5 Short-Distance Structure of Weak Decays

5.1 Operator Product Expansion in Weak Decays

5.1.1 Basic Idea

Weak decays of mesons are driven by weak interactions of quarks, whose strong interactions, binding the quarks into mesons, are characterized by a typical hadronic energy scale $\mu \leq O(1\,\text{GeV})$, much lower than the scale of weak interactions: $O(M_W)$. Evidently the problem of calculating the amplitudes for weak decays of mesons involves within the SM a number of energy scales to be denoted by μ:

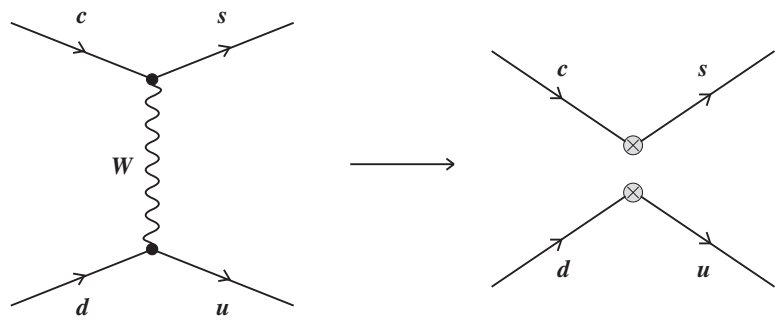
- Scales $\mu = O(M_W)$ at which the fundamental weak transition between quarks responsible for weak decays of mesons takes place and at which the QCD coupling $\alpha_s(\mu)$ is rather small.
- Scales $O(1\,\text{GeV}) \leq \mu \leq M_W$ in which significant variation of $\alpha_s(\mu)$ with μ has to be taken into account, and large logarithms multiplying $\alpha_s(\mu)$ must be summed to all orders in α_s.
- Scales $\mu \leq O(1\,\text{GeV})$ where confinement-effects binding quarks into mesons have to be taken into account.

Beyond the SM higher scales have to be considered, but the technology developed in this chapter can also be used there. For the time being we stay within the SM, which has the advantage that the model is well defined and moreover describes the data rather well.

Our goal is then to derive an effective low-energy theory that would allow us to calculate efficiently the amplitudes for weak decays, taking the dynamical effects at all scales listed earlier into account. The formal framework to achieve this is the Operator Product Expansion (OPE) [107–110]. I follow here to a large extent my Les Houches lectures [29] and the review in [111] making updates after twenty years. A very nice introduction to this field with many explicit calculations and references can also be found in [46]. Many aspects and technology of effective field theories are also presented in [112].

Let us first present the basic idea of the OPE. Consider the quark level transition $c \to su\bar{d}$. Disregarding QCD effects for the moment, the corresponding tree-level W-exchange amplitude (Figure 5.1a multiplied by "i") is given by

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(a) (b)

Figure 5.1 $c \to s\bar{u}d$ at tree level. From [29].

$$A = -\frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \frac{M_W^2}{k^2 - M_W^2} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A}$$
$$= \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} + O\left(\frac{k^2}{M_W^2}\right), \tag{5.1}$$

where k, the momentum transfer through the W propagator, is very small as compared to M_W. Consequently, terms of order $O(k^2/M_W^2)$ can safely be neglected, and the full amplitude A can be approximated by the first term on the r.h.s. of (5.1).

As in Section 3.1, we expressed the result in terms of the Fermi constant G_F and the product of quark currents $(\bar{s}c)_{V-A}$ and $(\bar{u}d)_{V-A}$ using the shorthand notations as

$$(\bar{s}c)_{V-A} \equiv \bar{s}\gamma_\mu(1-\gamma_5)c \equiv \bar{s}_\alpha \gamma_\mu (1-\gamma_5) c_\alpha \equiv (\bar{s}_\alpha c_\alpha)_{V-A}, \tag{5.2}$$

where $\alpha = 1, 2, 3$ is the color index. As the W boson is color blind, the summation over repeated color indices is straightforward, and we can just work with the first expression for the quark current. The index $V - A$ reminds us of the Dirac structure of this current.

Now the result in (5.1) may also be obtained from

$$\mathscr{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}c)_{V-A} (\bar{u}d)_{V-A} + \text{High D Operators}, \tag{5.3}$$

where the higher-dimension operators, typically involving derivative terms, correspond to the terms $O(k^2/M_W^2)$ in (5.1). Neglecting the latter terms corresponds to the neglect of higher-dimensional operators. In what follows we will always neglect the higher-dimensional operators keeping only the operators with dimension six or having lower dimensions. Then we can just write

$$\mathscr{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} C Q, \qquad Q = (\bar{s}c)_{V-A} (\bar{u}d)_{V-A}. \tag{5.4}$$

This simple example illustrates the basic idea of OPE: The product of two charged current operators is expanded into a series of local operators (Q), whose contributions are weighted by effective coupling constants C, the Wilson coefficients. In this particular

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example we have only one operator, represented by the diagram (b) in Figure 5.1, and its Wilson coefficient is simply equal unity: $C = 1$. This value will be changed by QCD corrections. Moreover, QCD corrections to the diagrams in Figure 5.1 will generate another operator.

The example presented here illustrates also the so-called Appelquist–Carazzone decoupling theorem [113]:

Heavy fields of mass M decouple at low-energy, generating operators suppressed by powers of $1/M$, except for their contributions to renormalization effects.

In our case $M = M_W$ and the suppression in question is hidden in G_F. The renormalization, or rather renormalization group effects, will be encountered soon. In the latter context the review by Wilson [114] can be strongly recommended.

5.1.2 Formal Approach

Let us be a bit more formal for a moment and investigate whether the same result can be obtained using the path integral formalism. We will see that this is indeed the case. Readers not familiar with this approach can skip this presentation and just read the last paragraph of this section, which will certainly delight them.

This discussion will on the one hand provide a formal basis for the simple procedure given earlier and on the other hand will give us more insight into the virtues of the OPE. Simultaneously we will discover that there is no need to be very formal for the rest of this book, and we can proceed by simply generalizing our simple procedure of just looking at diagrams to more complicated situations in which also QCD effects and more complicated diagrams are present.

Our formal discussion follows [111] and consists of four steps.

Step 1

Consider the generating functional for Green functions in the path integral formalism. The relevant part for our discussion is

$$Z_W \sim \int [dW^+][dW^-] \exp(i \int d^4x \mathscr{L}_W), \tag{5.5}$$

where

$$\mathscr{L}_W = -\frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) + M_W^2 W^+_\mu W^{-\mu} + \frac{g_2}{2\sqrt{2}}(J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu}), \tag{5.6}$$

$$J^+_\mu = V_{pn}\bar{p}\gamma_\mu(1-\gamma_5)n, \qquad p = (u,c,t), \qquad n = (d,s,b), \qquad J^-_\mu = (J^+_\mu)^\dagger. \tag{5.7}$$

Step 2

We use the unitary gauge for the W field. Introducing the operator:

$$K_{\mu\nu}(x,y) = \delta^{(4)}(x-y)\left[g_{\mu\nu}(\partial^2 + M_W^2) - \partial_\mu\partial_\nu\right], \tag{5.8}$$

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we have, after discarding a total derivative in the W kinetic term,

$$Z_W \sim \int [dW^+][dW^-] \exp\left[i \int d^4x d^4y W^+_\mu(x) K^{\mu\nu}(x,y) W^-_\nu(y) + i\frac{g_2}{2\sqrt{2}} \int d^4x (J^+_\mu W^{+\mu} + J^-_\mu W^{-\mu}) \right]. \tag{5.9}$$

The inverse of $K_{\mu\nu}$, denoted by $\Delta_{\mu\nu}$, and defined through

$$\int d^4y K_{\mu\nu}(x,y) \Delta^{\nu\lambda}(y,z) = g_\mu{}^\lambda \delta^{(4)}(x-z), \tag{5.10}$$

is given by

$$\Delta_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4} \Delta_{\mu\nu}(k) e^{-ik(x-y)}, \tag{5.11}$$

with

$$i\Delta_{\mu\nu}(k) = \frac{-i}{k^2 - M_W^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M_W^2} \right). \tag{5.12}$$

It is the W propagator in the unitary gauge

Step 3

Performing the Gaussian functional integration over $W^\pm(x)$ in (5.9) explicitly, we arrive at

$$Z_W \sim \exp\left[-i \int \frac{g_2^2}{8} J^-_\mu(x) \Delta^{\mu\nu}(x,y) J^+_\nu(y) d^4x d^4y \right]. \tag{5.13}$$

This result implies a nonlocal action functional for the quarks:

$$\mathcal{S}_{nl} = \int d^4x \mathscr{L}_{\text{kin}} - \frac{g_2^2}{8} \int d^4x d^4y J^-_\mu(x) \Delta^{\mu\nu}(x,y) J^+_\nu(y), \tag{5.14}$$

where the second term represents charged current interactions of quarks.

Step 4

Finally, we expand this second, nonlocal term in powers of $1/M_W^2$ to obtain a series of local interaction operators of dimensions that increase with the order in $1/M_W^2$. To lowest order

$$\Delta^{\mu\nu}(x,y) \approx \frac{g^{\mu\nu}}{M_W^2} \delta^{(4)}(x-y), \tag{5.15}$$

and the second term in (5.14) becomes

$$-\frac{g_2^2}{8M_W^2} \int d^4x J^-_\mu(x) J^{+\mu}(x), \tag{5.16}$$

corresponding to the usual effective charged current interaction Lagrangian

$$\mathscr{L}_{\text{int,eff}} = -\frac{G_F}{\sqrt{2}} J^-_\mu(x) J^{+\mu}(x) = -\frac{G_F}{\sqrt{2}} V^*_{pn} V_{p'n'} (\bar{n}p)_{V-A} (\bar{p}'n')_{V-A}, \tag{5.17}$$

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which contains, among other terms, the leading contribution to (5.3). The minus sign follows simply from $\mathscr{L} = -\mathscr{H}$.

Let us note several basic aspects of this approach:

- Formally, the procedure to approximate the interaction term in (5.14) by (5.16) is an example of short-distance OPE. The product of the local operators $J_\mu^-(x)$ and $J_\nu^+(y)$, to be taken at short distances due to the convolution with the massive, short-range W propagator $\Delta^{\mu\nu}(x, y)$, is expanded into a series of composite local operators. The leading term is shown in (5.16).
- The dominant contributions in the short-distance expansion come from the operators of lowest dimension (six in the present example). The operators of higher dimensions can usually be neglected in weak decays, although in the literature some studies of them can be found. Not in this book.
- OPE series is equivalent to the original theory, when considered to all orders in $1/M_W^2$. The truncation of the operator series yields a systematic approximation scheme for low-energy processes, neglecting contributions suppressed by powers of k^2/M_W^2.
- In going from the full to the effective theory the W boson is removed as an explicit, dynamical degree of freedom: It is "integrated out" in step 3 of our procedure. Alternatively in the canonical operator formalism the W field gets "contracted out" through the application of Wick's theorem. From the point of view of low-energy dynamics, the effects of a short-range exchange force mediated by a heavy boson approximately corresponds to a point interaction familiar from the Fermi theory.
- Similarly one can "integrate out" or "contract out" heavy quarks. This gives *effective f-quark theories* where f denotes the "light" quarks that have not been integrated out. We now understand what the effective number of flavors introduced in connection with the formula (4.147) really means. By going from higher to lower μ scales one integrates out systematically flavors with masses higher than the actual value of μ. However, as we will stress later, in connection with renormalization group ideas, there is some freedom at which μ a given flavor is integrated out. For instance, one can extend the five-flavor theory down to $\mu = m_b/2$. Such unphysical dependences on the choice of μ are eliminated by considering sufficient order in perturbation theory, but it is too early to discuss this important topic already here.

All this was a bit formal but fortunately we make still another observation. The approach of evaluating the relevant Green functions (or amplitudes) directly to construct the OPE, as in (5.1), gives the same result as the more formal technique employing path integrals. Consequently we can return, putting aside path integrals, to our Feynman diagram calculations. Our first task is to investigate how (5.1) or (5.3) changes when QCD effects are included.

5.1.3 OPE and Short-Distance QCD Effects

Preliminaries

Due to the asymptotic freedom of QCD, the short-distance QCD corrections to weak decays, that is the contribution of hard gluons at energies of the order $O(M_W)$ down to

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hadronic scales $O(1\,\text{GeV})$, can be treated in the renormalization group (RG) improved perturbation theory. We will illustrate this on our simple example of the $c \to su\bar{d}$ transition beginning with the ordinary perturbation theory, subsequently summing leading logarithms by the RG method and finally in the next section generalizing the result to include next-to-leading logarithms. We will do this in some detail, emphasizing certain characteristic features of this approach. In particular we will discuss at length the *scale and renormalization scheme dependences* of various elements of the calculations and their cancellations in physical quantities to be termed observables. Once all these features are well understood it will be straightforward to proceed to other transitions and to generalize this approach to more exciting situations involving penguin and box diagrams.

For the $c \to su\bar{d}$ transition we found without QCD effects the effective Hamiltonian (5.4), which we now rewrite as follows

$$\mathcal{H}_{\text{eff}}^{(0)} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A}. \tag{5.18}$$

Here the summation over repeated color indices is understood, but we exposed now these indices as gluons will soon enter the scene, and they are not color blind. The superscript (0) reminds us that QCD effects have not been included yet.

We will soon discover that when QCD effects are taken into account $\mathcal{H}_{\text{eff}}^{(0)}$ is generalized to

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_1(\mu) Q_1 + C_2(\mu) Q_2), \tag{5.19}$$

where

$$Q_1 = (\bar{s}_\alpha c_\beta)_{V-A} (\bar{u}_\beta d_\alpha)_{V-A}, \tag{5.20}$$

$$Q_2 = (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A}. \tag{5.21}$$

The essential features of this Hamiltonian are

- In addition to the original operator Q_2 (with index 2 for historical reasons) a new operator Q_1 with the *same flavor structure* but *different color structure* is generated. That a new operator has to be introduced is evident if we inspect the color structure of the diagrams (b) and (c) in Figure 5.2. They contain the product of the color charges $t^a_{\alpha\beta}$ and $t^a_{\gamma\delta}$, which, using the color algebra developed in Section 4, can be rewritten as follows

$$t^a_{\alpha\beta} t^a_{\gamma\delta} = -\frac{1}{2N} \delta_{\alpha\beta} \delta_{\gamma\delta} + \frac{1}{2} \delta_{\alpha\delta} \delta_{\gamma\beta}. \tag{5.22}$$

  The first term on the r.h.s. gives a correction to the coefficient of the operator Q_2, and the second term gives life to the new operator Q_1. Indeed δ and β have been interchanged in this term relative to the l.h.s. as is the case of Q_1 compared with Q_2. The diagram (a) in Figure 5.2 gives again Q_2.
- The Wilson coefficients C_1 and C_2, the coupling constants for the interaction terms Q_1 and Q_2, become calculable nontrivial functions of α_s, M_W, and the renormalization scale μ.
- If QCD is neglected, $C_1 = 0$, $C_2 = 1$, and (5.19) reduces to (5.18).

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Calculation of Wilson Coefficients

Our first task is the calculation of the coefficients $C_{1,2}$ in the ordinary perturbation theory. $C_{1,2}$ can be determined by the requirement that the amplitude A_{full} in the full theory be reproduced by the corresponding amplitude in the effective theory (5.19):

$$A_{\text{full}} = A_{\text{eff}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_1 \langle Q_1 \rangle + C_2 \langle Q_2 \rangle). \tag{5.23}$$

This procedure is called "the *matching* of the full theory onto the effective theory." We recall that the full theory is the one in which all particles appear as dynamical degrees of freedom. In the case at hand the effective theory is constructed by integrating out the W field only. The matching procedure, which gives the values of C_1 and C_2, proceeds in three steps and has been first outlined not in the context of weak decays but in the context of QCD effects in deep-inelastic electron-proton scattering [94]. The explicit three steps presented here are sufficient for the subsequent summation of the leading logarithms or equivalently for the leading order (LO) term of the RG improved perturbation theory. We will generalize these steps in the next section to be able to include also the next-to-leading order (NLO) term in this expansion. Here we go:

Step 1: Calculation of A_{full}

The current-current diagrams of Figure 5.2 (a)–(c) and their symmetric counterparts give for the full amplitude A_{full} to $O(\alpha_s)$ ($m_i = 0$, $p^2 < 0$):

$$A_{\text{full}} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \Big[\left(1 + 2C_F \frac{\alpha_s}{4\pi} \left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2} \right) \right) \right) S_2 + \frac{3}{N} \frac{\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{-p^2} \right) S_2 - 3 \frac{\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{-p^2} \right) S_1 \Big]. \tag{5.24}$$

Here:

$$S_1 \equiv \langle Q_1 \rangle_{tree} = (\bar{s}_\alpha c_\beta)_{V-A} (\bar{u}_\beta d_\alpha)_{V-A}, \tag{5.25}$$

$$S_2 \equiv \langle Q_2 \rangle_{tree} = (\bar{s}_\alpha c_\alpha)_{V-A} (\bar{u}_\beta d_\beta)_{V-A}, \tag{5.26}$$

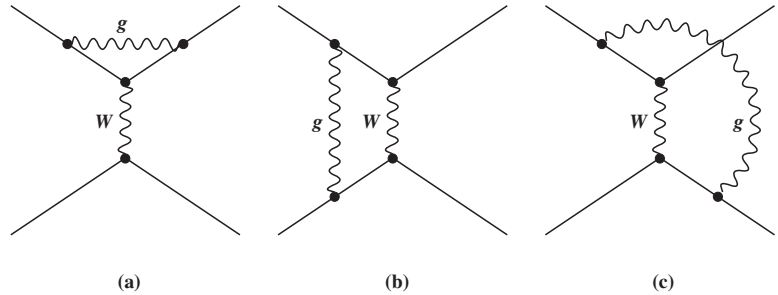


Figure 5.2 One-loop current-current diagrams in the full theory. From [29].

