E}\left(M_1^{(\alpha)}\right)$ as promised: for $r > 0$ and $0 \le \alpha < p_{\mathrm{a}}$, if we take $\alpha < \beta < p_{\mathrm{a}}$, then, using (5.5) again,

$$g_\alpha(r) \le \tilde{g}_\alpha(r) \le \exp\left(-(\beta-\alpha)\left(\frac{r}{a+\mathbb{E}\left(M_1^{(\beta)}\right)}-1\right)\right) \le e^{-\varphi(\alpha,\beta)r+\gamma(\alpha,\beta)},$$

for some constants $\varphi(\alpha, \beta), \gamma(\alpha, \beta) > 0$.

Now we perform a standard estimation, aiming to rule out the additive constant $\gamma(\alpha, \beta)$. For any $0 < \psi_1 < \varphi(\alpha, \beta)$, there exists $r_0 > 0$ such that for $r \ge r_0$,

$$-\varphi(\alpha,\beta)r + \gamma(\alpha,\beta) \le -\psi_1 r,$$

so

$$g_\alpha(r) \le e^{-\psi_1 r}.$$

On the other hand, for any $r > 0$, $g_p(r)$ is no greater than the probability of opening at least one edge adjacent to o, so $g_\alpha(r) \le 1 - (1-\alpha)^{\deg(o)} < 1$, where $\deg(o)$ is the degree of o in the graph G. Hence,

$$g_\alpha(r) \le e^{-\psi_2(\alpha)r}$$

for $r \le r_0$, for some sufficiently small $\psi_2(\alpha) > 0$. Taking $\psi = \min(\psi_1, \psi_2(\alpha))$ gives

$$g_\alpha(r) \le e^{-\psi r}$$

for any $r > 0$, completing the proof of Theorem 3.2. □

Remark 5.1. In order to prove Theorem 3.2, one could try to consider the percolation processes on the whole graph $G^{(h,R)} \cap L$ (without restricting it to L_δ) in order to obtain a functional inequality similar to (5.2), involving only one function. However, that approach caused the author many difficulties, some of which have not been overcome. Restricting the situation to L_δ makes the event A^δ depend on the states of only finitely many edges. This allows one e.g. to condition the event $A^\delta(r) \cap B$ on the family of events $\{\Gamma \text{ a witness for } B\}$ in the proof of Lemma 5.1, where Γ runs over a countable set, or to use the BK inequality and Russo's formula.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

5.1. Proof of Lemma 5.1—the chain of sausages

Definition 5.7. *(Recalled from Definition 5.6)* Let a denote the supremum of the projection distance (in the sense of $d_{\mathfrak{d}}$) between the endpoints of an edge, taken over all $(h, R) \in (0, 1] \times O(d)$ and all the edges of $G^{(h,R)}$ intersecting L.

Lemma 5.4. (Recalled from Lemma 5.1; cf. [9, Lemma (5.12)]) *Let $k \in \mathbb{N}_+$, and let $r_1, \ldots, r_k \geq 0$ be such that $\sum_{i=1}^{k} r_i \leq r - (k-1)a$. Then for $0 < p < 1$,*

$$\mathbb{P}_p\left(\varrho_k < r_k,\ \bigwedge_{i<k} \varrho_i = r_i \middle| A^\delta(r)\right) \geq (1 - g_p(r_k))\mathbb{P}_p\left(\bigwedge_{i<k} \varrho_i = r_i \middle| A^\delta(r)\right).$$

Before we start the proof, we state a few preliminaries.

Definition 5.8. For increasing events A and B in a percolation on any graph G, the event $A \circ B$ means that 'A and B occur on disjoint sets of edges'. Formally,

$$A \circ B = \{\omega_A \cup \omega_B : \omega_A, \omega_B \subseteq E(G) \wedge \omega_A \cap \omega_B = \emptyset \wedge \omega_A \in A \wedge \omega_B \in B\};$$

that is, $A \circ B$ is the set of configurations containing two disjoint sets of open edges (ω_A, ω_B above) which guarantee the occurrence of the events A and B, respectively.

Theorem 5.3. (BK inequality, [9, Theorems (2.12) and (2.15)]) *For any graph G and increasing events A and B depending on the states of only finitely many edges in p-Bernoulli bond percolation on G, we have*

$$\mathbb{P}_p(A \circ B) \leq \mathbb{P}_p(A)\mathbb{P}_p(B).$$

Recall that in the setting of the lemma to be proved we have fixed $(h, R) \in (0, 1] \times O(d)$, $p \in [0, 1]$, $r > 0$, and $\delta \in (0, h]$. We will use the following notation.

Definition 5.9. Let η denote the percolation configuration in $G^{(h,R)} \cap L_\delta$, i.e.

$$\eta = \Phi^{(h,R)}[\omega] \cap L_\delta.$$

Proof of the lemma. This proof mimics that of [9, Lemma (5.12)]. Let $k \geq 2$ (we defer the case of $k = 1$ to the end of the proof).

