Two students and a trigonometric integral

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Close friends Rebecca and Aisha are taking an integral calculus class. With their permission, we listen in on a conversation that they are having about an integration problem.

Rebecca: Would now be a good time to work on the problem that Dr Domath wants us to present in class?

Aisha: Yes, we should start working on it as we don't have much time and it seems to be a difficult problem.

Rebecca: (consulting her notebook) Great! She wants us to determine the integral

$$\int \sin^m rx \, \cos^n sx \, dx$$

for all nonnegative integers m and n and all positive integers r and s, and she emphasised that the evaluation should be valid for all such integers, and we could ignore constants of integration. Then, from the evaluation, she wants us to deduce the value of

$$\int \sin^2 3x \, \cos^3 2x \, dx.$$

Aisha: I remember in one class she showed us methods of determining

$$\int \sin^m x \, \cos^n x \, dx,$$

which is the special case $r = s = 1$ of the integral she has given to us.

Rebecca: Yes, that's right she showed us how to do it if one of m and n is odd. If n is odd, for example, you set $u = \sin x$ so $du = \cos x \, dx$ and as $\cos^2 x = 1 - \sin^2 x = 1 - u^2$ the integral becomes

$$\int u^m (1 - u^2)^{(n-1)/2} \, du.$$

Since $(n - 1)/2$ is a nonnegative integer the integrand $u^m (1 - u^2)^{(n-1)/2}$ is just a polynomial in u, which can be given explicitly by the binomial theorem and so easily integrated.

Aisha: But this method doesn't work when m and n are both even so it wouldn't work generally for the integral she has given us even if we could adapt the method to $\sin rx$ and $\cos sx$ rather than $\sin x$ and $\cos x$.

Rebecca: You're right. Another technique she showed us was that of reduction formulas.

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Aisha: That's right. She showed us that if you wanted to find a reduction formula for $\int \sin^m x \cos^n x\, dx$ you differentiate the integrand with m decreased by 1 and n increased by 1 and then integrate the resulting formula to obtain

$$\int \sin^m x \cos^n x\, dx =$$

$$-\frac{1}{n+1} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^{n+2} x\, dx$$

after rearranging and dividing by $n + 1$. But even if we could apply this reduction technique to her integral when m and n are both even by successively reducing the exponent m down to 0, we would still be left with having to integrate an even power of cosine.

Rebecca: We need another idea. What has she discussed in class recently?

Aisha: She has talked about exponentials and de Moivre's theorem. Perhaps we could use the formulas for $\sin x$ and $\cos x$ in terms of the exponential function that follow from de Moivre's theorem?

Rebecca: Yes, that's a good idea as we know how to integrate the exponential function.

Aisha: The formulas are $\sin rx = \left(e^{irx} - e^{-irx}\right)/2i$ and $\cos sx = \left(e^{isx} + e^{-isx}\right)/2$.

Rebecca: (writing furiously in her notebook) So for the integrand we need

$$\left(\frac{a^r - b^r}{2i}\right)^m \left(\frac{a^s + b^s}{2}\right)^n = \frac{1}{2^{m+n} i^m} \left(a^r - b^r\right)^m \left(a^s + b^s\right)^n,$$

in the special case $a = e^{ix}$ and $b = e^{-ix}$. By the binomial theorem, we have

$$\left(a^r - b^r\right)^m = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(a^r\right)^{m-j} \left(b^r\right)^j$$

and

$$\left(a^s + b^s\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(a^s\right)^{n-k} \left(b^s\right)^k.$$

Aisha: (looking at Rebecca's notes) Multiplying these two expressions together, we deduce

$$\left(a^r - b^r\right)^m \left(a^s + b^s\right)^n = \sum_{j=0}^{m} \sum_{k=0}^{n} (-1)^j \binom{m}{j} \binom{n}{k} a^{rm+sn-(rj+sk)} b^{rj+sk}.$$

Now what do we do?