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are just the tree-level matrix elements of Q_1 and Q_2. We recommend this calculation and the ones of Step 2 to motivated readers. With the technology developed in the previous chapter this is a rather easy task.

A few comments should be made.

- We use the term "amplitude" in the meaning of an "amputated Green function" (multiplied by "i"). Correspondingly operator matrix elements are amputated Green functions with operator insertion. Thus gluonic self-energy corrections on external legs are not included.
- For simplicity we have chosen all external momenta p to be equal and set all quark masses to zero. As we will see soon this choice has no impact on the coefficients C_i. We could, for instance, set all external quark momenta to zero and give all quarks a universal mass. We could also set all quark masses and momenta to zero but give the gluon a mass. All these tricks represent particular regularization of infrared divergences that have nothing to do with Wilson coefficients representing short-distance behavior of the theory. We stress this point because in the past in this part several mistakes have been done in the literature.
- We have kept only logarithmic corrections $\sim \alpha_s \cdot \log$ and discarded constant contributions of order $O(\alpha_s)$, which corresponds to the leading log approximation (LO). In the next section, which deals with NLO corrections, we will have to keep these terms.
- The singularity $1/\varepsilon$ can be removed by quark field renormalization. The interested reader can check it by using the procedure of renormalizing the amputated Green functions that we outlined in Section 4.2. This is, however, not necessary for finding C_i as we will see soon.

Step 2: Calculation of matrix elements $\langle Q_i \rangle$

The unrenormalized current-current matrix elements of Q_1 and Q_2 are found at $O(\alpha_s)$ by calculating the diagrams in Figure 5.3 (a)–(c) and their symmetric counterparts. Adding the contributions without QCD corrections (S_1 and S_2, respectively) and using the same assumptions about the external legs as in step 1 (in general the same infrared regularization), we find

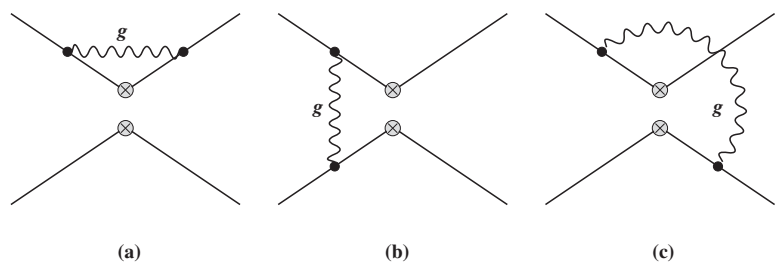


Figure 5.3 One-loop current-current diagrams in the effective theory. The four-vertex "⊗ ⊗" denotes the insertion of a four-fermion operator Q_i. From [29].

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$$\langle Q_1\rangle^{(0)} = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right)\right)\right) S_1 + \frac{3}{N}\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right)\right) S_1 - 3\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right)\right) S_2, \tag{5.27}$$

$$\langle Q_2\rangle^{(0)} = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right)\right)\right) S_2 + \frac{3}{N}\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right)\right) S_2 - 3\frac{\alpha_s}{4\pi}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\mu^2}{-p^2}\right)\right) S_1. \tag{5.28}$$

The divergences in the first terms can again be eliminated through the quark field renormalization. However, in contrast to the full amplitude in (5.24), the resulting expressions are still divergent after this renormalization. To remove these additional divergences *operator renormalization* is necessary:

$$Q_i^{(0)} = Z_{ij} Q_j \,. \tag{5.29}$$

We observe that the renormalization constant is in this case a 2×2 matrix $\hat{Z}$. Using (4.100) with $(n_F, n_G) = (4, 0)$, we find the relation between the unrenormalized ($\langle Q_i\rangle^{(0)}$) and the renormalized amputated Green functions ($\langle Q_i\rangle$):

$$\langle Q_i\rangle^{(0)} = Z_q^{-2} Z_{ij} \langle Q_j\rangle \,. \tag{5.30}$$

Z_q^{-2} removes the $1/\varepsilon$ divergences in the first terms in (5.27) and (5.28). Z_{ij} remove the remaining divergences. From (5.27) and (5.28) we read off ($\overline{\text{MS}}$-scheme)

$$\hat{Z} = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\begin{pmatrix} 3/N & -3 \\ -3 & 3/N \end{pmatrix}. \tag{5.31}$$

Thus the renormalized matrix elements $\langle Q_i\rangle$ are given by

$$\langle Q_1\rangle = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right)\right) S_1 + \frac{3}{N}\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right) S_1 - 3\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right) S_2 \,, \tag{5.32}$$

$$\langle Q_2\rangle = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right)\right) S_2 + \frac{3}{N}\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right) S_2 - 3\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right) S_1 \,. \tag{5.33}$$

Step 3: Extraction of C_i

Inserting $\langle Q_i\rangle$ into (5.23), and comparing with (5.24), we can now extract the coefficients C_1 and C_2. Yet, we have to be a bit careful. In the full theory we did not perform any quark field renormalization, whereas we did this renormalization in the effective theory as signaled by the first factor on the r.h.s. of (5.30). This is clearly inconsistent, and this inconsistency is signaled by the divergent Wilson coefficient, which is clearly wrong. To proceed correctly we have to either remove the divergence in (5.24) by performing quark field renormalization as in (5.30) or to leave (5.24) as it is and remove the quark field renormalization from (5.30). In both cases the matching (5.23) gives the same result

$$C_1(\mu) = -3\frac{\alpha_s}{4\pi}\ln\left(\frac{M_W^2}{\mu^2}\right), \qquad C_2(\mu) = 1 + \frac{3}{N}\frac{\alpha_s}{4\pi}\ln\left(\frac{M_W^2}{\mu^2}\right). \tag{5.34}$$

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In particular the infrared regulator $-p^2$ disappeared. It canceled out in the process of matching.

This simple example shows that it is essential in the process of matching to treat the external states in the full and the effective theory in the same manner to obtain the correct result for the Wilson coefficients. In this example we were lucky. The inconsistency, which we made for pedagogical reasons, was signaled by a leftover divergence. In the case of NLO calculations where also finite nonlogarithmic corrections have to be kept, a possible inconsistency in matching is much harder to see, and it is crucial that at all stages of matching the treatment of the external legs on both sides of (5.23) is the same. For this reason we are free to decide whether we perform external field renormalization or not. In the latter case the leftover divergences in the full and the effective theory will simply cancel each other in the process of matching. We discussed here the issue of the cancelation of the *ultraviolet* divergences related to external fields. The same comments, made already after (5.34), apply to the *infrared* divergences. For strategic reasons we will now discuss something else and will return to the issue of infrared divergences in the context of matching soon after.

A Different Look at Wilson Coefficients

The renormalization of the interaction terms C_iQ_i in the effective theory can also be achieved in a different, but equivalent, way by using the standard counter-term method. Here C_i are treated as coupling constants, which have to be renormalized. We follow here [111].

To this end let us consider $\mathscr{H}_{\rm eff}$ as the starting point with fields and "coupling constants" C_i regarded as bare quantities. They are renormalized according to ($q = s, c, u, d$)

$$q^{(0)} = Z_q^{1/2} q, \qquad C_i^{(0)} = Z_{ij}^c C_j, \tag{5.35}$$

where $\hat{Z}^c$ denotes the renormalization matrix for the couplings C_j. It is evident that $\hat{Z}^c$ must be somehow related to the renormalization matrix $\hat{Z}$ in (5.31). Let us find this relation.

Omitting the factor $(G_F/\sqrt{2})V_{cs}^* V_{ud}$ we have

$$\mathscr{H}_{\rm eff} = C_i^{(0)} Q_i(q^{(0)}) \equiv Z_q^2 Z_{ij}^c C_j Q_i \equiv C_i Q_i + (Z_q^2 Z_{ij}^c - \delta_{ij}) C_j Q_i, \tag{5.36}$$

where the first term on the r.h.s. is written in terms of renormalized couplings and fields (C_iQ_i) and the second term is a counter–term. The argument $q^{(0)}$ on the l.h.s. of (5.36) indicates that the interaction vertices Q_i are composed of bare fields. Using (5.36) we get the finite renormalized result

$$A_{\rm eff} = Z_q^2 Z_{ij}^c C_j \langle Q_i \rangle^{(0)}. \tag{5.37}$$

On the other hand, using (5.30), we have

$$A_{\rm eff} = C_j \langle Q_j \rangle = C_j Z_{ji}^{-1} Z_q^2 \langle Q_i \rangle^{(0)}. \tag{5.38}$$

Hence comparing the last two equations we finally find the relation

$$Z_{ij}^c = Z_{ji}^{-1}. \tag{5.39}$$

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This relation will turn out to be very useful in deriving the renormalization group equations for the couplings C_i.

Operator Mixing and Diagonalization

We have just seen that gluonic corrections to the matrix element of the original operator Q_2 are not just proportional to Q_2 itself, but involve the additional structure Q_1. Therefore, besides a Q_2-counter-term, a counter-term involving Q_1 is needed to renormalize this matrix element. Similarly the renormalization of Q_1 requires both Q_1 and Q_2 counter-terms. We say that the operators Q_1 and Q_2 *mix under renormalization*.

For the study of the renormalization group properties of the system (Q_1, Q_2), it is useful to diagonalize it by going to a different operator basis defined by

$$Q_\pm = \frac{Q_2 \pm Q_1}{2}, \qquad C_\pm = C_2 \pm C_1 \ . \tag{5.40}$$

The new operators Q_+ and Q_- are renormalized independent of each other:

$$Q_\pm^{(0)} = Z_\pm Q_\pm, \tag{5.41}$$

where

$$Z_\pm = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\left(\mp 3\frac{N \mp 1}{N}\right) \ . \tag{5.42}$$

In this new basis the OPE reads

$$A \equiv A_+ + A_- = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} (C_+(\mu)\langle Q_+(\mu)\rangle + C_-(\mu)\langle Q_-(\mu)\rangle) \ , \tag{5.43}$$

where $(S_\pm = (S_2 \pm S_1)/2)$,

$$A_\pm = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{ud} \left[\left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right)\right) S_\pm + \left(\frac{3}{N} \mp 3\right)\frac{\alpha_s}{4\pi}\ln\left(\frac{M_W^2}{-p^2}\right) S_\pm\right] \tag{5.44}$$

and

$$\langle Q_\pm(\mu)\rangle = \left(1 + 2C_F\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right)\right) S_\pm + \left(\frac{3}{N} \mp 3\right)\frac{\alpha_s}{4\pi}\ln\left(\frac{\mu^2}{-p^2}\right) S_\pm \ , \tag{5.45}$$

$$C_\pm(\mu) = 1 + \left(\frac{3}{N} \mp 3\right)\frac{\alpha_s}{4\pi}\ln\left(\frac{M_W^2}{\mu^2}\right) \ . \tag{5.46}$$

Factorization of SD and LD

We have just witnessed in explicit terms the most important feature of the OPE: factorization of short-distance (coefficients) and long-distance (operator matrix elements) contributions. Schematically, this factorization has the following structure at $O(\alpha_s)$ (r is a coefficient):

$$\left(1 + \alpha_s r \ln\left(\frac{M_W^2}{-p^2}\right)\right) \doteq \left(1 + \alpha_s r \ln\left(\frac{M_W^2}{\mu^2}\right)\right) \cdot \left(1 + \alpha_s r \ln\left(\frac{\mu^2}{-p^2}\right)\right), \tag{5.47}$$

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which is achieved by the following splitting of the logarithm

$$\ln \frac{M_W^2}{-p^2} = \ln\left(\frac{M_W^2}{\mu^2}\right) + \ln\left(\frac{\mu^2}{-p^2}\right), \tag{5.48}$$

or from the point of view of the integration over some virtual momenta through the splitting

$$\int_{-p^2}^{M_W^2} \frac{dk^2}{k^2} = \int_{\mu^2}^{M_W^2} \frac{dk^2}{k^2} + \int_{-p^2}^{\mu^2} \frac{dk^2}{k^2}. \tag{5.49}$$

In particular the last formula makes it clear that the Wilson coefficients contain the contributions from large virtual momenta of the loop correction from scales $\mu = O(1\,\text{GeV})$ to M_W, whereas the low-energy contributions are separated into the matrix elements. The renormalization scale μ acts as the scale at which the full contribution to the amplitude is separated into a low-energy and a high-energy part.

Independence of C_i from External States

Let us next return to the issue of the infrared divergences in the process of matching. In the matching discussed explicitly earlier, they are regulated by taking $p^2 \neq 0$. They appear both in A_full and A_eff. Yet as we have shown earlier, the dependence of A_full on p^2, representing the long-distance structure of A, is, from the point of view of the effective theory, fully contained in $\langle Q_i \rangle$, and the Wilson coefficients C_i are free from this dependence.

Because the coefficient functions do not depend on the external states, any external state can be used for their extraction, the only requirement being that the infrared (and mass) singularities are properly regularized. In our example an off-shell momentum p for massless external quarks has been used, but such a choice is clearly one of several possibilities. In general one could work with any other arbitrary momentum configuration, on-shell or off-shell, with or without external quark mass, with infrared divergences regulated by off-shell momenta, quark masses, a fictitious gluon mass, or by dimensional regularization. All these methods would give the same results for C_i.

In particular the dimensional regularization of infrared divergences is very convenient as many integrals simplify considerably. Older discussions of dimensional infrared regularization can be found in Muta's book [70] and also in a paper by Marciano [115]. This method has been used, for instance, in calculating NLO corrections to $K \to \pi\nu\bar{\nu}$ [116] and also for the matching conditions in $B \to X_s\gamma$ [117, 118]. In particular as we stressed in [118], the distinction of $1/\varepsilon$ ultraviolet divergences from the infrared ones is not necessary because after proper renormalization of ultraviolet singularities, the leftover divergences are of infrared origin only. These singularities cancel then automatically in the process of matching. To this end, however, it is essential to perform the matching at all stages in $D = 4 - 2\varepsilon$ dimensions. This implies that already at the NLO level, $O(\varepsilon)$ terms in Wilson coefficients have to be kept at the intermediate stages of the calculation. More details on this efficient technique can be found in the papers just quoted.

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5.1.4 OPE and the Renormalization Group

Preliminaries

So far we have computed

$$C_{\pm}(\mu) = 1 + \left(\frac{3}{N} \mp 3\right) \frac{\alpha_s}{4\pi} \ln\left(\frac{M_W^2}{\mu^2}\right), \tag{5.50}$$

in ordinary perturbation theory. Unfortunately for $\mu = 1\,\text{GeV}$ the first-order correction term amounts to 65 – 130%, although $\alpha_s/4\pi \approx 4\%$. This finding illustrates explicitly the breakdown of the naive perturbative expansion caused by the appearance of large logarithms originating in the presence of largely disparate scales M_W and $\mu = 1\,\text{GeV}$.

Clearly, the result in (5.50) can only be used for $\mu = O(M_W)$. For $\mu \ll M_W$ we have to sum the large logarithms to all orders of perturbation theory before we can trust our result for $C_{\pm}$. Fortunately we have developed in Section 4.3 a very powerful technique to sum such logarithms, and we know exactly what we have to do. Yes, to sum these large logs we have to find renormalization group equations for $C_{\pm}$ and solve them.

Renormalization Group Equations for $C_{\pm}$

The renormalization group equations for $C_{\pm}$ follow from the fact, that the unrenormalized Wilson coefficients $C_{\pm}^{(0)}$ do not depend on μ. Using the relations (5.35) and (5.39), properly adapted to the diagonal basis, we have first

$$C_{\pm} = Z_{\pm} C_{\pm}^{(0)}, \qquad Q_{\pm}^{(0)} = Z_{\pm} Q_{\pm}, \tag{5.51}$$

and subsequently

$$\boxed{\frac{dC_{\pm}(\mu)}{d\ln\mu} = \gamma_{\pm}(g) C_{\pm}(\mu).} \tag{5.52}$$

Here $\gamma_{\pm}$ is the anomalous dimension of the operator $Q_{\pm}$ and given by

$$\boxed{\gamma_{\pm}(g) = \frac{1}{Z_{\pm}} \frac{dZ_{\pm}}{d\ln\mu}.} \tag{5.53}$$

Comparing (5.52) and (5.53) with (4.121) and (4.123), respectively, we see great similarities with the case of the running quark mass. The only modification is the opposite sign in (5.52). Consequently, many relevant formulas of Section 4.3 can be immediately employed. Here we go:

- In the MS ($\overline{\text{MS}}$)-scheme

$$Z_{\pm} = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_{\pm,k}(g), \tag{5.54}$$

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and consequently

$$\gamma_\pm(g) = -2g^2 \frac{\partial Z_{\pm,1}(g)}{\partial g^2} \,. \tag{5.55}$$

- Using then

$$Z_\pm = 1 + \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\left(\mp 3\frac{N \mp 1}{N}\right), \tag{5.56}$$

as obtained in (5.42) gives the one-loop anomalous dimensions of $Q_\pm$:

$$\boxed{\gamma_\pm(\alpha_s) = \frac{\alpha_s}{4\pi}\gamma_\pm^{(0)}, \qquad \gamma_\pm^{(0)} = \pm 6\frac{N \mp 1}{N}.} \tag{5.57}$$

- The solution of (5.52) is given as for $m(\mu)$ in (4.152):

$$\boxed{C_\pm(\mu) = U_\pm(\mu, \mu_W)\, C_\pm(\mu_W),} \tag{5.58}$$

where $\mu_W = O(M_W)$ and $U_\pm(\mu, \mu_W)$ is the evolution function:

$$\boxed{U_\pm(\mu, \mu_W) = \exp\left[\int_{g(\mu_W)}^{g(\mu)} dg' \frac{\gamma_\pm(g')}{\beta(g')}\right] .} \tag{5.59}$$

- Using (5.57) and $\beta(g) = -\beta_0 g^3/16\pi^2$ we can now find $C_\pm(\mu)$ by using the leading term in the formula (4.153) for $m(\mu)$. Setting $\mu_0 = M_W$ and taking into account the relative sign between (4.151) and (5.52), we have

$$\boxed{C_\pm(\mu) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{\frac{\gamma_\pm^{(0)}}{2\beta_0}} C_\pm(M_W).} \tag{5.60}$$

- To complete the calculation, we use the fact that at $\mu = M_W$ no large logarithms are present, and $C_\pm(M_W)$ can be calculated in ordinary perturbation theory. From (5.46) we have in LO

$$C_\pm(M_W) = 1, \tag{5.61}$$

and consequently for $\mu = \mu_b = O(m_b)$

$$\boxed{C_\pm(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma_\pm^{(0)}}{2\beta_0}} .} \tag{5.62}$$

We have now summed all leading logarithms, and the important formula (5.62) gives the coefficients $C_\pm$ in the leading log approximation or in other words the leading term of

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the RG improved perturbation theory. For instance, specializing to the case of $f = 5$ and $\mu_b = O(m_b)$ we obtain

$$\boxed{C_+(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{6}{23}}, \qquad C_-(\mu_b) = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{-12}{23}},} \tag{5.63}$$

with α_s given by the leading expression (4.158). For $\mu_b = 5.0$ GeV and $\Lambda^{(5)}_{\overline{\rm MS}} = 225$ MeV, one finds $C_+(\mu_b) = 0.847$ and $C_-(\mu_b) = 1.395$, i.e., suppression of C_+ and an enhancement of C_- relative to $C_- = C_+ = 1$ without QCD corrections. The corresponding enhancements and suppressions for scales $O(1$ GeV$)$ reflect to some extent the dominance of the $\Delta I = 1/2$ transitions over $\Delta I = 3/2$ transitions in $K \to \pi\pi$ decays (the $\Delta I = 1/2$ rule) first analyzed in QCD in [119, 120]. These short-distance effects are insufficient, however, to explain the dominance of $\Delta I = 1/2$ transitions observed experimentally. We will return to this issue in Section 7.2.3.