For $e \in E\left(G^{(h,R)} \cap L_\delta\right)$, let D_e be the connected component of $o^{(h,R)}$ in $\eta \backslash \{e\}$. Let B_e denote the event that the following conditions are satisfied:

- e is open;
- exactly one end vertex of e lies in D_e—call it $x(e)$ and call the other $y(e)$;
- D_e is disjoint from S_r;
- there are exactly $k-1$ pivotal edges for the event $\left\{o^{(h,R)} \leftrightarrow y(e) \text{ in } \eta\right\}$ (i.e. the edges each of which separates $o^{(h,R)}$ from $y(e)$ in $D_e \cup \{e\}$)—call them $e'_1 = \left\{x'_1, y'_1\right\}, \ldots, e'_{k-1} = \left\{x'_{k-1}, y'_{k-1}\right\} = e$, where x'_i is closer to $o^{(h,R)}$ than y'_i, in the order from $o^{(h,R)}$ to $y(e)$ (as in Definition 5.5);
- $d_{\mathfrak{d}}\left(y'_{i-1}, x'_i\right) = r_i$ for $i < k$, where $y'_0 = o^{(h,R)}$.

Let $B = \bigcup_{e \in E\left(G^{(h,R)} \cap L_\delta\right)} B_e$. When B_e occurs, we say that $D_e \cup \{e\}$ with $y(e)$ marked, as a graph with distinguished vertex, is a *witness for B*.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Note that it may happen that there is more than one such witness (which means that B_e occurs for many different e). On the other hand, if $A^\delta(r)$ occurs, then B_e occurs for only one edge e, namely $e = e_{k-1}$ (in other words, $B \cap A^\delta(r) = \dot{\bigcup}_{e \in E\left(G^{(h,R)} \cap L_\delta\right)} (B_e \cap A^\delta(r))$), and there is only one witness for B. Hence,

$$\mathbb{P}_p\big(A^\delta(r) \cap B\big) = \sum_\Gamma \mathbb{P}_p(\Gamma \text{ a witness for } B)\mathbb{P}_p\Big(A^\delta(r) | \Gamma \text{ a witness for } B\Big),$$

where the sum is always over all Γ that are finite subgraphs of $G^{(h,R)} \cap L_\delta$ with distinguished vertices such that $\mathbb{P}_p(\Gamma \text{ a witness for } B) > 0$.

For Γ a graph with a distinguished vertex, let $y(\Gamma)$ denote that vertex. Under the condition that Γ is a witness for B, $A^\delta(r)$ is equivalent to the event that $y(\Gamma)$ is joined to S_r by an open path in η which is disjoint from $V(\Gamma)\backslash\{y(\Gamma)\}$. For brevity, we write the latter event as $\{y(\Gamma) \leftrightarrow S_r \text{ in } \eta \text{ off } \Gamma\}$. Now, the event $\{\Gamma \text{ a witness for } B\}$ depends only on the states of edges incident to vertices from $V(\Gamma)\backslash\{y(\Gamma)\}$, so it is independent of the event $\{y(\Gamma) \leftrightarrow S_r \text{ in } \eta \text{ off } \Gamma\}$. Hence,

$$\mathbb{P}_p\big(A^\delta(r) \cap B\big) = \sum_\Gamma \mathbb{P}_p(\Gamma \text{ a witness for } B)\mathbb{P}_p(y(\Gamma) \leftrightarrow S_r \text{ in } \eta \text{ off } \Gamma). \tag{5.6}$$

A similar argument, carried out below, gives us the estimate of $\mathbb{P}_p(\{\varrho_k \geq r_k\} \cap A^\delta(r) \cap B)$. Here we also use the following fact: conditioned on the event $\{\Gamma \text{ a witness for } B\}$, the event $A^\delta(r) \cap \{\varrho_k \geq r_k\}$ implies each of the following:

$$\begin{aligned}
&\big(A^\delta(r) \wedge e_k \text{ does not exist}\big) \vee \big(A^\delta(r) \wedge e_k \text{ exists } \wedge \varrho_k \geq r_k\big)\\
\Longrightarrow &(\exists \text{ two edge-disjoint paths joining } y(\Gamma) \text{ to } S_r \text{ in } \eta \text{ off } \Gamma) \vee\\
&\vee (\exists \text{ two edge-disjoint paths in } \eta \text{ off } \Gamma,\\
&\qquad \text{joining } y(\Gamma) \text{ to } S_r \text{ and to } S_{r_k}(y(\Gamma)), \text{ respectively})\\
\Longleftrightarrow\ &(\exists \text{ two edge-disjoint paths in } \eta \text{ off } \Gamma,\\
&\quad \text{joining } y(\Gamma) \text{ to } S_r \text{ and to } S_{r_k}(y(\Gamma)), \text{ respectively}),
\end{aligned}$$

because $S_{r_k}(y(\Gamma)) \subseteq B_r$ from the assumption on $\sum_{i=1}^k r_i$. So we estimate

$$\begin{aligned}
&\mathbb{P}_p\big(\{\varrho_k \geq r_k\} \cap A^\delta(r) \cap B\big)\\
&\quad = \sum_\Gamma \mathbb{P}_p(\Gamma \text{ a witn. for } B)\mathbb{P}_p\big(\{\varrho_k \geq r_k\} \cap A^\delta(r) \big| \Gamma \text{ a witn. for } B\big)\\
&\quad \leq \sum_\Gamma \mathbb{P}_p(\Gamma \text{ a witn. for } B)\, \mathbb{P}_p((y(\Gamma) \leftrightarrow S_r \text{ in } \eta \text{ off } \Gamma)\\
&\qquad \circ (y(\Gamma) \leftrightarrow S_{r_k}(y(\Gamma)) \text{ in } \eta \text{ off } \Gamma)).
\end{aligned} \tag{5.7}$$