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Rebecca: We collect together the terms having a common value t for $rj + sk$. If we do that we obtain

$$\left(a^r - b^r\right)^m \left(a^s + b^s\right)^n = \sum_{t=0}^{rm+sn} a(m, n, r, s; t)\, a^{rm+sn-t} b^t,$$

where

$$a(m, n, r, s; t) := \sum_{\substack{j,k=0 \\ rj+sk=t}}^{m,n} (-1)^j \binom{m}{j}\binom{n}{k}.$$

As j takes the values $0, 1, \ldots, m$ and k the values $0, 1, \ldots, n$, the possible values for $t = rj + sk$ occur among the numbers $0, 1, 2, \ldots, rm + sn$. However, not every number in this list necessarily occurs as a value of t. For example if r and s are both strictly larger than 1 then 1 is not a value of t. Thus if t is not of the form $rj + sk$ for some $j = 0, 1, \ldots, m$ and $k = 0, 1, \ldots, n$ then the value of $a(m, n, r, s; t)$ is zero. But do we know what the quantity $a(m, n, r, s; t)$ is when t is of the form $rj + sk$? Can we work it out in some way?

Aisha: Let's leave that for now. Later I'll have a look in the college library at some books on identities involving binomial coefficients to see if there is a formula for it.

Rebecca: OK. Replacing a by e^{ix} and b by e^{-ix}, we have shown that

$$\sin^m rx \cos^n sx = \frac{1}{2^{m+n} i^m} \sum_{t=0}^{rm+sn} a(m, n, r, s; t)\, e^{i(rm+sn-2t)x},$$

which is really great. We're getting close to what we want. As x is real, the quantity $\sin^m rx \cos^n sx$ is also real, so we can equate it to the real part of the right-hand side of this formula. We can do this by using de Moivre's theorem to replace $e^{i(rm+sn-2t)x}$ by $\cos(rm + sn - 2t)x + i \sin(rm + sn - 2t)x$.

Aisha: As $\dfrac{1}{i^m}$ is equal to $(-1)^{m/2}$ when m is even and to $\dfrac{(-1)^{(m-1)/2}}{i}$ when m is odd, we deduce that $\sin^m rx \cos^n sx$ is equal to

$$\begin{cases} \dfrac{(-1)^{m/2}}{2^{m+n}} \displaystyle\sum_{t=0}^{rm+sn} a(m,n,r,s;t) \cos(rm + sn - 2t)x & \text{if } m \text{ is even,} \\ \dfrac{(-1)^{(m-1)/2}}{2^{m+n}} \displaystyle\sum_{t=0}^{rm+sn} a(m,n,r,s;t) \sin(rm + sn - 2t)x & \text{if } m \text{ is odd.} \end{cases}$$

Rebecca: Great! Now to integrate $\sin^m rx \cos^n sx$, all we have to do is to integrate these sums term-by-term using

$$\int \cos(rm + sn - 2t)x\, dx = \begin{cases} \dfrac{\sin(rm + sn - 2t)x}{rm + sn - 2t} & \text{if } rm + sn - 2t \neq 0, \\ x & \text{if } rm + sn - 2t = 0, \end{cases}$$

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and

$$\int \sin(rm+sn-2t)x\,dx = \begin{cases} -\dfrac{\cos(rm+sn-2t)x}{rm+sn-2t} & \text{if } rm+sn-2t \neq 0, \\ 0 & \text{if } rm+sn-2t = 0. \end{cases}$$

Aisha: So we have to be careful when $rm + sn - 2t = 0$, that is when $(rm + sn)/2$ is an integer. If it is an integer it is certainly in the range between 0 and $rm + sn$.

Rebecca: (writing in her notebook) We can now give our evaluation. Here goes.