Choice of the Matching Scale

In calculating (5.60) we have set the high-energy matching scale to M_W. The choice of the high-energy matching scale, to be denoted by μ_W, is, of course, not unique. The only requirement is that $\mu_W = O(M_W)$ to avoid large logarithms $\ln(M_W/\mu_W)$. However, we know from (5.50) that in the LO approximation, in which $O(\alpha_s)$ terms are dropped in $C_\pm(\mu_W)$, we have using (5.61)

$$C_\pm(\mu_W) = C_\pm(M_{\rm W}) + O(\alpha_s) = 1. \tag{5.64}$$

Consequently in this approximation we also have

$$C_\pm(\mu_b) = \left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma^{(0)}_\pm}{2\beta_0}} = \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma^{(0)}_\pm}{2\beta_0}} (1 + O(\alpha_s)), \tag{5.65}$$

which differs from (5.62) by $O(\alpha_s)$ corrections.

We observe that a change of μ_W around the value of M_W causes an ambiguity of $O(\alpha_s)$ in the Wilson coefficient. This ambiguity represents a theoretical uncertainty in the determination of $C_\pm(\mu_b)$. To reduce it, it is necessary to go beyond the leading order. We will do this in the next section. Similar ambiguity exists in the choice of the low-energy scale $\mu_b = O(m_{\rm b})$, as we will discuss at various places in this book. In fact, the uncertainty here is larger as $\alpha_s(\mu_b)$ is larger than $\alpha_s(\mu_W)$.

Threshold Effects in LO

The evolution function U depends on f through $\alpha_s^{(f)}$, and β_0 in the exponent. One can generalize the renormalization group evolution from M_W down to say $\mu_c = O(m_c)$ to include the threshold effect of the b-quark as follows

$$\boxed{C_\pm(\mu_c) = U_\pm^{(f=4)}(\mu_c, \mu_b) U_\pm^{(f=5)}(\mu_b, M_W) C_\pm(M_{\rm W}),} \tag{5.66}$$

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which is valid in LO. Thus (5.63) generalizes to

$$\boxed{C_+(\mu_c)=\left[\frac{\alpha_s^{(4)}(\mu_b)}{\alpha_s^{(4)}(\mu_c)}\right]^{\frac{6}{25}}\left[\frac{\alpha_s^{(5)}(M_W)}{\alpha_s^{(5)}(\mu_b)}\right]^{\frac{6}{23}},\quad C_-(\mu_c)=\left[\frac{\alpha_s^{(4)}(\mu_b)}{\alpha_s^{(4)}(\mu_c)}\right]^{\frac{-12}{25}}\left[\frac{\alpha_s^{(5)}(M_W)}{\alpha_s^{(5)}(\mu_b)}\right]^{\frac{-12}{23}}.} \tag{5.67}$$

Again also here there is an ambiguity in μ_c, which can only be reduced by going to NLO.

RGE for C_i: The Case of Operator Mixing

The coefficients $C_i(\mu)$ can be now calculated by inverting (5.40) with the result

$$\boxed{C_1(\mu)=\frac{C_+(\mu)-C_-(\mu)}{2},\qquad C_2(\mu)=\frac{C_+(\mu)+C_-(\mu)}{2},} \tag{5.68}$$

where $C_\pm(\mu)$ are given in (5.62) or (5.67).

Yet, it is instructive to derive (5.68) by using a procedure that one can also apply to more complicated situations in which several operators mix under renormalization. To this end we write

$$\vec{C}^T=(C_1,C_2),\qquad \vec{Q}^T=(Q_1,Q_2). \tag{5.69}$$

Then

$$\vec{C}^{(0)}=\hat{Z}_c\vec{C},\qquad \vec{Q}^{(0)}=\hat{Z}\vec{Q}, \tag{5.70}$$

with $\hat{Z}_c^T=\hat{Z}^{-1}$. Defining next the anomalous dimension matrix $\hat{\gamma}$ by

$$\boxed{\hat{\gamma}=\hat{Z}^{-1}\frac{d\hat{Z}}{d\ln\mu},} \tag{5.71}$$

the μ-independence of $\vec{C}^{(0)}$ implies

$$\boxed{\frac{d\vec{C}(\mu)}{d\ln\mu}=\hat{\gamma}^T(\alpha_s)\vec{C}(\mu).} \tag{5.72}$$

The solution of this equation is

$$\boxed{\vec{C}(\mu)=\hat{U}(\mu,M_W)\vec{C}(M_W),} \tag{5.73}$$

where

$$\boxed{\hat{U}(\mu,M_{\rm W})=\exp\left[\int_{g(M_{\rm W})}^{g(\mu)}dg'\frac{\hat{\gamma}^T(g')}{\beta(g')}\right],} \tag{5.74}$$

is the RG-evolution matrix.

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In the MS ($\overline{\text{MS}}$)-scheme we have

$$\hat{Z} = \hat{1} + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} \hat{Z}_k(g), \tag{5.75}$$

and

$$\hat{\gamma}(g) = -2g^2 \frac{\partial \hat{Z}_1(g)}{\partial g^2}. \tag{5.76}$$

Consequently, using

$$\hat{Z} = 1 + \frac{\alpha_s}{4\pi} \frac{1}{\varepsilon} \begin{pmatrix} 3/N & -3 \\ -3 & 3/N \end{pmatrix} \tag{5.77}$$

we have to first order in α_s [119, 120]

$$\boxed{\hat{\gamma}(\alpha_s) = \frac{\alpha_s}{4\pi} \hat{\gamma}^{(0)} = \frac{\alpha_s}{4\pi} \begin{pmatrix} -6/N & 6 \\ 6 & -6/N \end{pmatrix}.} \tag{5.78}$$

To find $C_i(\mu)$ let us write the LO evolution matrix as

$$\boxed{\hat{U}^{(0)}(\mu, M_{\text{W}}) = \hat{V} \left(\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)} \right]^{\frac{\vec{\gamma}^{(0)}}{2\beta_0}} \right)_D \hat{V}^{-1},} \tag{5.79}$$

where $\hat{V}$ diagonalizes $\hat{\gamma}^{(0)T}$

$$\boxed{\hat{\gamma}_D^{(0)} = \hat{V}^{-1} \hat{\gamma}^{(0)T} \hat{V},} \tag{5.80}$$

and $\vec{\gamma}^{(0)}$ is the vector containing the diagonal elements of the diagonal matrix:

$$\boxed{\hat{\gamma}_D^{(0)} = \begin{pmatrix} \gamma_+^{(0)} & 0 \\ 0 & \gamma_-^{(0)} \end{pmatrix},} \tag{5.81}$$

with $\gamma_\pm^{(0)}$ given in (5.57). Using

$$\hat{V} = \hat{V}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{5.82}$$

and

$$\vec{C}^T(M_{\text{W}}) = (C_1(M_{\text{W}}), C_2(M_{\text{W}})) = (0, 1), \tag{5.83}$$

we reproduce (5.68) with $C_\pm(\mu)$ given by (5.63).

The threshold effects can be incorporated as in (5.66)

$$\boxed{\vec{C}(\mu_c) = \hat{U}^{(f=4)}(\mu_c, \mu_b) \hat{U}^{(f=5)}(\mu_b, \mu_W) \vec{C}(\mu_W).} \tag{5.84}$$

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It is evident that this procedure is valid for arbitrary number of operators mixing under renormalization. However, for more complicated situations one has to use computer programs like *Mathematica* to obtain analytic formulas like (5.68). We will give some examples later.

5.1.5 Summary of the Basic Formalism

It is a good moment to make a break and to summarize what we have achieved in our expedition so far. This will also allow us to make a strategy for the next steps, which as we will see are technically more advanced.

Ultimately, our goal is the evaluation of weak decay amplitudes involving hadrons in the framework of a low-energy effective theory, of the form

$$\langle \mathscr{H}_{\rm eff} \rangle = \frac{G_F}{\sqrt{2}} V_{CKM} \langle \vec{Q}^T(\mu) \rangle \vec{C}(\mu), \tag{5.85}$$

where μ denotes a scale of the order of the mass of the decaying hadron. The procedure for this calculation can be divided into the following three steps.

Step 1: Matching in perturbation theory

Calculate the Wilson coefficients $\vec{C}(\mu_W)$ at $\mu_W = O(M_W)$ to the desired order in α_s. Because logarithms of the form $\ln(\mu_W/M_W)$ are not large, this can be performed in ordinary perturbation theory. In the case of the operators $Q_{1,2}$ and in the LO approximation, we simply have $C_1(\mu_W) = 0$, $C_2(\mu_W) = 1$, or $C_\pm(\mu_W) = 1$.

This step amounts to matching the full theory onto a five-quark effective theory. In this process $W^\pm$, Z^0, the top-quark, and generally all heavy particles with masses higher than M_W are integrated out unless they have been already integrated out at a much higher scale. In the case of $Q_{1,2}$ analyzed so far, the effect of integrating out the top-quark is signaled by the use of $\alpha^{(5)}(\mu)$ instead of $\alpha^{(6)}(\mu)$ for $\mu \le M_W$. Later when we move to other decays, the effect of integrating out the top quark will be more profound.

The matching in question is achieved using the following procedure:

- Calculation of the amplitude in the full theory,
- Calculation of the operator matrix elements,
- Extraction of $C_i(\mu_W)$ from $A_{\rm full} = A_{\rm eff}$.

The resulting $C_i(\mu_W)$ depend generally on the masses of the heavy particles that have been integrated out. Again in the special case of $Q_{1,2}$, this dependence is absent in C_i, although it is present in G_F as G_F depends on M_W.

Step 2: RG improved perturbation theory

- Calculation of the anomalous dimensions of the operators,
- Solution of the renormalization group equation for $\vec{C}(\mu)$,
- Evolution of the coefficients from μ_W down to the appropriate low-energy scale μ

$$\vec{C}(\mu) = \hat{U}(\mu, \mu_W)\vec{C}(\mu_W)\,. \tag{5.86}$$

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Step 3: Nonperturbative regime

Calculation of the hadronic matrix elements $\langle \vec{Q}(\mu) \rangle$, normalized at the appropriate low-energy scale μ, by means of some nonperturbative method. These days lattice gauge theories dominate these calculations although other methods can still provide useful results in cases where lattice methods still have some difficulties. We will briefly discuss these methods in Chapter 7.

Important issues in this procedure are

- **Factorization** of short- and long-distance contributions:
  - $\vec{C}(\mu)$: contributions from scales *higher* than μ
  - $\langle \vec{Q}(\mu) \rangle$: contributions from scales *lower* than μ
  - Cancellation of the μ-dependence between $C_i(\mu)$ and $\langle Q_i(\mu) \rangle$.
- **Summation of large logs** by means of the RG method. More specifically, in the nth order of RG improved perturbation theory the terms

$$\alpha_s^n(\mu) \left(\alpha_s(\mu) \ln \frac{M_W}{\mu} \right)^k \tag{5.87}$$

are summed to all orders in k ($k = 0, 1, 2, \ldots$). This approach is justified as long as $\alpha_s(\mu)$ is small enough. The leading order corresponds in most cases to $n = 0$, the NLO to $n = 1$. In certain processes these canonical values of n may change.

5.1.6 Future Generalizations

Until now, our application of the basic formalism just summarized, concentrated on the current-current operators Q_1 and Q_2 in the LO approximation. In the following sections we will generalize this discussion in several aspects:

- We will generalize the calculation of the couplings $C_i(\mu)$ beyond the LO approximation.
- We will include new operators originating in penguin diagrams of various sort (gluon-penguins, photon-penguins, Z^0-penguins, chromo-magnetic penguins). These operators are generally called *penguin operators*. The new feature will be the m_t-dependence of $C_i(\mu_W)$ of these new operators.
- We will also include new operators originating in *box diagrams*. Also in this case the coefficients $C_i(\mu_W)$ will be m_t-dependent.
- In the process of including new operators it will turn out to be necessary in certain cases to consider also renormalization group equations involving simultaneously C_i of order $O(1)$, $O(\alpha_s)$, and $O(\alpha)$ with $\alpha = \alpha_{\rm QED}$ and even $O(\alpha\alpha_s)$.
- Finally, we will develop efficient methods for the calculation of the anomalous dimensions of the operators Q_i.

In the next section we will begin these generalizations by including NLO QCD corrections to $C_\pm(\mu)$. We will do this in such a manner that the generalization of the formulas, listed in the next section to more complicated processes involving penguin operators will be straightforward.

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5.1.7 Motivations for NLO Calculations

Going beyond the LO approximation is certainly an important but a nontrivial step. For this reason we need some motivations to perform this step. Here are the main reasons for going beyond LO:

- The NLO is first of all necessary to test the validity of the renormalization group improved perturbation theory.
- Without going to NLO the QCD scale $\Lambda_{\overline{\rm MS}}$ or equivalently $\alpha_s^{\overline{\rm MS}}$ extracted from various high-energy processes cannot be used meaningfully in weak decays.
- Due to renormalization group invariance the physical amplitudes do not depend on the scales μ present in α_s or in the running quark masses, in particular $m_t(\mu)$, $m_b(\mu)$ and $m_c(\mu)$. However, in perturbation theory this property is broken through the truncation of the perturbative series. Consequently one finds sizable scale ambiguities in the leading order, which can be reduced considerably by going to NLO. An example of such an ambiguity is the choice of the high-energy matching scale μ_W discussed earlier. Larger-scale ambiguities appear at the b-quark and charm-quark thresholds due to larger values of $\alpha_s(\mu)$ at these scales.
- The Wilson Coefficients are renormalization scheme dependent quantities. This scheme dependence appears first at NLO. For a proper matching of the short-distance contributions to the long-distance matrix elements obtained from lattice calculations, it is essential to calculate NLO corrections. The same is true for inclusive heavy quark decays in which the hadron decay can be modeled by a decay of a heavy quark and the matrix elements of Q_i can be effectively calculated in an expansion in $1/m_b$.
- In several cases the central issue of the top quark mass dependence is strictly an NLO effect.

The first NLO calculations for weak decays were performed in the 1980s, and an important progress in calculating them for most interesting decays has been done in 1990s. First NNLO calculations were performed around the year 2000. A history of the NLO and NNLO efforts has been summarized in [77] with the last update in 2014.

It should be remarked that in the 1990s such involved calculations were really not required because of large experimental errors and theoretical and parametric uncertainties. Therefore the early NLO calculations could not be fully appreciated at the time of their performance. But in the present days they are mandatory in view of flavor precision era that we entered few years ago and in view of the impressive progress made by lattice calculations. It is therefore mandatory to discuss this important topic next.

5.2 Current-Current Operators beyond Leading Order

5.2.1 Preliminaries

We will now generalize the formulas of the previous section beyond the LO approximation concentrating on the Wilson coefficients $C_\pm$ and $C_{1,2}$. We will begin with the case without

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operator mixing. Subsequently we will generalize our discussion to the case of operator mixing. Next we will develop methods for the calculation of anomalous dimensions both at the one-loop level and the two-loop level. While we do not have space to present an explicit two-loop calculation of anomalous dimensions, we will derive explicitly the one-loop anomalous dimension matrix (5.78). A detailed discussion of renormalization scheme and renormalization scale dependences and of their cancellations in physical amplitudes is an important part of this section. We will also discuss the issue of the so-called evanescent operators, which have to be taken into account in a proper calculation of the anomalous dimensions at the two-loop level. This section ends with a compendium of NLO formulas, which allow the calculation of Wilson coefficients of current-current four-quark operators at the NLO level.

5.2.2 The Case without Operator Mixing

Let us consider the coefficients $C_\pm(\mu)$ for which we have the general expression:

$$\boxed{C_\pm(\mu) = U_\pm(\mu, M_{\rm W}) C_\pm(M_{\rm W}),} \tag{5.88}$$

where

$$\boxed{U_\pm(\mu, M_{\rm W}) = \exp\left[\int_{g(M_{\rm W})}^{g(\mu)} dg' \frac{\gamma_\pm(g')}{\beta(g')}\right],} \tag{5.89}$$

and we have set $\mu_W = M_W$ to simplify the formulas listed here. This restriction will be relaxed whenever it will turn out to be important.