Now we use the BK inequality for the last term (as the events involved are increasing and defined in terms of only the edges from $E\big(G^{(h,R)} \cap (L_\delta \cap B_r)\big)$), obtaining

$$\begin{aligned}
&\mathbb{P}_p(\{\varrho_k \geq r_k\} \cap A^\delta(r) \cap B)\\
&\quad \leq \sum_\Gamma \mathbb{P}_p(\Gamma \text{ a witness for } B) \cdot \mathbb{P}_p(y(\Gamma) \leftrightarrow S_r \text{ in } \eta \text{ off } \Gamma)\\
&\qquad \cdot \mathbb{P}_p(y(\Gamma) \leftrightarrow S_{r_k}(y(\Gamma)) \text{ in } \eta \text{ off } \Gamma)
\end{aligned}$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

$$\leq \left(\sum_{\Gamma} \mathbb{P}_p(\Gamma \text{ a witness for } B) \mathbb{P}_p(y(\Gamma) \leftrightarrow S_r \text{ in } \eta \text{ off } \Gamma) \right) g_p(r_k)$$
$$= \mathbb{P}_p\big(A^\delta(r) \cap B\big) g_p(r_k)$$

(by (5.6)). Dividing by $\mathbb{P}_p\big(A^\delta(r)\big)$ (which is positive by Assumption 5.1) gives

$$\mathbb{P}_p\big(\{\varrho_k \geq r_k\} \cap B | A^\delta(r)\big) \leq \mathbb{P}_p\big(B | A^\delta(r)\big) g_p(r_k), \tag{5.8}$$

and subtracting both sides of the inequality from $\mathbb{P}_p\big(B|A^\delta(r)\big)$ gives

$$\mathbb{P}_p\big(\{\varrho_k < r_k\} \cap B | A^\delta(r)\big) \geq \mathbb{P}_p\big(B | A^\delta(r)\big)(1 - g_p(r_k)). \tag{5.9}$$

Note that, conditioned on $A^\delta(r)$, B is equivalent to the event $\{\varrho_i = r_i \text{ for } i < k\}$, so the above amounts to

$$\mathbb{P}_p\left(\varrho_k < r_k, \bigwedge_{i<k} \varrho_i = r_i \middle| A^\delta(r) \right) \geq \mathbb{P}_p\left(\bigwedge_{i<k} \varrho_i = r_i \middle| A^\delta(r) \right)(1 - g_p(r_k)),$$

which is the desired conclusion.

Now consider the case of $k = 1$. In this case, similarly to (5.7) and thanks to the assumption $r_1 \leq r$,

$$\mathbb{P}_p\big(\{\varrho_1 \geq r_1\} \cap A^\delta(r)\big) \leq \mathbb{P}_p\big(\big(o^{(h,R)} \leftrightarrow S_{r_1} \text{ in } \eta\big) \circ \big(o^{(h,R)} \leftrightarrow S_r \text{ in } \eta\big)\big)$$
$$\leq g_p(r_1) \mathbb{P}_p\big(A^\delta(r)\big).$$

Further, similarly to (5.8),

$$\mathbb{P}_p\big(\varrho_1 < r_1 | A^\delta(r)\big) \geq 1 - g_p(r_1),$$

which is the lemma's conclusion for $k = 1$. □

5.2. Proof of Corollary 5.1

Corollary 5.2. (Recalled from Corollary 5.1) *For any $r > 0$, positive integer k, and $0 < p < 1$,*

$$\mathbb{P}_p\big(\varrho_1 + \cdots + \varrho_k < r - (k-1)a | A^\delta(r)\big) \geq \mathbb{P}_p(M_1 + \cdots + M_k < r - (k-1)a).$$

Proof. We compose the proof of the intermediate inequalities

$$\begin{aligned}
&\mathbb{P}_p\big(\varrho_1 + \cdots + \varrho_k < r - (k-1)a | A^\delta(r)\big) \\
&\geq \mathbb{P}_p\big(\varrho_1 + \cdots + \varrho_{k-1} + M_k < r - (k-1)a | A^\delta(r)\big) \\
&\cdots \\
&\geq \mathbb{P}_p\big(\varrho_1 + M_2 + \cdots + M_k < r - (k-1)a | A^\delta(r)\big) \\
&\geq \mathbb{P}_p\big(M_1 + \cdots + M_k < r - (k-1)a | A^\delta(r)\big) \\
&= \mathbb{P}_p\big(M_1 + \cdots + M_k < r - (k-1)a\big)
\end{aligned}$$

using the step

$$\begin{aligned}
&\mathbb{P}_p\big(\varrho_1 + \cdots + \varrho_j + M_{j+1} + \cdots + M_k < r - (k-1)a | A^\delta(r)\big) \\
&\geq \mathbb{P}_p\big(\varrho_1 + \cdots + \varrho_{j-1} + M_j + \cdots + M_k < r - (k-1)a | A^\delta(r)\big),
\end{aligned}$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

for $j = k, k-1, \dots, 2, 1$. Now we prove this step: let $j \in \{1, 2, \dots k\}$. Put

$$\mathcal{R}^{(h,R)} = \{d_{\partial}(x, y) : x, y \in V(G^{(h,R)})\}.$$

Note that this is a countable set of all possible values of ϱ_i for $i = 1, \dots, N$.