At NLO we have to include nonlogarithmic $O(\alpha_s)$ correction to $C_\pm(M_W)$, which we have neglected in our one-loop calculations in the previous section. Therefore we write

$$\boxed{C_\pm(M_W) = 1 + \frac{\alpha_s(M_W)}{4\pi} B_\pm,} \tag{5.90}$$

with the coefficients $B_\pm$ still to be determined.

Moreover, we have to include two-loop contributions to $\gamma_\pm(\alpha_s)$ and $\beta(g)$:

$$\boxed{\gamma_\pm(\alpha_s) = \gamma_\pm^{(0)} \frac{\alpha_s}{4\pi} + \gamma_\pm^{(1)} \left(\frac{\alpha_s}{4\pi}\right)^2,} \tag{5.91}$$

$$\boxed{\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} - \beta_1 \frac{g^5}{(16\pi^2)^2},} \tag{5.92}$$

which are represented by the second terms on the r.h.s. of these two equations. The fact that for the NLO analysis we need simultaneously one-loop contributions to $C_\pm(M_W)$ but two-loop contributions for $\gamma_\pm(\alpha_s)$ and $\beta(g)$ shows that we are going beyond the usual perturbation theory. But this should not be surprising. At LO we used $\gamma_\pm(\alpha_s)$ and $\beta(g)$ at one-loop level and could set $B_\pm = 0$. The one-loop calculations we performed in the

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previous section served only to determine $\gamma_\pm(\alpha_s)$. A more efficient method for calculating $\gamma_\pm(\alpha_s)$ will be developed in this section.

Inserting the last two formulas into (5.89) and expanding in α_s we find

$$\boxed{U_\pm(\mu, M_W) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J_\pm\right] \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_\pm} \left[1 - \frac{\alpha_s(M_W)}{4\pi} J_\pm\right]} \tag{5.93}$$

with

$$\boxed{J_\pm = \frac{d_\pm}{\beta_0}\beta_1 - \frac{\gamma_\pm^{(1)}}{2\beta_0}, \qquad d_\pm = \frac{\gamma_\pm^{(0)}}{2\beta_0}.} \tag{5.94}$$

This is similar to the μ-dependence of the quark mass discussed previously and given in (4.153) except that we have written the evolution function in a particular way: The couplings $\alpha_s(\mu)$ increase by going from right to left. This is clearly not necessary in the absence of operator mixing, but it will turn out to be useful for the future generalization to the case of operator mixing. One deals then with matrices $\hat{\gamma}$ and $\hat{J}$, and the order of matrices in a product matters.

Inserting (5.93) and (5.90) into (5.88), we find an important formula for $C_\pm(\mu)$ in the NLO approximation:

$$\boxed{C_\pm(\mu) = \left[1 + \frac{\alpha_s(\mu)}{4\pi} J_\pm\right] \left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_\pm} \left[1 + \frac{\alpha_s(M_W)}{4\pi}(B_\pm - J_\pm)\right].} \tag{5.95}$$

Let us next outline the procedure for finding $B_\pm$. Because the operators Q_+ and Q_- do not mix under renormalization, B_+ and B_- can be found separately. The procedure for finding $B_\pm$ amounts to the generalization of the matching procedure in LO to include in addition to logarithms also constant $O(\alpha_s)$ terms. The diagrams to be calculated are again the ones in Figures 5.2 and 5.3.

Step 1

$$A^\pm_{\text{full}} = \frac{G_F}{\sqrt{2}} \left(1 + \frac{\alpha_s(\mu_W)}{4\pi} \left[-\frac{\gamma_\pm^{(0)}}{2} \ln \frac{M_W^2}{-p^2} + \tilde{A}_\pm^{(1)}\right]\right) S_\pm\,, \tag{5.96}$$

where $S_\pm$ are the tree matrix elements and $\tilde{A}_\pm^{(1)}$ the terms that we dropped in the LO calculation.

Step 2

$$\begin{aligned} A^\pm_{\text{eff}} &= \frac{G_F}{\sqrt{2}} C_\pm(\mu_W) \langle Q_\pm(\mu_W)\rangle \\ &= \frac{G_F}{\sqrt{2}} C_\pm(\mu_W) \left(1 + \frac{\alpha_s(\mu_W)}{4\pi} \left[\frac{\gamma_\pm^{(0)}}{2} \ln \frac{-p^2}{\mu_W^2} + \tilde{r}_\pm\right]\right) S_\pm\,, \end{aligned} \tag{5.97}$$

where $\tilde{r}_\pm$ are nonlogarithmic $O(\alpha_s)$ terms in $\langle Q_\pm(\mu_W)\rangle$, which we have to keep now.

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Step 3

Comparison of (5.96) and (5.97) yields

$$\boxed{C_\pm(\mu_W) = 1 + \frac{\alpha_s(\mu_W)}{4\pi}\left[-\frac{\gamma_\pm^{(0)}}{2}\ln\frac{M_W^2}{\mu_W^2} + B_\pm\right]}\,, \tag{5.98}$$

where

$$\boxed{B_\pm = \tilde{A}_\pm^{(1)} - \tilde{r}_\pm\,.} \tag{5.99}$$

Setting $\mu_W = M_{\rm W}$, we reproduce (5.90). Any infrared dependence like $\ln(-p^2)$ or any special properties of the external quark states present in $\tilde{A}_\pm^{(1)}$ and $\tilde{r}_\pm$ cancel in the difference (5.99) so that $B_\pm$ are just numerical constants independent of external states. We will give the numerical values of $B_\pm$ later.

Needless to say these formulas apply to any operator that does not mix with other operators under renormalization. The operator dependence enters only through the values of $\gamma_\pm^{(0)}$ and $B_\pm$.

5.2.3 The Case of Operator Mixing

Preliminaries

Let us generalize the preceding discussion to the case of operator mixing and any number of operators that mix under renormalization. Now

$$\mathscr{H}_{\rm eff} = \frac{G_F}{\sqrt{2}}\sum_i C_i(\mu)Q_i(\mu) \equiv \frac{G_F}{\sqrt{2}}\vec{Q}^T(\mu)\vec{C}(\mu)\,, \tag{5.100}$$

where the index i runs over all contributing operators, in our case Q_1 and Q_2.

The Wilson coefficient functions are given then by

$$\boxed{\vec{C}(\mu) = \hat{U}(\mu,\mu_W)\vec{C}(\mu_W)\,.} \tag{5.101}$$

Our goal is to find $\vec{C}(\mu_W)$ and the evolution matrix $\hat{U}(\mu,\mu_W)$ keeping NLO corrections.

Determination of $\vec{C}(\mu_W)$

The procedure for finding $\vec{C}(\mu_W)$ proceeds again in three steps:

Step 1

The amplitude in the full theory after field renormalization is given by:

$$A_{\rm full} = \frac{G_F}{\sqrt{2}}\vec{S}^T\left(\vec{A}^{(0)} + \frac{\alpha_s(\mu_W)}{4\pi}\vec{A}^{(1)}\right). \tag{5.102}$$

Here $\vec{S}$ denotes the tree-level matrix elements of the operators $\vec{Q}$. To simplify the presentation we have absorbed the logarithms in the $O(\alpha_s)$ term $\vec{A}^{(1)}$, which also contains

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nonlogarithmic terms as in (5.96). Later we will discuss a specific example that will exhibit the detailed structure of (5.102) more transparently.

Step 2
In the effective theory, after quark field renormalization and the renormalization of the operators through

$$\vec{Q}^{(0)} = \hat{Z}\vec{Q}, \tag{5.103}$$

the renormalized matrix elements of the operators are

$$\langle \vec{Q}(\mu_W)\rangle = \left(\hat{1} + \frac{\alpha_s(\mu_W)}{4\pi}\hat{r}\right)\vec{S}, \tag{5.104}$$

and consequently

$$A_{\text{eff}} = \frac{G_F}{\sqrt{2}}\vec{S}^T\left(\hat{1} + \frac{\alpha_s(\mu_W)}{4\pi}\hat{r}^T\right)\vec{C}(\mu_W). \tag{5.105}$$

Again $\hat{r}$ contains the relevant logarithms together with the nonlogarithmic terms as in (5.97).

Step 3
Equating (5.102) and (5.105), we obtain

$$\boxed{\vec{C}(\mu_W) = \vec{A}^{(0)} + \frac{\alpha_s(\mu_W)}{4\pi}\left(\vec{A}^{(1)} - \hat{r}^T\vec{A}^{(0)}\right).} \tag{5.106}$$

Renormalization Group Evolution

The next goal is the calculation of the evolution matrix $\hat{U}(\mu, \mu_W)$, including NLO corrections. We will do it for a general anomalous dimension matrix, so that our formulas can also be used in later sections in which new operators, in particular QCD penguin operators, will enter the scene.

The renormalization group equation for $\vec{C}$,

$$\boxed{\frac{d\vec{C}(\mu)}{d\ln\mu} = \hat{\gamma}^T(g)\vec{C}(\mu),} \tag{5.107}$$

has to be solved now with the boundary condition (5.106).

The general solution can be written down iteratively

$$\hat{U}(\mu, \mu_W) = 1 + \int_{g(\mu_W)}^{g(\mu)} dg_1 \frac{\hat{\gamma}^T(g_1)}{\beta(g_1)} + \int_{g(\mu_W)}^{g(\mu)} dg_1 \int_{g(\mu_W)}^{g_1} dg_2 \frac{\hat{\gamma}^T(g_1)}{\beta(g_1)}\frac{\hat{\gamma}^T(g_2)}{\beta(g_2)} + \cdots \tag{5.108}$$

which using $dg/d\ln\mu = \beta(g)$ solves

$$\frac{d}{d\ln\mu}\hat{U}(\mu, \mu_W) = \hat{\gamma}^T(g)\hat{U}(\mu, \mu_W)\,. \tag{5.109}$$

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The series in (5.108) can be written more compactly:

$$\boxed{\hat{U}(\mu,\mu_W) = T_g \exp\left[\int_{g(\mu_W)}^{g(\mu)} dg' \frac{\hat{\gamma}^T(g')}{\beta(g')}\right],} \tag{5.110}$$

where in the case $g(\mu) > g(\mu_W)$ the g-ordering operator T_g is defined through

$$T_g f(g_1)\dots f(g_n) = \sum_{perm} \Theta(g_{i_1} - g_{i_2})\dots\Theta(g_{i_{n-1}} - g_{i_n}) f(g_{i_1})\dots f(g_{i_n}). \tag{5.111}$$

It brings ordering of the functions $f(g_i)$ such that the coupling constants increase from right to left. The sum in (5.111) runs over all permutations $\{i_1,\dots,i_n\}$ of $\{1,2,\dots,n\}$. The T_g ordering is necessary because at NLO $[\hat{\gamma}(g_1),\hat{\gamma}(g_2)] \neq 0$. Indeed the matrices $\hat{\gamma}^{(0)}$ and $\hat{\gamma}^{(1)}$ in the perturbative expansion of the anomalous dimension matrix

$$\hat{\gamma}(\alpha_s) = \hat{\gamma}^{(0)}\frac{\alpha_s}{4\pi} + \hat{\gamma}^{(1)}\left(\frac{\alpha_s}{4\pi}\right)^2, \tag{5.112}$$

do not commute with each other.

Inserting (5.112) and the expansion (5.92) for $\beta(g)$ into (5.110), we can write the evolution matrix in analogy to (5.93) as

$$\boxed{\hat{U}(\mu,\mu_W) = \left[\hat{1} + \frac{\alpha_s(\mu)}{4\pi}\hat{J}\right]\hat{U}^{(0)}(\mu,\mu_W)\left[\hat{1} - \frac{\alpha_s(\mu_W)}{4\pi}\hat{J}\right].} \tag{5.113}$$

Now it is clear why we have written (5.93) in a special manner. It can be nicely generalized to the mixing case where the ordering of matrices matters. $\hat{U}^{(0)}$ in (5.113) is the leading evolution matrix, which we already encountered in (5.79):

$$\boxed{\hat{U}^{(0)}(\mu,\mu_W) = \hat{V}\left(\left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu)}\right]^{\frac{\vec{\gamma}^{(0)}}{2\beta_0}}\right)_D \hat{V}^{-1},} \tag{5.114}$$

where $\hat{V}$ diagonalizes $\hat{\gamma}^{(0)T}$

$$\boxed{\hat{\gamma}_D^{(0)} = \hat{V}^{-1}\gamma^{(0)T}\hat{V},} \tag{5.115}$$

and $\vec{\gamma}^{(0)}$ is the vector containing the diagonal elements of the diagonal matrix $\hat{\gamma}_D^{(0)}$.

The derivation of the analytic expression for the matrix $\hat{J}$ follows [121] and is also given in the appendix of [122]. As it is rather involved, we give here only the final result. To write down the expression for the matrix $\hat{J}$, we define the matrix

$$\boxed{\hat{G} = \hat{V}^{-1}\hat{\gamma}^{(1)T}\hat{V},} \tag{5.116}$$

and a matrix $\hat{H}$ whose elements are

$$\boxed{H_{ij} = \delta_{ij}\gamma_i^{(0)}\frac{\beta_1}{2\beta_0^2} - \frac{G_{ij}}{2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)}}.} \tag{5.117}$$

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Then $\hat{J}$ can be written compactly as follows

$$\boxed{\hat{J} = \hat{V}\hat{H}\hat{V}^{-1}.} \tag{5.118}$$

The formula for H_{ij} is not valid for $2\beta_0 + \gamma_i^{(0)} - \gamma_j^{(0)} = 0$, which happens for certain values of (i, j) for $f = 3$. For this case the correct formula can be found in [123].

Final Result for $\vec{C}(\mu)$

Putting all things together, we obtain the final result

$$\boxed{\vec{C}(\mu) = \left(1 + \frac{\alpha_s(\mu)}{4\pi}\hat{J}\right)\hat{U}^{(0)}(\mu, \mu_W)\left(\vec{A}^{(0)} + \frac{\alpha_s(\mu_W)}{4\pi}\left[\vec{A}^{(1)} - (\hat{r}^T + \hat{J})\,\vec{A}^{(0)}\right]\right).} \tag{5.119}$$

We will discuss this important result in more detail later stressing in particular its μ and renormalization scheme dependences and their cancellations by the ones of the hadronic matrix elements.

In the case of (Q_1, Q_2) the inclusion of the flavor thresholds in (5.101) is very similar to the LO case:

$$\vec{C}(\mu) = \hat{U}_3(\mu, \mu_c)\hat{U}_4(\mu_c, \mu_b)\hat{U}_5(\mu_b, \mu_W)\vec{C}(\mu_W), \tag{5.120}$$

where $\hat{U}_f$ is the evolution matrix for f effective flavors given in (5.113). This formula has to be slightly modified if the penguin operators are present. We will return to this point in Section 6.4.

The expression (5.120) applies to $\mu \le m_c$ and could be used for K meson decays. When the D meson system is considered, the first factor on the r.h.s. of this equation is absent. For B meson physics only $\hat{U}_5(\mu_b, \mu_W)$ is relevant.

5.2.4 Calculation of Anomalous Dimensions

Master Formulas

In the previous section we have calculated the anomalous dimension matrix in the process of the matching of the full and effective theories. In fact, comparing (5.31) with the last two terms in (5.32) and (5.33) one can see that the anomalous dimension matrix for $Q_{1,2}$ can be read off from the coefficients of the logarithms in the matrix elements $\langle Q_{1,2}\rangle$ in (5.32) and (5.33). In more complicated situations, where many operators are present and two-loop and higher-loop diagrams are involved, such a method is not very useful, and it is important to develop an efficient method for the calculation of anomalous dimensions. Here it comes:

- The evaluation of the amputated Green functions with insertion of the operators $\vec{Q}$ as in Figure 5.3 gives the relation

$$\langle\vec{Q}\rangle^{(0)} = Z_q^{-2}\hat{Z}\langle\vec{Q}\rangle \equiv \hat{Z}_{GF}\langle\vec{Q}\rangle, \tag{5.121}$$

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where $\hat{Z}$ is the renormalization constant matrix for the operators $\vec{Q}$, Z_q for the external quark fields, and $\hat{Z}_{GF}$ is just defined earlier. As the matrix elements $\langle\vec{Q}\rangle$ are finite, $\hat{Z}_{GF}$ collects all the divergences present in $\langle\vec{Q}\rangle^{(0)}$.

- Next, the anomalous dimension matrix is given by

$$\hat{\gamma}(g) = \hat{Z}^{-1}\frac{d\hat{Z}}{d\ln\mu}. \tag{5.122}$$

- In the MS (or $\overline{\text{MS}}$) scheme we have

$$\hat{Z} = \hat{1} + \sum_{k=1}^{\infty}\frac{1}{\varepsilon^k}\hat{Z}_k(g), \tag{5.123}$$

and consequently as derived in the previous section

$$\hat{\gamma}(g) = -2g^2\frac{\partial\hat{Z}_1(g)}{\partial g^2} = -2\alpha_s\frac{\partial\hat{Z}_1(\alpha_s)}{\partial\alpha_s}. \tag{5.124}$$

- For Z_q and $\hat{Z}_{GF}$ we have

$$Z_q = 1 + \sum_{k=1}^{\infty}\frac{1}{\varepsilon^k}Z_{q,k}(g), \qquad \hat{Z}_{GF} = \hat{1} + \sum_{k=1}^{\infty}\frac{1}{\varepsilon^k}\hat{Z}_{GF,k}(g). \tag{5.125}$$

As the matrix elements $\langle\vec{Q}\rangle$ are finite, the singularities in $\hat{Z}_{GF}$ are found directly from the calculation of the unrenormalized Green functions (5.121).

- From (5.121), (5.123), and (5.125) we find

$$\hat{Z}_1 = 2Z_{q,1}\hat{1} + \hat{Z}_{GF,1}. \tag{5.126}$$

- With

$$Z_{q,1} = a_1\frac{\alpha_s}{4\pi} + a_2\left(\frac{\alpha_s}{4\pi}\right)^2, \qquad \hat{Z}_{GF,1} = \hat{b}_1\frac{\alpha_s}{4\pi} + \hat{b}_2\left(\frac{\alpha_s}{4\pi}\right)^2, \tag{5.127}$$

we obtain by means of (5.124) *master formulas* for the elements of the one- and two-loop anomalous dimension matrices:

$$\boxed{\gamma_{ij}^{(0)} = -2[2a_1\delta_{ij} + (b_1)_{ij}], \qquad \gamma_{ij}^{(1)} = -4[2a_2\delta_{ij} + (b_2)_{ij}]\,.} \tag{5.128}$$

- In the case without mixing between operators, these expressions reduce to:

$$\boxed{\gamma^{(0)} = -2[2a_1 + b_1], \qquad \gamma^{(1)} = -4[2a_2 + b_2]\,.} \tag{5.129}$$

How to Use One-Loop Master Formulas

Let us illustrate how the first formula in (5.128) can be used to obtain the one-loop anomalous dimension matrix (5.78). From (5.27) we extract the coefficients of the $1/\varepsilon$ singularities to be

$$(b_1)_{11} = 2C_F + \frac{3}{N}, \qquad (b_1)_{12} = -3\,. \tag{5.130}$$

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Similarly from (5.28) we find

$$(b_1)_{21} = -3, \qquad (b_1)_{22} = 2C_F + \frac{3}{N}\,. \tag{5.131}$$

Now $a_1 = -C_F$. Consequently, the term $2a_1$ in (5.128) cancels precisely the $2C_F$ term present in $(b_1)_{11}$ and $(b_1)_{22}$. The leftover entries give the one-loop anomalous dimension matrix (5.78).

The fact that the renormalization of the external quark fields cancels the terms $2C_F$ in $(b_1)_{11}$ and $(b_1)_{22}$ is by no means accidental. It is a consequence of the vanishing of the anomalous dimension of the conserved weak current. Indeed, the master formula for the anomalous dimension of a current can be obtained by considering a two-point Green function instead of four-point functions considered in deriving the master formulas (5.129). One finds this time

$$\gamma_c^{(0)} = -2[a_1 + b_1^c], \qquad \gamma_c^{(1)} = -4[a_2 + b_2^c], \tag{5.132}$$

where b_1^c and b_2^c are obtained by calculating the relevant one-loop and two-loop diagrams, respectively. b_1^c is simply obtained by calculating the one-loop upper vertex of the diagram (a) in Figure 5.3. b_2^c is found by calculating the corresponding two-loop generalization of this vertex. One finds $b_1^c = C_F$. The factor of two in $(b_1)_{11}$ in front of C_F in (5.130) comes from a symmetric diagram to the diagram (a) in Figure 5.3 with the gluon exchanged between the lower quark legs. With $a_1 = -C_F$ we find $\gamma_c^{(0)} = 0$ as it should be. At two loops $\gamma_c^{(1)}$ is renormalization scheme dependent and in certain schemes can be nonzero. As we will see later, it vanishes in the NDR scheme but is nonzero in the HV scheme.

We get the following useful message from this discussion that applies to schemes in which the anomalous dimension of a quark current vanishes at any order of perturbation theory. The only diagrams responsible for the nonvanishing anomalous dimensions of current-current operators Q_1 and Q_2 are the diagrams in which the gluons connect the quark legs belonging to different weak currents. At the one-loop level these are the diagrams (b) and (c) in Figure 5.3 and the corresponding symmetric diagrams. One should stress that this simple rule is not valid for the insertion of penguin operators into current-current diagrams. We will see this explicitly in Section 6.4.

How to Use the Two-Loop Master Formulas

The calculation of $(b_2)_{ij}$ in (5.128) is a bit trickier and technically more difficult. To this end one has to calculate first two-loop diagrams with Q_i insertions. Examples are given in Figure 5.4. Next the corresponding two-loop counter-diagrams have to be *subtracted*. In the MS-like schemes the latter are obtained by retaining only the $1/\varepsilon$ parts in the subdiagrams. The counter-diagrams corresponding to the two-loop diagrams in Figure 5.4 are shown in Figure 5.5, where the small boxes stand for the singular parts of the corresponding subdiagrams in Figure 5.4. For instance, the Feynman rule for the small box in diagram (b) of Figure 5.5 is, in accordance with (4.42), given by

$$iC_F\delta_{\alpha\beta}\frac{\alpha_s}{4\pi}\;\not{p}\frac{1}{\varepsilon}, \tag{5.133}$$

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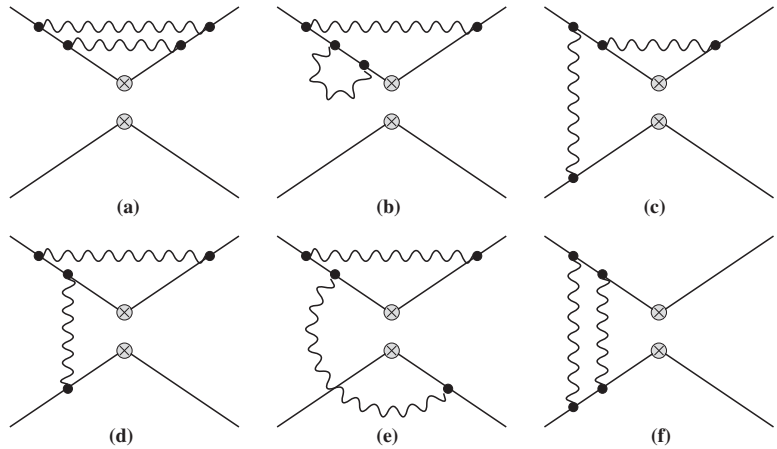


Figure 5.4 Examples of two-loop current-current diagrams contributing to the NLO anomalous dimensions of the operators Q_1 and Q_2. From [29].

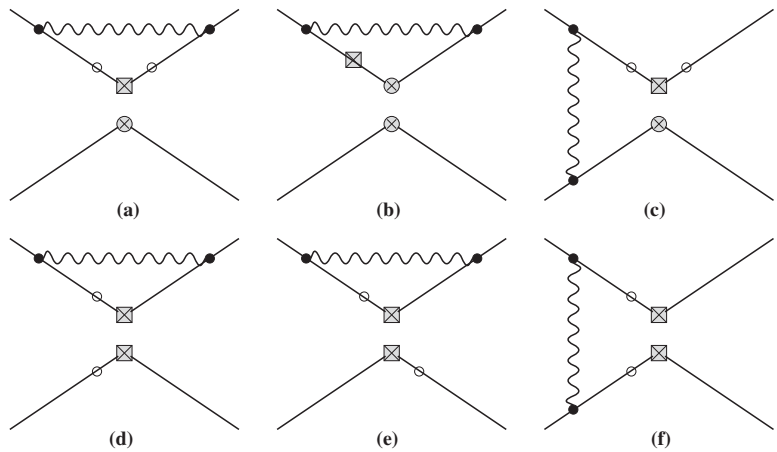


Figure 5.5 Counter-diagrams to the diagrams of Figure 5.4. From [29].

where p is the four-momentum of the quark. Because MS-like schemes are the so-called mass-independent renormalization schemes, the quark masses can be set to zero in evaluating anomalous dimensions.

Dropping color factors and Dirac tensors the result for each diagram including the counter-diagram has the structure

$$I - I_C = \left(\frac{\alpha}{4\pi}\right)^2 \left[\frac{\mu^2}{-p^2}\right]^{2\varepsilon} \left[\frac{F}{\varepsilon^2} + \frac{G}{\varepsilon} + \cdots\right] - \left(\frac{\alpha}{4\pi}\right)^2 \left[\frac{\mu^2}{-p^2}\right]^{\varepsilon} \left[\frac{F_C}{\varepsilon^2} + \frac{G_C}{\varepsilon} + \cdots\right], \tag{5.134}$$

where the second term represents the counter-diagram. Note that the power of μ^2 in the counter-term is ε and not 2ε as in I. It turns out that for diagrams with nonvanishing F

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one has diagram by diagram the relation $F_C = 2F$, and consequently the pole part does not depend on μ as it should be:

$$I - I_C = \left(\frac{\alpha}{4\pi}\right)^2 \left[-\frac{F}{\varepsilon^2} + \frac{G - G_C}{\varepsilon} + \cdots\right] \quad \rightarrow \quad b_2 = G - G_C. \tag{5.135}$$

The coefficient $G-G_C$ can then be identified with the contribution of a given diagram (after the inclusion of color factors) to the coefficient $(b_2)_{ij}$ entering the master formula (5.128).

Again, as in the one-loop case, the diagrams with gluons exchanged only between quark legs belonging to the same weak current can be omitted in the evaluation of two-loop anomalous dimensions of the operators Q_1 and Q_2, provided the anomalous dimension of the weak current vanishes at two-loop level. In this case their contributions to $(b_2)_{ij}$ are canceled by the a_2 term in the master formula (5.128). It should be stressed that this feature might not be preserved by some regularization schemes. In particular, it depends on the treatment of γ_5 in $D \neq 4$ dimensions. In the NDR scheme, in which γ_5 anticommutes with γ_μ in $D \neq 4$ dimensions, one has indeed $\gamma_c^{(1)} = 0$. But this is not true in the HV scheme, where γ_5 has more complicated properties. Indeed one finds [71]

$$[\gamma_c^{(1)}]_{HV} = 4C_F\beta_0. \tag{5.136}$$

We will return to this issue in a moment. For the time being we give the two-loop generalizations of the one-loop anomalous dimension matrix (5.78) in the NDR scheme [71]:

$$\hat{\gamma}^{(1)}_{NDR} = \begin{pmatrix} -\frac{22}{3} - \frac{57}{2N^2} - \frac{2f}{3N} & \frac{39}{N} - \frac{19N}{6} + \frac{2f}{3} \\ \frac{39}{N} - \frac{19N}{6} + \frac{2f}{3} & -\frac{22}{3} - \frac{57}{2N^2} - \frac{2f}{3N} \end{pmatrix}, \tag{5.137}$$

and in the HV scheme [71]:

$$\hat{\gamma}^{(1)}_{HV} = \begin{pmatrix} -\frac{110}{3} - \frac{57}{2N^2} + \frac{44N^2}{3} + \left(\frac{14}{3N} - \frac{8N}{3}\right) f & \frac{39}{N} + \frac{23N}{2} - 2f \\ \frac{39}{N} + \frac{23N}{2} - 2f & -\frac{110}{3} - \frac{57}{2N^2} + \frac{44N^2}{3} + \left(\frac{14}{3N} - \frac{8N}{3}\right) f \end{pmatrix}. \tag{5.138}$$

The corresponding result in the DRED scheme has been first calculated in [79] and confirmed in [71]. We observe substantial renormalization scheme dependence in $\hat{\gamma}^{(1)}$. In particular the diagonal elements in (5.138) contain terms $O(N^2)$, whereas such terms are absent in (5.137). The origin of these terms can be traced back to the nonvanishing of $[\gamma_c^{(1)}]_{HV}$. Indeed these terms cancel in the difference

$$\hat{\gamma}^{(1)}_{HV} - 2[\gamma_c^{(1)}]_{HV}\hat{1} \equiv [\hat{\gamma}^{(1)}_{HV}]_{\text{eff}}. \tag{5.139}$$

As we will discuss later, such a shift of two-loop anomalous dimensions is always possible provided also the matching conditions for the Wilson coefficients at μ_W are appropriately changed. Yet eventually this shift modifies the Wilson coefficients, and this modification will be compensated by the corresponding change in the matrix elements of the operators so that physical quantities are independent of these manipulations. All this should be clearer after we discussed cancellation of these scheme dependences in physical quantities in explicit terms.

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In this context we should warn the reader that the numerical values of the Wilson coefficients in the HV scheme presented here and in [111, 123] correspond to the choice $[\hat{\gamma}^{(1)}_{HV}]_{\rm eff}$. This differs from the treatment in [124, 125], who use $\hat{\gamma}^{(1)}_{HV}$ of (5.138) instead. For this reason the NLO corrections to Wilson coefficients in the HV scheme presented here are generally smaller than the ones found by the Rome group. The final physical results for decay amplitudes, and observables are, however, the same.

5.2.5 Explicit Calculation of 2×2 Anomalous Dimension Matrix

Current-Current Insertions: Generalities

Until now we have discussed the structure of the calculation of the anomalous dimensions of current-current operators, but it is time to enter some details. The set of six diagrams contributing to one-loop anomalous dimension matrix through operator insertions into current-current topologies is given by the diagrams in Figure 5.3 and their symmetric counterparts. We begin by developing the technology for the calculation of insertions of any operator with arbitrary color and Dirac structure into the diagrams of Figure 5.3. This will allow us to calculate later also insertions of penguin operators into the current-current topologies of Figure 5.3.

Let us then denote the color and Dirac structure of any operator by

$$\hat{V}_1 \otimes \hat{V}_2, \qquad \Gamma_1 \otimes \Gamma_2 \,, \tag{5.140}$$

respectively, so that an operator can be generally written as follows:

$$O = (\bar{s}_\alpha \Gamma_1 \hat{V}_1^{\alpha\beta} c_\beta) \otimes (\bar{u}_\gamma \Gamma_2 \hat{V}_2^{\gamma\delta} d_\delta). \tag{5.141}$$

Here we have made specific choice of quark flavors adapted to the operators discussed in the previous section, but it is trivial to generalize the following discussion to any other choice of flavors.

Let us consider a few examples of the color structures:

$$\hat{V}_1^{\alpha\beta} \otimes \hat{V}_2^{\gamma\delta} = \delta_{\alpha\beta} \otimes \delta_{\gamma\delta} \equiv \mathbf{1}_{\alpha\beta,\gamma\delta} \,, \tag{5.142}$$

$$\hat{V}_1^{\alpha\beta} \otimes \hat{V}_2^{\gamma\delta} = \delta_{\alpha\delta} \otimes \delta_{\gamma\beta} \equiv \tilde{\mathbf{1}}_{\alpha\beta,\gamma\delta} \,, \tag{5.143}$$

$$\hat{V}_1^{\alpha\beta} \otimes \hat{V}_2^{\gamma\delta} = (t^a)_{\alpha\beta} \otimes (t^a)_{\gamma\delta} \equiv \Pi_{\alpha\beta,\gamma\delta} \,. \tag{5.144}$$

Then the color identity (5.22) is simply given by

$$\Pi = \frac{1}{2}\left(\tilde{\mathbf{1}} - \frac{1}{N}\mathbf{1}\right) \,. \tag{5.145}$$

In this notation the operators Q_1 and Q_2 have both the Dirac structure $\Gamma_1 = \Gamma_2 = \gamma_\mu(1-\gamma_5)$. With the ordering of flavors as in (5.141), the color structure of Q_1 is $\tilde{\mathbf{1}}$. The one of Q_2 is $\mathbf{1}$.

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To gain some insight into the calculation of the diagrams in Figure 5.3, let us consider diagram (a) with the insertion of the operator Q_2. The flavor labels are as in Figure 5.1. For this diagram we have then

$$\mathcal{D}_a^{(1)} = -ig^2\mu^{2\varepsilon}C_F \int \frac{d^Dk}{(2\pi)^D} \frac{[\bar{s}T_\nu c]\otimes[\bar{u}\Gamma^\nu d]}{k^2[(k+p)^2]^2}, \tag{5.146}$$

where

$$T_\nu = \gamma_\mu(\not{k}+\not{p})\Gamma_\nu(\not{k}+\not{p})\gamma^\mu = \gamma_\mu\gamma_\rho\Gamma_\nu\gamma_\sigma\gamma^\mu(k+p)^\rho(k+p)^\sigma, \tag{5.147}$$

and $\Gamma_\nu = \gamma_\nu(1-\gamma_5)$. We have used $t^at^a = C_F$.