We express the probability under consideration as an integral, thinking of the whole probability space as the Cartesian product of the space on which the percolation processes are defined and the space used for defining $M_1, M_2, \dots$, and using a version of Fubini's theorem for events:

$$\begin{aligned}&\mathbb{P}_p(\varrho_1 + \cdots + \varrho_j + M_{j+1} + \cdots + M_k < r - (k-1)a | A^\delta(r))\\ &= \int \mathbb{P}_p(\varrho_1 + \cdots + \varrho_j + S_M < r - (k-1)a | A^\delta(r))\, \mathrm{d}\mathcal{L}^k_{j+1}(S_M),\end{aligned}$$

where $\mathcal{L}^k_{j+1}$ denotes the distribution of the random variable $M_{j+1} + \cdots + M_k$,

$$= \int \sum_{(r_1,\dots,r_{j-1})} \mathbb{P}_p\left(\bigwedge_{i<j} \varrho_i = r_i \wedge \varrho_j < r - (k-1)a - \sum_{i=1}^{j-1} r_i - S_M \middle| A^\delta(r)\right) \mathrm{d}\mathcal{L}^k_{j+1}(S_M),$$

where the sum is taken over all $(r_1, \dots, r_{j-1}) \in (\mathcal{R}^{(h,R)})^{j-1}$ such that $r_1 + \cdots + r_{j-1} < r - (k-1)a - S_M$,

$$\geq \int \sum_{(r_1,\dots,r_{j-1})} \left(1 - \tilde{g}_p\left(r - (k-1)a - \sum_{i=1}^{j-1} r_i - S_M\right)\right) \cdot \mathbb{P}_p\left(\bigwedge_{i<j} \varrho_i = r_i \middle| A^\delta(r)\right) \mathrm{d}\mathcal{L}^k_{j+1}(S_M)$$

from Lemma 5.1 $\left(\text{with } k = j \text{ and } r_j = r - (k-1)a - \sum_{i=1}^{j-1} r_i - S_M\right)$ and because $g_p \leq \tilde{g}_p$,

$$\begin{aligned}&= \int \sum_{(r_1,\dots,r_{j-1})} \mathbb{P}_p\left(M_j < r - (k-1)a - \sum_{i=1}^{j-1} r_i - S_M \wedge \bigwedge_{i<j} \varrho_i = r_i \middle| A^\delta(r)\right) \mathrm{d}\mathcal{L}^k_{j+1}(S_M)\\ &= \int \mathbb{P}_p(\varrho_1 + \cdots + \varrho_{j-1} + M_j + S_M < r - (k-1)a | A^\delta(r))\, \mathrm{d}\mathcal{L}^k_{j+1}(S_M)\\ &= \mathbb{P}_p(\varrho_1 + \cdots + \varrho_{j-1} + M_j + M_{j+1} + \cdots + M_k < r - (k-1)a | A^\delta(r)).\end{aligned}$$

That completes the proof. □

5.3. Proof of Lemma 5.2

Lemma 5.5. (Recalled from Lemma 5.2; cf. [9, Lemma (5.17)]) *For* $0 < p < 1$, $r > 0$,

$$\mathbb{E}_p(N(A^\delta(r)) | A^\delta(r)) \geq \frac{r}{a + \int_0^r \tilde{g}_p(m)\, \mathrm{d}m} - 1.$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Proof. For any $k \in \mathbb{N}_+$, if $\varrho_1 + \cdots + \varrho_k < r - (k-1)a$, then $e_1, \ldots, e_k$ exist and $N\big(A^\delta(r)\big) \geq k$. So, from Corollary 5.1,

$$\begin{aligned}\mathbb{P}_p\big(N\big(A^\delta(r)\big) \geq k | A^\delta(r)\big) &\geq \mathbb{P}_p\left(\sum_{i=1}^{k} \varrho_i < r - (k-1)a\right) \\ &\geq \mathbb{P}_p\left(\sum_{i=1}^{k} M_i < r - (k-1)a\right). \end{aligned} \tag{5.10}$$

Now we use a calculation which relates $a + \int_0^r \tilde{g}_p(m)\,\mathrm{d}m$ to the distribution of M_1. Namely, we replace the variables M_i by

$$M_i' = a + \min(M_i, r)$$

for $i = 1, 2, \ldots$ (a kind of truncated version of M_i). In this setting,

$$\sum_{i=1}^{k} M_i < r - (k-1)a \iff \sum_{i=1}^{k} M_i' < r + a.$$

For the '$\Leftarrow$' implication, note that the right-hand side implies

$$r - (k-1)a > \sum_{i=1}^{k} \big(M_i' - a\big) = \sum_{i=1}^{k} \min(M_i, r)$$

and then each $M_i < r$; hence $M_i = \min(M_i, r)$. So from (5.11),

$$\begin{aligned}\mathbb{E}_p\big(N\big(A^\delta(r)\big) | A^\delta(r)\big) &= \sum_{k=1}^{\infty} \mathbb{P}_p\big(N\big(A^\delta(r)\big) \geq k | A^\delta(r)\big) \\ &\geq \sum_{k=1}^{\infty} \mathbb{P}_p\left(\sum_{i=1}^{k} M_i' < r + a\right) = \sum_{k=1}^{\infty} \mathbb{P}_p(K \geq k+1) \\ &= \mathbb{E}_p(K) - 1,\end{aligned}$$