Keeping only the divergent part in the relevant D-dimensional integral, we find

$$\int \frac{d^Dk}{(2\pi)^D} \frac{(k+p)^\rho(k+p)^\sigma}{k^2[(k+p)^2]^2} = ig^{\rho\sigma}\frac{1}{16\pi^2}\frac{1}{4}\frac{1}{\varepsilon} + \text{finite}. \tag{5.148}$$

Thus the divergent part of $\mathcal{D}_a^{(1)}$ is given by

$$\mathcal{D}_a^{(1)} = C_F\frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right][\bar{s}\gamma_\mu\gamma_\rho\Gamma_\nu\gamma^\rho\gamma^\mu c]\otimes[\bar{u}\Gamma^\nu d]. \tag{5.149}$$

It is straightforward to extend this calculation to other diagrams in Figure 5.3 and to the arbitrary operator given in (5.141). To this end we fix the ordering of the four flavors as in (5.141) and drop the external spinors. We find then

$$\mathcal{D}_a = \frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right]\left(C_a^{(1)}\gamma_\mu\gamma_\rho\Gamma_1\gamma^\rho\gamma^\mu\otimes\Gamma_2 + C_a^{(2)}\Gamma_1\otimes\gamma_\mu\gamma_\rho\Gamma_2\gamma^\rho\gamma^\mu\right), \tag{5.150}$$

$$\mathcal{D}_b = -\frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right]\left(C_b^{(1)}\Gamma_1\gamma_\rho\gamma_\mu\otimes\Gamma_2\gamma^\rho\gamma^\mu + C_b^{(2)}\gamma_\mu\gamma_\rho\Gamma_1\otimes\gamma^\mu\gamma^\rho\Gamma_2\right), \tag{5.151}$$

$$\mathcal{D}_c = \frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right]\left(C_c^{(1)}\Gamma_1\gamma_\rho\gamma_\mu\otimes\gamma^\mu\gamma^\rho\Gamma_2 + C_c^{(2)}\gamma_\mu\gamma_\rho\Gamma_1\otimes\Gamma_2\gamma^\rho\gamma^\mu\right), \tag{5.152}$$

where the index (1) stands for the diagrams shown in Figure 5.3 and the index (2) for their symmetric counterparts. The color factors are given by

$$C_a^{(1)} = t^a\hat{V}_1t^a\otimes\hat{V}_2, \qquad C_a^{(2)} = \hat{V}_1\otimes t^a\hat{V}_2t^a, \tag{5.153}$$

$$C_b^{(1)} = \hat{V}_1t^a\otimes\hat{V}_2t^a, \qquad C_b^{(2)} = t^a\hat{V}_1\otimes t^a\hat{V}_2, \tag{5.154}$$

$$C_c^{(1)} = \hat{V}_1t^a\otimes t^a\hat{V}_2, \qquad C_c^{(2)} = t^a\hat{V}_1\otimes\hat{V}_2t^a. \tag{5.155}$$

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Anomalous Dimensions of Q_1 and Q_2

Let us apply these general formulas to the case of the operator Q_2 for which we have

$$\hat{V}_1 \otimes \hat{V}_2 = \mathbf{1}, \qquad \Gamma_1 = \Gamma_2 = \gamma_\nu(1-\gamma_5) \equiv \Gamma\,. \tag{5.156}$$

Because we are interested only in the $1/\varepsilon$, singularity we can use in one-loop formulas the rules for γ-algebra valid in four dimensions. Then

$$\gamma_\mu\gamma_\rho\Gamma\gamma^\rho\gamma^\mu \otimes \Gamma = \Gamma \otimes \gamma_\mu\gamma_\rho\Gamma\gamma^\rho\gamma^\mu = 4\Gamma \otimes \Gamma, \tag{5.157}$$

$$\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu = \gamma_\mu\gamma_\rho\Gamma \otimes \gamma^\mu\gamma^\rho\Gamma = 16\Gamma \otimes \Gamma, \tag{5.158}$$

$$\Gamma\gamma_\rho\gamma_\mu \otimes \gamma^\mu\gamma^\rho\Gamma = \gamma_\mu\gamma_\rho\Gamma \otimes \Gamma\gamma^\rho\gamma^\mu = 4\Gamma \otimes \Gamma. \tag{5.159}$$

These results can be most efficiently reproduced by using a trick that we will call the *Greek Method* [126] from now on. Let us illustrate this method by deriving (5.158). Following [126] let us write

$$\gamma_\mu\gamma_\rho\Gamma \otimes \gamma^\mu\gamma^\rho\Gamma = \mathrm{A}\,\Gamma \otimes \Gamma\,, \tag{5.160}$$

where A is the coefficient we are looking for. To find it we replace $\otimes$ in (5.160) by a matrix γ_τ to obtain

$$\gamma_\mu\gamma_\rho\Gamma\gamma_\tau\gamma^\mu\gamma^\rho\Gamma = A\,\Gamma\gamma_\tau\Gamma. \tag{5.161}$$

Inserting Γ of (5.156) into this equality and contracting indices, we determine A to be 16. The reader can check that the same result is obtained by using the Clifford algebra, which allows the reduction of three γ_μ matrices to a single one.

Using (5.157)–(5.159) in our master formulas (5.150)–(5.152) and summing all diagrams, we find

$$\sum_i \mathcal{D}_i = \frac{\alpha_s}{4\pi}\frac{1}{\varepsilon}\Gamma \otimes \Gamma\left[C_a^{(1)} + C_a^{(2)} - 4(C_b^{(1)} + C_b^{(2)}) + C_c^{(1)} + C_c^{(2)}\right]. \tag{5.162}$$

For the color structure in (5.156), we then find

$$C_a^{(1)} = C_a^{(2)} = C_F\mathbf{1}, \tag{5.163}$$

$$C_b^{(1)} = C_b^{(2)} = C_c^{(1)} = C_c^{(2)} = \frac{1}{2}\left(\tilde{\mathbf{1}} - \frac{1}{N}\mathbf{1}\right). \tag{5.164}$$

Now $\tilde{\mathbf{1}}$ stands for the operator Q_1. Consequently, inserting (5.163) and (5.164) into (5.162) and comparing the coefficient of $1/\varepsilon$ with $\hat{Z}_{GF,1}$ in (5.127), we extract

$$(b_1)_{21} = -3, \qquad (b_1)_{22} = 2C_F + \frac{3}{N}. \tag{5.165}$$

The insertion of the operator Q_1 represented by $\tilde{\mathbf{1}}$ into diagrams of Figure 5.3 can be evaluated in an analogous manner by using the master formulas (5.150)–(5.152). Because

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the color structure is more complicated, the calculation is now a bit more involved. To avoid this complication it is useful to make a Fierz reordering in Q_1 and Q_2 so that

$$Q_1 = (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta c_\beta)_{V-A} \qquad Q_2 = (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta c_\alpha)_{V-A} \,. \tag{5.166}$$

Now the roles of Q_1 and Q_2 are interchanged: Q_1 is $\mathbf{1}$, and Q_2 is $\tilde{\mathbf{1}}$. Because gluons are flavor blind, we find immediately

$$(b_1)_{11} = 2C_F + \frac{3}{N}, \qquad (b_1)_{12} = -3 \,. \tag{5.167}$$

Equations (5.165) and (5.167) are precisely the values given in (5.131) and (5.130), respectively. Upon inserting them into the one-loop master formula (5.128) and using $a_1 = -C_F$, we finally reproduce the anomalous dimension matrix (5.78). We will extend this calculation to penguin operators in Section 6.4.

5.2.6 Mixing of Operators with Different Dimensions

It is useful to know the following properties of mixing of operators with different canonical dimensions

- The operators of a given dimension mix only into operators of the same or lower dimension. In a more formal terminology: To renormalize an operator of a given dimension one needs only operators as counter-terms of the same or lower dimension.
- This means in particular that the operators of dimension six, as Q_1 and Q_2, can mix into other six-dimensional operators and five-dimensional magnetic penguin operators, which we will encounter in the next section. On the other hand, the magnetic penguin operators cannot mix into dimension six operators.
- Consequently, whereas the Q_1 and Q_2 operators influence the Wilson coefficients of the magnetic penguin operators, the latter operators have no impact on C_1 and C_2.

The proof of these properties is based essentially on dimensional analysis. It can be found on page 149 of the book by Collins [90].

Here comes another useful remark. As we will discuss in Section 6.5, the mixing between the operators (Q_1, Q_2) and the magnetic penguin operators appears first at the two-loop level. That is, the leading anomalous dimension is obtained by calculating two-loop diagrams and not one-loop diagrams as discussed so far. The next-to-leading anomalous dimensions are then obtained from three-loop calculations. In this particular case our master formulas in (5.128) change to $(i \neq j)$

$$\gamma_{ij}^{(0)} = -4[(b_2)_{ij}], \qquad \gamma_{ij}^{(1)} = -6[(b_3)_{ij}], \tag{5.168}$$

with $(b_2)_{ij}$ and $(b_3)_{ij}$ obtained from $1/\varepsilon$ singularities in two-loop and three-loop diagrams, respectively. Motivated readers should be able to derive these formulas without any efforts.

5.2.7 Renormalization Scheme Dependence

At NLO various quantities like the Wilson coefficients and the anomalous dimensions depend on the renormalization scheme (RS) for operators. This dependence arises because

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the renormalization prescription involves an arbitrariness in the finite parts to be subtracted along with the ultraviolet singularities. Two different schemes are then related by a finite renormalization.

A particular example of the RS dependence is the dependence on the treatment of γ_5 in D-dimensions. We have seen in (5.137) and (5.138) that the two-loop anomalous dimension matrix for the operators (Q_1, Q_2) in the NDR scheme differs from the one in the HV scheme.

Returning back to our discussion of the NLO corrections to the Wilson coefficients of Section 5.2.3, we find that

$$\beta_0, \quad \beta_1, \quad \gamma^{(0)}, \quad \vec{A}^{(0)}, \quad \vec{A}^{(1)}, \quad \hat{r}^T + \hat{J}, \quad \langle \vec{Q} \rangle^T \vec{C} \tag{5.169}$$

are *scheme independent*, whereas

$$\hat{r}, \quad \gamma^{(1)}, \quad \hat{J}, \quad \vec{C}, \quad \langle \vec{Q} \rangle \tag{5.170}$$

are *scheme dependent.* Let us demonstrate this.

First of all, it is clear that the product

$$\langle \vec{Q}(\mu) \rangle^T \vec{C}(\mu), \tag{5.171}$$

representing the full amplitude, is independent of the RS. The factorization of the amplitude into $\vec{C}$ and $\langle \vec{Q} \rangle$ makes them, however, scheme dependent. Explicitly, for two different schemes (primed and unprimed), we have

$$\langle \vec{Q} \rangle' = \left(\hat{1} + \frac{\alpha_s}{4\pi} \hat{s} \right) \langle \vec{Q} \rangle \,, \qquad \vec{C}' = \left(\hat{1} - \frac{\alpha_s}{4\pi} \hat{s}^T \right) \vec{C} \,, \tag{5.172}$$

where $\hat{s}$ is a constant matrix representing a finite renormalization of $\vec{C}$ and $\langle \vec{Q} \rangle$.

Having the relations (5.172) at hand it is straightforward to find relations between various quantities in the primed and unprimed schemes. From

$$\langle \vec{Q}(\mu_W) \rangle = \left(\hat{1} + \frac{\alpha_s(\mu_W)}{4\pi} \hat{r} \right) \vec{S}, \tag{5.173}$$

where $\vec{S}$ is a vector of tree-level matrix elements, we immediately obtain

$$\hat{r}' = \hat{r} + \hat{s} \,. \tag{5.174}$$

Next, from

$$\langle \vec{Q}(\mu) \rangle^T \vec{C}(\mu) \equiv \langle \vec{Q}(\mu) \rangle^T \hat{U}(\mu, M_W) \vec{C}(M_W) \tag{5.175}$$

and (5.172), we have

$$\hat{U}'(\mu, M_W) = \left(\hat{1} - \frac{\alpha_s(\mu)}{4\pi} \hat{s}^T \right) \hat{U}(\mu, M_W) \left(\hat{1} + \frac{\alpha_s(M_W)}{4\pi} \hat{s}^T \right). \tag{5.176}$$

A comparison with

$$\hat{U}(\mu, \mu_W) = \left(\hat{1} + \frac{\alpha_s(\mu)}{4\pi} \hat{J} \right) \hat{U}^{(0)}(\mu, \mu_W) \left(\hat{1} - \frac{\alpha_s(\mu_W)}{4\pi} \hat{J} \right) \tag{5.177}$$

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yields then

$$\hat{J}' = \hat{J} - \hat{s}^T. \tag{5.178}$$

Next from (5.30) and (5.172) we clearly have

$$\hat{Z}' = \hat{Z}\left(\hat{1} - \frac{\alpha_s}{4\pi}\hat{s}\right). \tag{5.179}$$

Using next the definition of the anomalous dimension matrix (5.122) and the expansion (5.112), we find

$$\hat{\gamma}^{(0)\prime} = \hat{\gamma}^{(0)}, \qquad \hat{\gamma}^{(1)\prime} = \hat{\gamma}^{(1)} + [\hat{s}, \hat{\gamma}^{(0)}] + 2\beta_0\hat{s}. \tag{5.180}$$

Let us make a few observations:

- From (5.174) and (5.178) follows the scheme independence of $\hat{r}^T + \hat{J}$. Next, $\vec{A}^{(0)}$ and $\vec{A}^{(1)}$, obtained from the calculation in the full theory, are clearly independent of the renormalization of operators. Consequently, the factor on the right-hand side of $\hat{U}^{(0)}$ in $\vec{C}(\mu)$ in (5.119), related to the "upper end" of the evolution, is independent of the RS.
- The same is true for $\hat{U}^{(0)}$, as $\hat{\gamma}^{(0)}$ and β_0 are scheme independent.
- $\vec{C}$ depends on the RS through $\hat{J}$ to the left of $\hat{U}^{(0)}$. This dependence is compensated by the corresponding scheme dependence of $\langle\vec{Q}\rangle$ in (5.172).

In the absence of operator mixing the relations between various quantities in two different schemes simplify. Going back to $C_\pm(\mu)$ in (5.95), we have

$$\gamma_\pm^{(1)\prime} = \gamma_\pm^{(1)} + 2\beta_0 s_\pm\,, \qquad B'_\pm = B_\pm - s_\pm\,, \qquad J'_\pm = J_\pm - s_\pm\,, \tag{5.181}$$

where $s_\pm$ are constant numbers analogous to $\hat{s}$ in (5.172).

Recalling

$$J_\pm = \frac{1}{2\beta_0}\left(\frac{\beta_1}{\beta_0}\gamma_\pm^{(0)} - \gamma_\pm^{(1)}\right), \tag{5.182}$$

we verify the scheme independence of $B_\pm - J_\pm$ in (5.95). Again the scheme dependence of $C_\pm(\mu)$ originates in the scheme dependence of $J_\pm$ present in the first factor in (5.95).

We should emphasize that the renormalization scheme dependence discussed here refers to the renormalization of operators and should be distinguished from the renormalization scheme dependence of α_s. The issue of the latter scheme dependence in the context of OPE is discussed in detail at the end of section III in [111] and will not be repeated here.

5.2.8 Renormalization Scale Dependence

A physical amplitude cannot depend on the arbitrary renormalization scale μ. The μ-dependence of the Wilson coefficients has to be canceled by the μ-dependence of the matrix elements $\langle Q_i(\mu)\rangle$. Due to the mixing under renormalization this cancellation may involve simultaneously several operators. Now, whereas $C_i(\mu)$ can be calculated in perturbation theory, this is not the case for the matrix elements $\langle Q_i(\mu)\rangle$. Yet, assuming that

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we know these matrix elements at a given scale, their μ–dependence is again governed by renormalization group equations with the solution given by

$$\langle \vec{Q}(M_W)\rangle^T = \langle \vec{Q}(\mu)\rangle^T \hat{U}(\mu, M_W). \tag{5.183}$$

Here $\hat{U}(\mu, M_W)$ is the same evolution matrix used for the evolution of WCs. But it should be noted that whereas the evolution of WCs, as seen in (5.107), is governed by $\hat{\gamma}^T$, the one of operator matrix elements by $\hat{\gamma}$.

Moreover, in inclusive decays of heavy mesons like $B \to X_s\gamma$ and $B \to X_s e^+e^-$, one can analyze the cancellation of the μ-dependence using perturbative calculations of the relevant matrix elements $\langle Q_i(\mu)\rangle$. We refer to [127, 128] for the full exposition of this issue in $B \to X_s\gamma$ and $B \to X_s e^+e^-$. Here it suffices to illustrate the cancellation of the μ dependence by considering a toy model in which only a single operator Q is present, and its matrix element $\langle Q(\mu)\rangle$ is calculated in perturbation theory. In fact, the μ-dependence of this matrix element is correctly described within perturbation theory and can be obtained from the more general formula (5.183).

Let us consider then the amplitude

$$A = \langle \mathscr{H}_{\rm eff}\rangle = \frac{G_F}{\sqrt{2}}\langle Q(\mu_b)\rangle C(\mu_b) \tag{5.184}$$

with

$$C(\mu_b) = U(\mu_b, \mu_W)C(\mu_W), \tag{5.185}$$

where $\mu_b = O(m_b)$ and $\mu_W = O(M_{\rm W})$. We want to discuss the cancellation of the μ_b and μ_W dependences in (5.184) and (5.185) in explicit terms.

Beginning with the leading logarithmic approximation, we have

$$U(\mu_b, \mu_W) = U^{(0)}(\mu_b, \mu_W) = \left[\frac{\alpha_s(\mu_W)}{\alpha_s(\mu_b)}\right]^{\frac{\gamma^{(0)}}{2\beta_0}}, \qquad C(\mu_W) = 1\,. \tag{5.186}$$

Moreover, $\langle Q(\mu_b)\rangle = \langle Q\rangle_{\rm tree}$ carries no μ_b dependence. Consequently, in LO the amplitude depends on μ_b and μ_W. Because $\alpha_s(\mu_b) \gg \alpha_s(\mu_W)$, the μ_b-dependence is stronger than the μ_W-dependence. If $\gamma^{(0)}/(2\beta_0)$ is $O(1)$, the μ_b-dependence of the resulting amplitudes and branching ratios may be rather disturbing. A known example of such a situation is the strong μ_b-dependence of the branching ratio $\mathcal{B}(B \to X_s\gamma)$ at LO. We will discuss this in detail in Section 6.5.

Let us next include NLO corrections. Now the various entries in (5.186) are generalized as follows:

$$U(\mu_b, \mu_W) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi}J\right) U^{(0)}(\mu_b, \mu_W)\left(1 - \frac{\alpha_s(\mu_W)}{4\pi}J\right), \tag{5.187}$$

$$C(\mu_W) = 1 + \frac{\alpha_s(\mu_W)}{4\pi}\left(\frac{\gamma^{(0)}}{2}\ln\left(\frac{\mu_W^2}{M_{\rm W}^2}\right) + B\right), \tag{5.188}$$

$$\langle Q(\mu_b)\rangle = \langle Q\rangle_{\rm tree}\left[1 + \frac{\alpha_s(\mu_b)}{4\pi}\left(\frac{\gamma^{(0)}}{2}\ln\left(\frac{m_b^2}{\mu_b^2}\right) + \tilde{r}\right)\right]. \tag{5.189}$$

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We will now show that the amplitude A in (5.184) is independent of μ_b and μ_W at $O(\alpha_s)$. Using the following useful formula,

$$\frac{\alpha_s(m_1)}{\alpha_s(m_2)} = 1 + \frac{\alpha_s}{4\pi}\beta_0 \ln\left(\frac{m_2^2}{m_1^2}\right), \tag{5.190}$$

where only $O(\alpha_s)$ terms have been retained, and keeping only logarithmic terms we can rewrite (5.187) as

$$U(\mu_b,\mu_W) = \left(1 + \frac{\alpha_s(\mu_b)}{4\pi}\frac{\gamma^{(0)}}{2}\ln\left(\frac{\mu_b^2}{m_b^2}\right)\right)\left[\frac{\alpha_s(M_{\rm W})}{\alpha_s(m_{\rm b})}\right]^{\frac{\gamma^{(0)}}{2\beta_0}}\left(1 + \frac{\alpha_s(\mu_W)}{4\pi}\frac{\gamma^{(0)}}{2}\ln\left(\frac{M_{\rm W}^2}{\mu_W^2}\right)\right). \tag{5.191}$$

Inserting (5.188), (5.189), and (5.191) into (5.184), we find that μ_b and μ_W dependences cancel at $O(\alpha_s)$.

This simple example illustrates very clearly the virtue of NLO corrections. They reduce considerably various μ-dependences present in the LO approximation. On the other hand, we recover the well-known fact that at fixed order of perturbation theory there remain unphysical μ-dependences, which are of the order of the neglected higher-order contributions. In our simple example the leftover μ_b and μ_W dependences can be investigated numerically by inserting expressions (5.187)–(5.189) into (5.184) and varying μ_b and μ_W say in the ranges $m_{\rm b}/2 \le \mu_b \le 2m_{\rm b}$ and $M_{\rm W}/2 \le \mu_W \le 2M_{\rm W}$, respectively. By comparing the result of this exercise with an analogous exercise in LO, one can on the one hand appreciate the importance of NLO calculations. On the other hand, the leftover μ_W and μ_b dependences at NLO give a rough estimate of the theoretical uncertainty due to the truncation of the perturbative series. Such estimates can be found in any NLO analysis presented in the literature and will not be presented here except for a few examples in later chapters.

The μ-dependences discussed here are related to the renormalization group evolution of the Wilson coefficients from high- to low-energy scales. This evolution originates in the nonvanishing of the anomalous dimensions of the corresponding operators Q_i. On the other hand, as we have seen in Section 4.3, the nonvanishing of the anomalous dimension γ_m of the mass operator implies the μ-dependence of the quark masses, in particular $m_t(\mu_t)$, $m_b(\mu_b)$, and $m_c(\mu_c)$. These μ-dependences and their cancellation in decay amplitudes will be discussed in later chapters of this book.

The following section is rather technical and can be skipped by the readers who do not plan to do two-loop calculations.

5.2.9 Evanescent Operators

Origin of Evanescent Operators

In evaluating the anomalous dimensions of Q_1 and Q_2, we have used the Greek Method to reduce the complicated Dirac structures given in (5.157)–(5.159) to $\Gamma \otimes \Gamma$. Because we were only interested in the $1/\varepsilon$ singularity in a one-loop diagram, this reduction has been performed in $D = 4$ dimensions. In the case of two-loop calculations, in which the

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diagrams of Figure 5.3 are subdiagrams of the diagrams in Figure 5.4, this reduction has to be performed in arbitrary D-dimensions. Indeed, in a two-loop diagram, the leading singularity is $1/\varepsilon^2$. The $O(\varepsilon)$ terms arising from reductions like (5.157)–(5.159) in arbitrary D-dimensions, when multiplied by $1/\varepsilon^2$, will contribute to the $1/\varepsilon$ singularity relevant for the calculation of the two-loop anomalous dimensions.

The question then arises of how to find $O(\varepsilon)$ corrections to (5.157)–(5.159). We will follow here a simple method for finding these terms that has been proposed in [71]. Although more general methods have been developed subsequently, this method appears to be most useful for practical purposes. Yet other methods [129, 130], in particular the one of Herrlich and Nierste [130], give a deeper insight into these matters, and we will briefly discuss them at the end of this section.

The simplest method to find the $O(\varepsilon)$ terms in question would be to apply the Greek Method in D-dimensions. That is, evaluate (5.161) in D-dimensions to determine the coefficient A. For instance in the case of (5.158), we would find $16 - 4\varepsilon$ instead of 16 when using the NDR scheme for γ_5. This is what has been done in [126]. Yet as pointed out in [71], the mere replacement of 16 in (5.158) by $16 - 4\varepsilon$ with analogous replacements in (5.157) and (5.159) would eventually give incorrect two-loop anomalous dimensions. As demonstrated in [71] the correct procedure is to supplement the Greek Method in D-dimensions by the addition of other operators to the r.h.s. of (5.157)–(5.159), which vanish in $D = 4$ dimensions. Such operators are called *evanescent* operators.

We will soon explain the role of evanescent operators in the calculation of two-loop anomalous dimensions. While these matters are rather technical, we think it is useful to present them here to illustrate how careful one has to be already at two-loop level to obtain correct results. First, however, let us generalize (5.157)–(5.159) to $D \neq 4$ dimensions. In the case of the NDR scheme for γ_5, one finds [71]

$$\gamma_\mu\gamma_\rho\Gamma\gamma^\rho\gamma^\mu \otimes \Gamma = 4(1-2\varepsilon)\Gamma \otimes \Gamma, \tag{5.192}$$

$$\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu = 4(4-\varepsilon)\Gamma \otimes \Gamma + E^{\rm NDR}, \tag{5.193}$$

$$\Gamma\gamma_\rho\gamma_\mu \otimes \gamma^\mu\gamma^\rho\Gamma = 4(1-2\varepsilon)\Gamma \otimes \Gamma - E^{\rm NDR}, \tag{5.194}$$

where $O(\varepsilon^2)$ terms have been dropped. Identical results are found for the structures in the second column of the set (5.157)–(5.159). $E^{\rm NDR}$ stands for the evanescent operator given explicitly by

$$E^{\rm NDR} = \frac{1}{2}[\gamma_\mu\gamma_\rho\Gamma\gamma^\rho\gamma^\mu \otimes \Gamma + \Gamma \otimes \gamma_\mu\gamma_\rho\Gamma\gamma^\rho\gamma^\mu - \Gamma\gamma_\rho\gamma_\mu \otimes \gamma^\mu\gamma^\rho\Gamma - \gamma_\mu\gamma_\rho\Gamma \otimes \Gamma\gamma^\rho\gamma^\mu]. \tag{5.195}$$

As one can verify using the Greek Method, $E^{\rm NDR}$ vanishes in $D = 4$. In the case of HV and DRED schemes, the formulas (5.192)–(5.194) are modified and the evanescent operators have more complicated structures. They can be found in [71]. It should be noted that there is no contribution from evanescent operators to (5.192) in the NDR scheme.

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From calculational point of view the insertions of evanescent operators into the relevant diagrams are most efficiently evaluated by defining $E^{\rm NDR}$ simply as the difference between the structures on the l.h.s. of (5.193) and (5.194) and the respective terms on the r.h.s. involving $\Gamma \otimes \Gamma$. We will demonstrate this explicitly later.

Including Evanescent Operators in the Master Formulas

In Section 5.2.4 we have derived the master formulas (5.128) and (5.129) for the computation of two loop anomalous dimensions. This derivation did not take into account the presence of evanescent operators.[1] Therefore in cases in which the contributions of these operators matter, our formulas are strictly speaking incomplete. It is the purpose of the next few pages to correct for it and to derive a procedure for the calculation of two-loop anomalous dimensions, which takes into account the evanescent operators. We follow here again [71].

Let us go back to (5.134), which involves the coefficients (F, G) and (F_C, G_C) in the singularities of the two-loop diagrams and the corresponding counter-diagrams, respectively. The evaluation of F and G is still straightforward. Having the final result for a two-loop diagram with complicated Dirac structure, one can simply project on the space of physical operators, denoted generically by $\Gamma \otimes \Gamma$, by using the Greek method. In this way one can easily deduce the coefficients of the terms proportional to $\Gamma \otimes \Gamma$. As an example, let us consider the Dirac structure resulting from the diagram (f) in Figure 5.4. Then the projection by means of the Greek Method gives:

$$\Gamma\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\tau \otimes \Gamma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau = 16\,(16 - 14\varepsilon)\,\Gamma \otimes \Gamma\,. \tag{5.196}$$

The treatment of counter-diagrams needs more care. After the evaluation of the subdiagrams, the $1/\varepsilon$ contributions are multiplied by the structures in (5.192)–(5.194), i.e., they include evanescent (E) operators. Making the projection onto $\Gamma \otimes \Gamma$ already at this stage would be incorrect. Indeed, inserting E into counter-diagrams of Figure 5.5, generates back the original operator $\Gamma \otimes \Gamma$ and introduces a correction to G_C and consequently a correction to the two-loop anomalous dimension of the original operator. It is precisely this correction that we have neglected in our master formulas. We will now find how our method has to be modified to include the effects of E-operators. The next two pages are very technical, and some readers may prefer to skip them and go directly to the three-step procedure presented subsequently.

For a two-loop computation, it is sufficient to consider the effects of mixing with the evanescent operators specified in the previous section. However, higher-loop computations would require, in the NDR and HV schemes, consideration of an ever-increasing number of independent operators. Thus generally the renormalized operators in generic D- dimensions are given by

$$O_i = (\hat{Z}^{-1})_{ij} O_j^{(0)}\,. \tag{5.197}$$

[1] The results in (5.137) and (5.138) include these contributions.

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Let us consider the case where the set $O_i^{(0)}$ includes the initial bare operators Q_+ and Q_- introduced in the previous section, which are expected not to mix under renormalization. In what follows we will denote them here by $O_j^{(0)}$ with $j = 1, 2$, respectively. All operators O_j with $j > 2$ correspond to evanescent operators. It is convenient to choose the basis such that the operators $O_j^{(0)}$ with $j = 3, 4$, respectively are the evanescent operators E^+, E^-. As mentioned earlier, it is, for our purposes, not necessary to specify the basis further nor to give explicit formulas of $E^\pm$.

The renormalization matrix $\hat{Z}$ has a perturbative expansion of the form,

$$\hat{Z} = \hat{1} + \frac{\alpha_s}{4\pi}\hat{Z}^{(1)} + \frac{\alpha_s^2}{(4\pi)^2}\hat{Z}^{(2)} + \cdots \tag{5.198}$$

Only the first four columns and first four rows of these (a priori infinite dimensional) matrices are of interest here. The understanding of the form of the matrices $\hat{Z}^{(1)}, \hat{Z}^{(2)}$ is crucial. First we have

$$Z^{(1)} = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & * \\ * & * & - & - \\ * & * & - & - \end{pmatrix}, \tag{5.199}$$

where a * denotes nonzero entries and the elements "–" are of no interest to us. In particular we have $Z_{12}^{(1)} = Z_{21}^{(1)} = 0$. This situation need not, however, continue at higher loops because in generic D-dimensions the bare operators $Q_\pm^{(0)}$ do not have definite Fierz transformation properties in the NDR and HV schemes. Hence it can, and in fact does in the NDR and HV schemes, happen that,

$$Z_{12}^{(2)} \neq 0, \qquad Z_{21}^{(2)} \neq 0\,. \tag{5.200}$$

At the same time, at the one-loop level not only do we have $Z_{31}^{(1)} \neq 0$, $Z_{42}^{(1)} \neq 0$ but to define renormalized evanescent operators, which can really be neglected in precisely four dimensions one must, in general, take into account the mixing with operators of differing "naive" Fierz symmetry, i.e., it can happen that

$$\hat{Z}_{32}^{(1)} \neq 0, \qquad \hat{Z}_{41}^{(1)} \neq 0\,. \tag{5.201}$$

We will soon see that this is necessary so that the renormalized operators when restricted to precisely four dimensions have the correct Fierz symmetry.

Consider now the renormalization group equations for Green functions containing one renormalized operator O_j insertion, in the regularized D-dimensional theory. They take the standard form but due to the mixing with the evanescent bare operators, an anomalous dimension matrix occurs

$$\hat{\gamma} = \hat{Z}^{-1}\mu\frac{\partial}{\partial\mu}\hat{Z} = \hat{Z}^{-1}(-\varepsilon g + \beta(g))\frac{\partial}{\partial g}\hat{Z}. \tag{5.202}$$

Expanding $\beta(g)$, $\hat{\gamma}$ and $\hat{Z}$ in powers of the renormalized coupling g as in the previous sections we obtain from (5.202)

$$\hat{\gamma}^{(0)} = -2\varepsilon\hat{Z}^{(1)}, \tag{5.203}$$

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and at two loops,

$$\hat{\gamma}^{(1)} = -4\varepsilon\hat{Z}^{(2)} - 2\beta_0\hat{Z}^{(1)} + 2\varepsilon\hat{Z}^{(1)}\hat{Z}^{(1)}. \tag{5.204}$$

Expanding the $\hat{Z}^{(r)}$ in inverse powers of ε,

$$\hat{Z}^{(1)} = \hat{Z}_0^{(1)} + \frac{1}{\varepsilon}\hat{Z}_1^{(1)}, \qquad \hat{Z}^{(2)} = \hat{Z}_0^{(2)} + \frac{1}{\varepsilon}\hat{Z}_1^{(2)} + \frac{1}{\varepsilon^2}\hat{Z}_2^{(2)} \tag{5.205}$$

and using the fact that the anomalous dimension matrix has a finite limit for $\varepsilon \to 0$, we must have the relation,

$$4\hat{Z}_2^{(2)} + 2\beta_0\hat{Z}_1^{(1)} - 2\hat{Z}_1^{(1)}\hat{Z}_1^{(1)} = 0, \tag{5.206}$$

which has been explicitly checked for the physical ± submatrix in [71]. We also get

$$\gamma^{(1)} = -4\hat{Z}_1^{(2)} - 2\beta_0\hat{Z}_0^{(1)} + 2(\hat{Z}_1^{(1)}\hat{Z}_0^{(1)} + \hat{Z}_0^{(1)}\hat{Z}_1^{(1)}) \,. \tag{5.207}$$

Note that we have introduced in (5.205) the nonsingular terms $\hat{Z}_0^{(1)}$ and $\hat{Z}_0^{(2)}$ to be able to incorporate the effects of evanescent operators. In particular the presence of the finite renormalization $\hat{Z}_0^{(1)}$ allows in the approach of [71] to remove the finite contributions from evanescent operators to the matrix elements of physical operators. On the other hand, as we will see in a moment, this finite renormalization has an impact on the two-loop anomalous dimensions of the physical operators and consequently on their Wilson coefficients. In this context we note that

$$(\hat{Z}_0^{(1)})_{ij} = 0 \quad (i,j = 1,2) \,, \qquad (\hat{Z}_1^{(1)})_{31} = (\hat{Z}_1^{(1)})_{42} = 0. \tag{5.208}$$

The latter property ensures that $1/\varepsilon^2$ terms are not affected by the evanescent operators at the two-loop level. Finally using the properties (5.199) and (5.208) in (5.207), we find

$$\gamma_+^{(1)} = \gamma_{11}^{(1)} = -4(\hat{Z}_1^{(2)})_{11} + 2(\hat{Z}_1^{(1)})_{13}(\hat{Z}_0^{(1)})_{31}, \tag{5.209}$$

$$\gamma_-^{(1)} = \gamma_{22}^{(1)} = -4(\hat{Z}_1^{(2)})_{22} + 2(\hat{Z}_1^{(1)})_{24}(\hat{Z}_0^{(1)})_{42}, \tag{5.210}$$

$$\gamma_{+-}^{(1)} = \gamma_{12}^{(1)} = -4(\hat{Z}_1^{(2)})_{12} + 2(\hat{Z}_1^{(1)})_{13}(\hat{Z}_0^{(1)})_{32}, \tag{5.211}$$

$$\gamma_{-+}^{(1)} = \gamma_{21}^{(1)} = -4(\hat{Z}_1^{(2)})_{21} + 2(\hat{Z}_1^{(1)})_{24}(\hat{Z}_0^{(1)})_{41}. \tag{5.212}$$

The first term in (5.209) and (5.210) represents (after addition of wave function renormalization) our master formula (5.128), which was obtained neglecting the mixing with the E-operators. The remaining terms reflecting the mixing in question are the corrections we were looking for.

Without these corrections and corresponding corrections in (5.211) and (5.212), the renormalized operators $Q_\pm$ would not have the correct Fierz symmetry, and they would mix under renormalization at the two-loop level, i.e., $\gamma_{+-}^{(1)}$ and $\gamma_{-+}^{(1)}$ would be nonzero. The inclusion of E-operators restores the Fierz symmetry and removes this mixing, i.e., $\gamma_{+-}^{(1)} = \gamma_{-+}^{(1)} = 0$. This is explicitly demonstrated in [71].

Looking at the "extra contribution" from the evanescent operators, one realizes that it is proportional to the contribution that the counter-terms involving an evanescent operator insertion yield to the computation of $\hat{Z}^{(2)}$. This is precisely what we stated at the beginning

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of this section. Note, however, that the correction terms in (5.209) and (5.210) are by a factor of 2 smaller than the corresponding counter-terms (involving $Q_\pm$ operators) present in the main terms. In the language of diagrams the result just means that in calculating $\gamma^{(1)}$ the contributions to counter-term diagrams involving an evanescent operator should be multiplied by a factor of 1/2.

How to Use the Improved Master Formulas

We can now summarize the improved procedure for the calculation of the two-loop anomalous dimensions.

Step 1

Calculate the full two-loop diagrams and project the Dirac structures onto the physical operators by means of the Greek Method. This gives in particular the coefficient G in (5.134).

Step 2

Calculate the usual contribution to the counter-term by taking the relevant subdiagram of a given two-loop diagram, projecting it onto the physical operators $\Gamma \otimes \Gamma$ by means of the Greek Method, inserting the result of this projection into the remaining subdiagram of a given two-loop diagram, and projecting the resulting expression again onto the physical operators by means of the Greek Method. This step gives the first part of G_C in (5.134). We will denote it by G_C^a.