where

$$K = \min\big\{k : M_1' + \cdots + M_k' \geq r + a\big\}.$$

For $k \in \mathbb{N}$, let

$$S_k = M_1' + \cdots + M_k'.$$

By Wald's equation (see e.g. [11, p. 396]) for the random variable S_K,

$$r + a \leq \mathbb{E}_p(S_K) = \mathbb{E}_p(K)\mathbb{E}_p\big(M_1'\big).$$

For Wald's equation to be valid for S_K, the random variable K has to satisfy $\mathbb{E}_p\big(M_i' | K \geq i\big) = \mathbb{E}_p\big(M_i'\big)$ for $i \in \mathbb{N}_+$. But we have

$$K \geq i \iff M_1' + \cdots + M_{i-1}' < r + a,$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

so M_i' is independent of the event $\{K \geq i\}$ for $i \in \mathbb{N}_+$, which allows us to use Wald's equation. (In fact, K is a stopping time for the sequence $(M_i')_{i=1}^{\infty}$.) Hence,

$$\begin{aligned}\mathbb{E}_p\big(N\big(A^\delta(r)\big)|A^\delta(r)\big) &\geq \mathbb{E}_p(K) - 1 \geq \frac{r+a}{\mathbb{E}_p\big(M_1'\big)} - 1 \\ &= \frac{r+a}{a + \int_0^\infty \mathbb{P}_p(\min(M_1, r) \geq m)\,\mathrm{d}m} - 1 \\ &\geq \frac{r}{a + \int_0^r \tilde{g}_p(m)\,\mathrm{d}m} - 1,\end{aligned}$$

which finishes the proof. □

5.4. Proof of Lemma 5.3

Lemma 5.6. (Recalled from Lemma 5.3; cf. [9, Lemma (5.24)]) *For any $p < p_{\mathrm{a}}$, there exists $\delta(p)$ such that*

$$\tilde{g}_p(r) \leq \delta(p) \cdot \frac{1}{\sqrt{r}} \quad \textit{for } r > 0.$$

One fact about $\tilde{g}_p(r)$ that we will use in the proof is that it just converges to 0 with $r \to \infty$.

Proposition 5.1. *For any $p \in \mathcal{N}$,*

$$\tilde{g}_p(r) \xrightarrow[r\to\infty]{} 0.$$

Proof of the proposition. Put

$$M^{(h,R)} = r_{\partial}\big(C^{(h,R)}\big)$$

for $(h, R) \in (0, 1] \times O(d)$. Note that it is sufficient to prove

$$\sup_{R\in O(d)} \mathbb{P}_p\big(M^{(1,R)} \geq r\big) \xrightarrow[r\to\infty]{} 0, \tag{5.11}$$

because

$$\begin{aligned} g_p(r) &= \sup_{(h,R)\in(0,1]\times O(d)} \mathbb{P}_p\big(o^{(h,R)} \leftrightarrow S_r \text{ in } G^{(h,R)} \cap L\big) \\ &\leq \sup_{(h,R)\in(0,1]\times O(d)} \mathbb{P}_p\big(o^{(h,R)} \leftrightarrow S_{hr} \text{ in } G^{(h,R)}\big) && \text{(because } hr \leq r\text{)} \\ &= \sup_{R\in O(d)} \mathbb{P}_p\big(o \leftrightarrow S_r \text{ in } G^{(1,R)}\big) && \text{(by scaling the situation)} \\ &\leq \sup_{R\in O(d)} \mathbb{P}_p\big(M^{(1,R)} \geq r\big), \end{aligned}$$

so $g_p(r) \xrightarrow[r\to\infty]{} 0$ and, equivalently, $\tilde{g}_p(r) \xrightarrow[r\to\infty]{} 0$ will be implied. To prove (5.11), we use the upper semi-continuity of the function

$$O(d) \ni R \mapsto \mathbb{P}_p\big(M^{(1,R)} \geq r\big)$$

for any $p \in \mathcal{N}$ and $r > 0$. In order to verify the latter, let us fix such p and r and let $(R_n)_n$ be a sequence of elements of $O(d)$ convergent to some R. Assume without loss of generality that

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

the cluster $C^{(1,R)}$ is bounded in the Euclidean metric, and throughout this proof, condition on it for all the events by default. We are going to show that

$$\limsup_{n\to\infty} \{M^{(1,R_n)} \geq r\} \subseteq \{M^{(1,R)} \geq r\}. \tag{5.12}$$

For any isometry Φ of $\mathbb{H}^d$, let $\hat{\Phi}$ denote the unique continuous extension of Φ to $\hat{\mathbb{H}}^d$ (which is a homeomorphism of $\hat{\mathbb{H}}^d$—see [2, Corollary II.8.9]). Put $\Phi_n = \Phi^{(1,R)} \circ \left(\Phi^{(1,R_n)}\right)^{-1}$ and assume that the event $\limsup_{n\to\infty} \{M^{(1,R_n)} \geq r\}$ occurs. Then, for infinitely many values of n, all the following occur:

$$M^{(1,R_n)} \geq r \Longrightarrow \hat{C}^{(1,R_n)} \text{ intersects } \hat{S}_r,$$

and, by applying $\hat{\Phi}_n$,

$$\hat{C}^{(1,R)} = \widehat{C}^{(1,R)} \text{ intersects } \hat{\Phi}_n(\hat{S}_r).$$