Step 3

Calculate the contribution of the evanescent operator to the counter-diagram by simply inserting the difference of the two structures in (5.193) or (5.194), which define E, into the remaining one-loop subdiagram of a given two-loop diagram and project the result onto the physical operators by means of the Greek Method. This step gives the correction to the counter-term. We will denote the coefficient of $1/\varepsilon$ from this part by G_C^b.

Then the two-loop anomalous dimension matrix is found by calculating

$$\gamma_{ij}^{(1)} = -4[2a_2\delta_{ij} + (b_2)_{ij}], \qquad (b_2)_{ij} = \left(G - G_C^a - \frac{1}{2}G_C^b\right)_{ij}. \tag{5.213}$$

Note the factor 1/2 in the evanescent contribution. Formula (5.213) generalizes the master formula (5.128) with b_{ij} in (5.135) to include the contributions of evanescent operators.

Let us illustrate this procedure by calculating the contribution of the diagram (f) in Figure 5.4 and of its counter-diagram (f) in Figure 5.5 to the two-loop anomalous dimension of the operator with the Dirac structure $\gamma_\mu(1-\gamma_5) \otimes \gamma^\mu(1-\gamma_5)$. We drop the color factors in what follows.

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Step 1

Calculating the diagram (f) in Figure 5.4, using the projection (5.196) and multiplying by two (inclusion of the symmetric counterpart) one finds [71]

$$F = 16\,, \qquad G = 66 \tag{5.214}$$

with (F, G) defined in (5.134).

Step 2

We first calculate the diagram b in Figure 5.3, as in (5.151). We find

$$\mathcal{D}_b^{(1)} = -\frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right](1+2\varepsilon)\left[\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu\right], \tag{5.215}$$

where $(1+2\varepsilon)$ is an additional correction to the integral (5.148), which has to be kept now.

To find G_C^a we first project $\mathcal{D}_b^{(1)}$ on $\Gamma\otimes\Gamma$ by using the Greek Method and keeping only the divergent part:

$$[\mathcal{D}_b^{(1)}]_{div} = -\frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right] 16\,\Gamma\otimes\Gamma\,. \tag{5.216}$$

Inserting this into diagram f in Figure 5.5 gives by means of (5.215)

$$I_C^{(a)} = 2\left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{4}\frac{1}{\varepsilon}\right]^2(1+2\varepsilon)\left[\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu\right], \tag{5.217}$$

where the overall factor 2 takes into account the symmetric counterpart of this diagram. Using next the projection

$$\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu = 4(4-\varepsilon)\Gamma\otimes\Gamma \tag{5.218}$$

gives

$$I_C^{(a)} = \left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{32}{\varepsilon^2}+\frac{56}{\varepsilon}\right]\Gamma\otimes\Gamma, \tag{5.219}$$

where finite terms have been dropped. Consequently,

$$F_C^a = 32\,, \qquad G_C^a = 56. \tag{5.220}$$

Step 3

To calculate G_C^b we take first the evanescent part of $\mathcal{D}_b^{(1)}$. Dropping the $O(\varepsilon)$ from the integral (it contributes only to finite parts of $I_C^{(b)}$), we have

$$[\mathcal{D}_b^{(1)}]_{ev} = -\frac{\alpha_s}{4\pi}\left[\frac{1}{4}\frac{1}{\varepsilon}\right]\left[\Gamma\gamma_\rho\gamma_\mu \otimes \Gamma\gamma^\rho\gamma^\mu - 4(4-\varepsilon)\Gamma\otimes\Gamma\right], \tag{5.221}$$

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where we have used (5.193) to express $E^{\rm NDR}$ in terms of the difference of Dirac structures between the square brackets in (5.221). Inserting $[\mathcal{D}_b^{(1)}]_{ev}$ into the diagram f of Figure 5.5 and multiplying by two for the symmetric counterpart, we get

$$I_C^{(b)} = 2\left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{1}{4}\frac{1}{\varepsilon}\right]^2(1+2\varepsilon)\left[\Gamma\gamma_\rho\gamma_\mu\gamma_\nu\gamma_\tau\otimes\Gamma\gamma^\rho\gamma^\mu\gamma^\nu\gamma^\tau - 4(4-\varepsilon)\Gamma\gamma_\nu\gamma_\tau\otimes\Gamma\gamma^\nu\gamma^\tau\right]. \tag{5.222}$$

Projecting the two Dirac structures on $\Gamma\otimes\Gamma$ by means of (5.196) and (5.193), respectively, we obtain

$$I_C^{(b)} = \left(\frac{\alpha_s}{4\pi}\right)^2\left[-\frac{12}{\varepsilon}\right]\Gamma\otimes\Gamma \tag{5.223}$$

and

$$F_C^b = 0\,, \qquad G_C^b = -12. \tag{5.224}$$

We observe that the $1/\varepsilon^2$ singularity is unaffected by the evanescent contribution. We can now calculate the relevant combination in (5.213) to be

$$G - G_C^a - \frac{1}{2}G_C^b = 16\,, \tag{5.225}$$

which is precisely the $1/\varepsilon$ singularity given in table 3 (diagram 5) of [71]. Also the $1/\varepsilon^2$ singularity $F - F_C^a = -16$ agrees with [71] and $F_C = 2F$ as promised after (5.134). Great! Everything works! We hope the reader is now motivated to calculate the remaining twenty-six two-loop diagrams and corresponding counter-diagrams necessary to reproduce the matrix (5.137). Actually the calculation of counter-diagrams is rather straightforward. The difficult part is the calculation of the two-loop diagrams, like the ones in Figure 5.4.

Evanescent Scheme Dependences

The definition of evanescent operators is not unique as stressed by Dugan and Grinstein [129] and in particular by Herrlich and Nierste [130]. As an example consider the Dirac structure on the l.h.s. of (5.193). Following [130] we can generalize this formula to

$$\Gamma\gamma_\rho\gamma_\mu\otimes\Gamma\gamma^\rho\gamma^\mu = (16+a\varepsilon)\Gamma\otimes\Gamma + E^{\rm NDR}(a), \tag{5.226}$$

where "a" is an arbitrary parameter, which defines the evanescent operator $E^{\rm NDR}(a)$. For $a=-4$ the definition in (5.193) is chosen.

Now as the preceding discussion has shown, the presence of evanescent operators influences the two-loop anomalous dimensions of physical operators $\Gamma\otimes\Gamma$. Consequently, as emphasized in [130], the arbitrariness in the definition of the evanescent operators translates into an additional scheme dependence of two-loop anomalous dimensions, which can be effectively parametrized by "a" in (5.226). Therefore, when giving the results for two-loop anomalous dimensions, it is not sufficient to state simply that they correspond to NDR, HV or any other renormalization scheme. One has to specify in addition the definition of evanescent operators. This is essential as this scheme dependence of two-loop anomalous dimensions can only be canceled in physical amplitudes by the corresponding

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scheme dependences present in the matching conditions (for instance, $B_\pm$) at scales $O(M_W)$ and by the one present in the finite matrix elements of operators at scales $O(\mu)$.

This means that the treatment of evanescent operators in the process of matching and in the calculation of matrix elements of operators at scales $O(\mu)$ must be consistent with the one used in the calculation of two-loop anomalous dimensions. This issue is elaborated at length in [130] and in appendix B of [131].

There are two virtues of the definition of evanescent operators proposed in [71] and discussed in detail by us:

- The evanescent operators defined in this manner influence only two-loop anomalous dimensions. By definition they do not contribute to the matching and to the finite corrections to matrix elements at scales $O(\mu)$. They are simply subtracted away in the process of renormalization.
- As a consequence of this, the Fierz symmetry is preserved separately in two-loop anomalous dimensions, matching conditions, and matrix elements at scales $O(\mu)$.

The second property ensures that the operators Q_+ and Q_- do not mix under renormalization separately in two-loop anomalous dimensions, in the matching conditions, and in the matrix elements so that objects like B_{+-}, B_{-+}, $\gamma^{(1)}_{+-}$, $\gamma^{(1)}_{-+}$ are ensured to vanish in this scheme. In other schemes (see [129]) this is not the case, and the Fierz symmetry is only recovered after the two-loop anomalous dimensions are combined with the matching conditions, which makes the calculations unnecessarily rather involved.

Now comes the most important message of this section: Most of the existing NLO calculations adopt the definition of evanescent operators in [71], and all the two-loop anomalous dimensions and matching conditions given in this book, in my Les Houches lectures [29], and in the review [111] correspond to this definition.

5.2.10 The Effective $\Delta F = 1$ Hamiltonian: Current-Current Operators

Basic Formalism

Let us summarize the results for the coefficients $C_{1,2}(\mu)$ of the current-current operators $Q_{1,2}$ discussed extensively in this section, and let us evaluate them for the cases of $\Delta B = 1$, $\Delta C = 1$ and $\Delta S = 1$ decays.

To be specific, let us consider

$$Q_1 = (\bar{b}_\alpha c_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A}, \qquad Q_2 = (\bar{b}_\alpha c_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}, \tag{5.227}$$

$$Q_1 = (\bar{s}_\alpha c_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A}, \qquad Q_2 = (\bar{s}_\alpha c_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}, \tag{5.228}$$

$$Q_1 = (\bar{s}_\alpha u_\beta)_{V-A}(\bar{u}_\beta d_\alpha)_{V-A}, \qquad Q_2 = (\bar{s}_\alpha u_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A}, \tag{5.229}$$

for $\Delta B = 1$, $\Delta C = 1$, and $\Delta S = 1$ decays, respectively. Note, that these operators are responsible for the decays of mesons with b quark, c quark, and s quark, respectively. Looking at the flavor structure of Q_2 operators, one can easily reconstruct Feynman diagrams for various decays analogous to the ones in Chapter 3. We leave it as a useful exercise.

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The corresponding effective Hamiltonians, from which various decay amplitudes including NLO QCD corrections can be calculated, are given by

$$\mathscr{H}_{\rm eff}(\Delta B=1)=\frac{G_F}{\sqrt{2}}V_{cb}^*V_{ud}[C_1(\mu)Q_1+C_2(\mu)Q_2],\qquad (\mu=O(m_b)),\tag{5.230}$$

$$\mathscr{H}_{\rm eff}(\Delta C=1)=\frac{G_F}{\sqrt{2}}V_{cs}^*V_{ud}[C_1(\mu)Q_1+C_2(\mu)Q_2],\qquad (\mu=O(m_c)),\tag{5.231}$$

$$\mathscr{H}_{\rm eff}(\Delta S=1)=\frac{G_F}{\sqrt{2}}V_{us}^*V_{ud}[C_1(\mu)Q_1+C_2(\mu)Q_2].\qquad (\mu=O(1\,\text{GeV})).\tag{5.232}$$

In the next section the Hamiltonian (5.232) will be generalized to include also penguin operators. The Hamiltonians (5.230) and (5.231) having operators built out of four different flavors are unaffected by penguin contributions. On the other hand there are other $\Delta B=1$ and $\Delta C=1$ Hamiltonians, which contain important penguin contributions. A well-known example is the Hamiltonian for $B\to X_s\gamma$ decay, which will be analyzed in detail in Section 6.5. However, the inclusion of penguin operators does not change the Wilson coefficients $C_{1,2}(\mu)$.

In a numerical analysis, which involves only current-current operators, it is convenient to work with the operators $Q_\pm$ and their coefficients $C_\pm$ defined by

$$\boxed{Q_\pm=\frac{1}{2}(Q_2\pm Q_1),\qquad C_\pm=C_2\pm C_1.}\tag{5.233}$$

Q_+ and Q_- do not mix under renormalization and

$$\boxed{C_\pm(\mu)=\left[1+\frac{\alpha_s(\mu)}{4\pi}J_\pm\right]\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{d_\pm}\left[1+\frac{\alpha_s(M_W)}{4\pi}(B_\pm-J_\pm)\right]}\tag{5.234}$$

with

$$\boxed{J_\pm=\frac{d_\pm}{\beta_0}\beta_1-\frac{\gamma_\pm^{(1)}}{2\beta_0},\qquad d_\pm=\frac{\gamma_\pm^{(0)}}{2\beta_0},}\tag{5.235}$$

$$\boxed{B_\pm=\frac{N\mp1}{2N}\left[\pm11+\kappa_\pm\right],\qquad \gamma_\pm^{(0)}=\pm12\frac{N\mp1}{2N},}\tag{5.236}$$

$$\boxed{\gamma_\pm^{(1)}=\frac{N\mp1}{2N}\left[-21\pm\frac{57}{N}\mp\frac{19}{3}N\pm\frac{4}{3}f-2\beta_0\kappa_\pm\right].}\tag{5.237}$$

The parameters $\kappa_\pm$ distinguish between various renormalization schemes. In particular

$$\kappa_\pm=\begin{cases}0 & \text{NDR}\\ \mp4 & \text{HV}\\ \mp6-3 & \text{DRED}\end{cases}.\tag{5.238}$$

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We recall that $B_\pm - J_\pm$ is scheme independent, and the scheme dependence of $C_\pm(\mu)$ originates entirely from the scheme dependence of $J_\pm$ at the lower end of the evolution in (5.234). Using $N = 3$ one has

$$J_\pm = (J_\pm)_{NDR} + \frac{3 \mp 1}{6}\kappa_\pm = (J_\pm)_{NDR} \pm \frac{\gamma_\pm^{(0)}}{12}\kappa_\pm \,. \tag{5.239}$$

To exhibit the μ dependence at the same footing as the renormalization scheme dependence, it is useful to rewrite (5.234) in the case of B decays as follows:

$$C_\pm(\mu) = \left[1 + \frac{\alpha_s(m_b)}{4\pi}\tilde{J}_\pm(\mu)\right]\left[\frac{\alpha_s(M_W)}{\alpha_s(m_b)}\right]^{d_\pm}\left[1 + \frac{\alpha_s(M_W)}{4\pi}(B_\pm - J_\pm)\right], \tag{5.240}$$

with

$$\tilde{J}_\pm(\mu) = (J_\pm)_{NDR} \pm \frac{\gamma_\pm^{(0)}}{12}\kappa_\pm + \frac{\gamma_\pm^{(0)}}{2}\ln\left(\frac{\mu^2}{m_b^2}\right) \tag{5.241}$$

summarizing both the renormalization scheme dependence and the μ-dependence. Note that in the first parenthesis in (5.240), we have set $\alpha_s(\mu) = \alpha_s(m_b)$ as the difference in the scales in this correction is still of a higher order. We also note that the scheme and the μ-dependent terms are both proportional to $\gamma_\pm^{(0)}$. This implies that the change of renormalization scheme can be compensated by the change in μ. From (5.241) we find generally

$$\mu_i^\pm = \mu_{NDR}\exp\left(\mp\frac{\kappa_\pm^{(i)}}{12}\right), \tag{5.242}$$

where i denotes a given scheme. From (5.238) we have then

$$\mu_{HV} = \mu_{NDR}\exp\left(\frac{1}{3}\right), \qquad \mu_{DRED}^\pm = \mu_{NDR}\exp\left(\frac{2 \pm 1}{4}\right). \tag{5.243}$$

Evidently whereas the change in μ relating HV and NDR is the same for C_+ and C_- and consequently for $C_i(\mu)$, the relation between NDR and DRED is more involved. In any case μ_{HV} and $\mu_{DRED}^\pm$ are larger than μ_{NDR}. This discussion shows that a meaningful analysis of the μ dependence of $C_i(\mu)$ can only be made simultaneously with the analysis of the renormalization scheme dependence.

Numerical Results at NLO

Let us make simple calculations to get an idea about the size of $C_{1,2}(\mu)$ and their renormalization scheme dependence.

For B-decays we have simply:

$$C_1(\mu) = \frac{C_+(\mu) - C_-(\mu)}{2}, \qquad C_2(\mu) = \frac{C_+(\mu) + C_-(\mu)}{2}. \tag{5.244}$$

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We set $f = 5$ in the preceding formulas and use the two-loop $\alpha_s(\mu)$ of (4.143) with $\Lambda^{(5)}_{\overline{\rm MS}} = 225\,{\rm MeV}$ corresponding to $\alpha_s(M_Z) = 0.118$. Choosing $\mu_b = 4.40\,{\rm MeV}$, we find

$$C_1(\mu_b) = -0.184, \qquad C_2(\mu_b) = 1.078, \qquad \text{(NDR)}, \tag{5.245}$$

$$C_1(\mu_b) = -0.226, \qquad C_2(\mu_b) = 1.100, \qquad \text{(HV)}. \tag{5.246}$$

In the case of D decays and K decays, the relevant scales are $\mu = O(m_c)$ and $\mu = O(1\,{\rm GeV})$, respectively, and consequently a more complicated formula with thresholds should in principle be used. Yet it is possible to use the following trick, which avoids these complications. We can simply use our master formulas with $\Lambda^{(5)}_{\overline{\rm MS}}$ replaced by $\Lambda^{(4)}_{\overline{\rm MS}}$ and an "effective" number of active flavors $f = 4.15$. The latter effective value for f allows to obtain an agreement with the exact results to better than 1.5%. We find then with $\Lambda^{(4)}_{\overline{\rm MS}} = 325\,{\rm MeV}$, $\mu_K = 1.0\,{\rm GeV}$, and $\mu_c = 1.50\,{\rm GeV}$

$$C_1(\mu_K) = -0.510, \qquad C_2(\mu_K) = 1.275, \qquad \text{(NDR)}, \tag{5.247}$$

$$C_1(\mu_K) = -0.631, \qquad C_2(\mu_K) = 1.358, \qquad \text{(HV)}, \tag{5.248}$$

$$C_1(\mu_c) = -0.378, \qquad C_2(\mu_c) = 1.188, \qquad \text{(NDR)}, \tag{5.249}$$

$$C_1(\mu_c) = -0.457, \qquad C_2(\mu_c) = 1.237, \qquad \text{(HV)}. \tag{5.250}$$

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