For any such n, let x_n be chosen from the set $\widehat{C}^{(1,R)} \cap \hat{\Phi}_n(\hat{S}_r)$. Because $\widehat{C}^{(1,R)}$ is compact, the sequence $(x_n)_n$ (indexed by a subset of $\mathbb{N}_+$) has an (infinite) subsequence $(x_{n_k})_{k=1}^{\infty}$ convergent to some point in $\widehat{C}^{(1,R)}$. On the other hand, note that $\hat{\Phi}_n \xrightarrow[n\to\infty]{} \mathrm{Id}_{\hat{\mathbb{H}}^d}$ uniformly in the Euclidean metric of the disc model (see Definition 2.2). Hence, the distance in that metric between $x_{n_k} \in \hat{\Phi}_{n_k}(\hat{S}_r)$ and $\hat{S}_r$ tends to 0 as $k \to \infty$, so

$$\lim_{k\to\infty} x_{n_k} \in \hat{S}_r \cap \widehat{C}^{(1,R)} = \widehat{S}_r \cap \widehat{C}^{(1,R)},$$

which shows that $M^{(1,R)} \geq r$, as desired in (5.12). Now,

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}_p\big(M^{(1,R_n)} \geq r\big) &\leq \mathbb{P}_p\big(\limsup_{n\to\infty} \{M^{(1,R_n)} \geq r\}\big) \\ &\leq \mathbb{P}_p\big(M^{(1,R)} \geq r\big),\end{aligned}$$

which means exactly the upper semi-continuity of $R \mapsto \mathbb{P}_p\big(M^{(1,R)} \geq r\big)$.

Next, note that for $p \in \mathcal{N}$ and $R \in O(d)$

$$\mathbb{P}_p\big(M^{(1,R)} \geq r\big) \xrightarrow[r\to\infty]{} 0 \quad \text{(decreasingly)},$$

because $\mathbb{P}_p$-a.s. $M^{(1,R)} < \infty$ by Remark 3.2. Hence, if for $r > 0$ and $\varepsilon > 0$ we put

$$U_\varepsilon(r) = \big\{R \in O(d) : \mathbb{P}_p\big(M^{(1,R)} \geq r\big) < \varepsilon\big\},$$

then for any fixed $\varepsilon > 0$,

$$\bigcup_{r\nearrow\infty} U_\varepsilon(r) = O(d). \tag{5.13}$$

$U_\varepsilon(r)$ is always an open subset of $O(d)$ by the upper semi-continuity of $R \mapsto \mathbb{P}_p\big(M^{(1,R)} \geq r\big)$, so by the compactness of $O(d)$, the union (5.13) is indeed finite. Moreover, because $U_\varepsilon(r)$ increases as r increases, it equals $O(d)$ for some $r > 0$. This means that $\sup_{R\in O(d)} \mathbb{P}_p\big(M^{(1,R)} \geq r\big) \leq \varepsilon$, whence $\sup_{R\in O(d)} \mathbb{P}_p\big(M^{(1,R)} \geq r\big) \xrightarrow[r\to\infty]{} 0$, as desired. □

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

Proof of the lemma. Assume without loss of generality that $\tilde{g}_p(r) > 0$ for $r > 0$. We are going to construct sequences $(p_i)_{i=1}^{\infty}$ and $(r_i)_{i=1}^{\infty}$ such that

$$p_{\mathrm{a}} > p_1 > p_2 > \cdots > p, \quad 0 < r_1 \le r_2 \le \cdots,$$

and such that the sequence $\big(\tilde{g}_{p_i}(r_i)\big)_{i=1}^{\infty}$ decays rapidly. The construction is by recursion: for $i \ge 1$, having constructed $p_1, \dots, p_i$ and $r_1, \dots, r_i$, we put

$$r_{i+1} = r_i/g_i \quad \text{and} \quad p_{i+1} = p_i - 3g_i(1 - \ln g_i), \tag{5.14}$$

where $g_i = \tilde{g}_{p_i}(r_i)$. (Note that, indeed, $r_{i+1} \le r_i$ and $p_{i+1} < p_i$.) The above formula may give an incorrect value of p_{i+1}, i.e. a value not satisfying $p_{i+1} > p$ (this condition is needed because we want to bound values of $\tilde{g}_p$). In order to prevent that, we use the following inequality and choose appropriate values of p_1, r_1.

Fix $i \ge 1$ and assume that $r_1 \ge a$ and $p_1, \dots, p_i > p$ are defined by (5.14). Let $j \in \{1, \dots, i-1\}$. Then

$$g_{j+1} \le g_j^2. \tag{5.15}$$

We now prove this inequality. From (5.5),

$$\begin{aligned} g_{j+1} &\le \tilde{g}_{p_j}(r_{j+1}) \exp\left(-(p_j - p_{j+1})\left(\frac{r_{j+1}}{a + \int_0^{r_{j+1}} \tilde{g}_{p_j}(m)\,\mathrm{d}m} - 1\right)\right) \\ &\le g_j \exp\left(1 - (p_j - p_{j+1})\frac{r_{j+1}}{a + \int_0^{r_{j+1}} \tilde{g}_{p_j}(m)\,\mathrm{d}m}\right). \end{aligned}$$

The inverse of the fraction above is estimated as follows:

$$\begin{aligned} \frac{1}{r_{j+1}}\left(a + \int_0^{r_{j+1}} \tilde{g}_{p_j}(m)\,\mathrm{d}m\right) &\le \frac{a}{r_{j+1}} + \frac{r_j}{r_{j+1}} + \frac{1}{r_{j+1}}\int_{r_j}^{r_{j+1}} \tilde{g}_{p_j}(m)\,\mathrm{d}m \\ &\le \frac{a}{r_{j+1}} + g_j + \frac{r_{j+1} - r_j}{r_{j+1}}\tilde{g}_{p_j}(r_j), \end{aligned}$$

using $r_{j+1} = r_j/g_j$ and the monotonicity of $\tilde{g}_{p_j}(\cdot)$,

$$\le \frac{a}{r_{j+1}} + 2g_j.$$

Now, by the assumption, $r_j \ge r_1 \ge a$, so $r_{j+1} = r_j/g_j \ge a/g_j$ and

$$\frac{a}{r_{j+1}} + 2g_j \le 3g_j.$$

That gives

$$g_{j+1} \le g_j \exp\left(1 - \frac{p_j - p_{j+1}}{3g_j}\right) = g_j^2$$

by the definition of p_{j+1}.

We define a sequence $(x_i)_{i=1}^{\infty}$ by $x_{i+1} = x_i^2$ for $i \ge 1$ $\left(\text{i.e. } x_i = x_1^{2^{i-1}}\right)$, where $0 < x_1 < 1$. It is an exercise to prove that for such a sequence,

$$s(x_1) := \sum_{i=1}^{\infty} 3x_i(1 - \ln x_i) \tag{5.16}$$

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

is finite and $s(x_1) \xrightarrow[x_1 \to 0]{} 0$. (The idea of the proof of this fact is similar to that of estimating the sum in (3.4).)

Now, taking any $p_1 \in (p, p_a)$ and $1 > x_1 > 0$ in (5.16) such that $s(x_1) \le p_1 - p$, and taking $r_1 \ge a$ so large that $\tilde{g}_{p_1}(r_1) < x_1$ (thanks to Proposition 5.1), we obtain $g_i < x_i$ for $j \in \{1, \dots, i\}$ (by induction). Then, in the setting of (5.14),

$$p_{i+1} = p_1 - \sum_{j=1}^{i} 3g_i(1 - \ln g_i) > p_1 - \sum_{j=1}^{i} 3x_i(1 - \ln x_i)$$

because $x \mapsto 3x(1 - \ln x)$ is increasing for $x \in (0, 1]$,

$$\ge p_1 - s(x_1) \ge p.$$

Now we know that the recursion (5.14) is well-defined, and we can use the constructed sequences to prove the lemma. First, note that for $k \ge 1$,

$$r_k = r_1/(g_1 g_2 \cdots g_{k-1}).$$

Furthermore, (5.15) implies

$$g_{k-1}^2 \le g_{k-1} g_{k-2}^2 \le \cdots \le g_{k-1} g_{k-2} \cdots g_2 g_1^2 = \frac{r_1}{r_k} g_1 = \frac{\delta^2}{r_k}, \tag{5.17}$$

where $\delta = \sqrt{r_1 g_1}$. Now, let $r \ge r_1$. We have $r_k \xrightarrow[k \to \infty]{} \infty$ because $\frac{r_k}{r_{k+1}} = g_k \xrightarrow[k \to \infty]{} 0$, so for some k, $r_{k-1} \le r < r_k$. Then

$$\tilde{g}_p(r) \le \tilde{g}_{p_{k-1}}(r) \le \tilde{g}_{p_{k-1}}(r_{k-1}) = g_{k-1} \le \frac{\delta}{\sqrt{r_k}} < \frac{\delta}{\sqrt{r}}$$

(from (5.17) and the monotonicity of $\tilde{g}_p(r)$ with respect to each of p and r), which finishes the proof. □

Remark 5.2. As promised in Section 5, in this remark we summarize the differences between the proof of Theorem 3.2 and the proof of Theorem (5.4) in [9]:

1. First of all, the skeleton structure and most of the notation of the proof here is borrowed from [9]. The major difference in notation here is the use of $A^{\delta}(r)$ and S_r (corresponding respectively to A_n and $\partial S(s)$ in [9]).
2. Strictly speaking, the proper line of the proof borrowed from [9] starts by considering the functions f_p instead of g_p or $\tilde{g}_p$, although the functional inequality (5.1) involves both functions $f_\cdot(\cdot)$ and $\tilde{g}_\cdot(\cdot)$. In fact, each of the functions $f_\cdot(\cdot)$, $\tilde{g}_\cdot(\cdot)$, and $g_\cdot(\cdot)$ is a counterpart of the function $g_\cdot(\cdot)$ from [9] at some stage of the proof. After proving the inequality (5.1), we pass to a couple of limits with it in order to obtain the inequality (5.2), which involves only $\tilde{g}_\cdot(\cdot)$ (the step not present in [9]). This is necessary to enable the repeated use of the inequality (5.2) at the end of the proof of Theorem 3.2.
3. Obviously, the geometry used here is much different from that in [9]. In fact, we analyse the percolation cluster in $G^{(h,R)} \cap L_\delta$ using the pseudometric $d_{\mathfrak{F}}$ (in place of the graph metric δ in [9]). Consequently, the set $\mathcal{R}^{(h,R)}$ of possible values of the random variables

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

ϱ in Lemma 5.1 is much richer than $\mathbb{N}$, the respective set for the graph $\mathbb{Z}^d$. Moreover, the functions $\tilde{g}_p$ arise from the percolation process on the whole $G^{(h,R)} \cap L$, so the distribution of the random variables M_i is not necessarily discrete. That necessitates the use of integrals instead of sums to handle those random variables, especially in the proof of Corollary 5.1. All of this also leads to a few other minor technical differences between the proof here and the proof in [9].

4. The author has tried to clarify the use of the assumption on $\sum_{i=1}^{k} r_i$ in Lemma 5.1 and to explain why Wald's equation can be used in the proof of Lemma 5.2, a consideration that is somewhat hidden in [9].

5. The proof of Lemma 5.3 has been reorganized a little (compared to the proof of Lemma (5.24) in [9]) and contains a proof of the convergence $\tilde{g}_p(r) \underset{r}{\to} 0$ (Proposition 5.1).

Acknowledgement

This paper is essentially the second part of my doctoral thesis. I express my gratitude to my advisor, Jan Dymara, for his supervision and words of advice.

Funding information

There are no funding bodies to thank in relation to the creation of this article.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

[1] Babson, E. and Benjamini, I. (1999). Cut sets and normed cohomology with applications to percolation. *Proc. Amer. Math. Soc*. **127**, 589–597.
[2] Bridson, M. R. and Haefliger, A. (1999). *Metric Spaces of Non-Positive Curvature*. Springer, Berlin.
[3] Burton, R. M.and Keane, M. (1989). Density and uniqueness in percolation. *Commun. Math. Phys.* **121**, 501–505.
[4] Benjamini, I. and Schramm, O. (1996). Percolation beyond $\mathbf{Z}^d$, many questions and a few answers. *Electron. Commun. Prob.* 1, 71–82.
[5] Benjamini, I. and Schramm, O. (2001). Percolation in the hyperbolic plane. *J. Amer. Math. Soc.* **14**, 487–507.
[6] Czajkowski, J. (2012). Clusters in middle-phase percolation on hyperbolic plane. In *Noncommutative Harmonic Analysis with Applications to Probability III* (Banach Center Publications **96**), Institute of Mathematics, Polish Academy of Sciences, Warsaw, pp. 99–113.
[7] Czajkowski, J. (2013). Non-uniqueness phase of percolation on reflection groups in $\mathbb{H}^3$. Preprint. Available at http://arxiv.org/abs/1303.5624.
[8] Diestel, R. (2010). *Graph Theory*, 4th edn. Springer, New York.
[9] Grimmett, G. (1999). *Percolation*. Springer, New York.
[10] Grimmett, G. and Newman, C. M. (1990). Percolation in $\infty + 1$ dimensions. In *Disorder in Physical Systems*, Oxford University Press, New York, pp. 167–190.
[11] Grimmett, G. and Stirzaker, D. R. (1992). *Probability and Random Processes*, 2nd edn. Oxford University Press.
[12] Hutchcroft, T. (2019). Percolation on hyperbolic graphs. *Geom. Funct. Anal.* **29**, 766–810.
[13] Kapovich, I. and Benakli, N. (2002). Boundaries of hyperbolic groups. In *Combinatorial and Geometric Group Theory*, American Mathematical Society, Providence, RI, pp. 39–93.
[14] Lalley, S. P. Percolation clusters in hyperbolic tessellations. *Geom. Funct. Anal.* **11**, 971–1030.
[15] Lalley, S. P. (1998). Percolation on Fuchsian groups. *Ann. Inst. H. Poincaré Prob. Statist.* **34**, 151–177.
[16] Lyons, R. and Peres, Y. (2013). *Probability on Trees and Networks*. Cambridge University Press.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press

[17] MENSHIKOV, M. V. (1986). Coincidence of critical points in percolation problems. *Dokl. Akad. Nauk SSSR* **288**, 1308–1311 (in Russian).
[18] NEWMAN, C. M. AND SCHULMAN, L. S. (1981). Infinite clusters in percolation models. *J. Statist. Phys.* **26**, 613–628.
[19] PAK, I. AND SMIRNOVA-NAGNIBEDA, T. (2000). On non-uniqueness of percolation on nonamenable Cayley graphs. *C. R. Acad. Sci. Paris* **330**, 495–500.
[20] SIDORAVICIUS, V., WANG, L. AND XIANG, K. (2020). Limit set of branching random walks on hyperbolic groups. Preprint. Available at http://arxiv.org/abs/2007.13267v1.
[21] WOESS, W. (2000). *Random Walks on Infinite Graphs and Groups*. Cambridge University Press.

https://doi.org/10.1017/apr.2022.43 Published online by Cambridge University Press