1. If m is even and sn is odd (so that $rm + sn$ is odd) then

$$\int \sin^m rx \cos^n sx\,dx = \frac{(-1)^{m/2}}{2^{m+n}} \sum_{t=0}^{rm+sn} a(m, n, r, s; t) \frac{\sin(rm + sn - 2t)x}{rm + sn - 2t}.$$

2. If m is even and sn is even (so that $rm + sn$ is even) then

$$\int \sin^m rx \cos^n sx\,dx = \frac{(-1)^{m/2}}{2^{m+n}} \sum_{\substack{t=0 \\ t \neq (rm+sn)/2}}^{rm+sn} a(m,n,r,s;t) \frac{\sin(rm+sn-2t)x}{rm+sn-2t} + x\frac{(-1)^{m/2}}{2^{m+n}} a\left(m, n, r, s; \frac{(rm+sn)}{2}\right).$$

3. If m is odd and $r + sn$ is odd (so that $rm + sn$ is odd) then

$$\int \sin^m rx \cos^n sx\,dx = \frac{(-1)^{(m+1)/2}}{2^{m+n}} \sum_{t=0}^{rm+sn} a(m,n,r,s;t) \frac{\cos(rm+sn-2t)x}{rm+sn-2t}.$$

4. If m is odd and $r + sn$ is even (so that $rm + sn$ is even) then

$$\int \sin^m rx \cos^n sx\,dx = \frac{(-1)^{(m+1)/2}}{2^{m+n}} \sum_{\substack{t=0 \\ t \neq (rm+sn)/2}}^{rm+sn} a(m,n,r,s;t) \frac{\cos(rm+sn-2t)x}{rm+sn-2t}.$$

Aisha: Great, we've done it! Now we must determine the particular integral that Dr Domath gave us. It has $m = 2$, $n = 3$, $r = 3$ and $s = 2$ so m and sn are both even so it is in case 2. Thus we need $a(2, 3, 3, 2; t)$ for $t = 0, 1, \ldots, 12$.

Rebecca: From what we did these values are given by

$$(a^3 - b^3)^2 (a^2 + b^2)^3 = \sum_{t=0}^{12} a(2, 3, 3, 2; t)\, a^{12-t} b^t.$$

Aisha: We can use *MAPLE* to expand $(a^3 - b^3)^2 (a^2 + b^2)^3$ in powers of a and b.

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Rebecca: (working on her computer) *MAPLE* gives

$$(a^3 - b^3)^2(a^2 + b^2)^3 = a^{12} + 3a^{10}b^2 - 2a^9b^3 + 3a^8b^4 - 6a^7b^5$$
$$+2a^6b^6 - 6a^5b^7 + 3a^4b^8 - 2a^3b^9 + 3a^2b^{10} + b^{12}.$$

Aisha: (reading the screen of Rebecca's computer and writing quickly in Rebecca's notebook) Here is a table of the values.

t	$a(2,\ 3,\ 3,\ 2;\ t)$
0, 12	1
1, 11	0
2, 10	3
3, 9	−2
4, 8	3
5, 7	−6
6	2

Rebecca: Putting these values into case 2 we get

$$\int \sin^2 3x \cos^3 2x\, dx =$$
$$-\frac{x}{16} - \frac{1}{192}\sin 12x - \frac{3}{128}\sin 8x + \frac{1}{48}\sin 6x - \frac{3}{64}\sin 4x + \frac{3}{16}\sin 2x.$$

Aisha: (nervously) How can we be sure that this is right?

Rebecca: All we have to do is to show that the derivative of the right-hand side is equal to $\sin^2 3x \cos^3 2x$.

Aisha: (writing in Rebecca's notebook) The derivative of the right-hand side is equal to

$$-\frac{1}{16} - \frac{1}{16}\cos 12x - \frac{3}{16}\cos 8x + \frac{1}{8}\cos 6x - \frac{3}{16}\cos 4x + \frac{3}{8}\cos 2x.$$

Now what do we do?

Rebecca: (after completing some trigonometric calculations) We can expand $\sin^2 3x \cos^3 2x$ using the formulas

$$\sin^2 3x = \tfrac{1}{2}(1 - \cos 6x),\ \cos^3 2x = \tfrac{1}{4}(3\cos 2x + \cos 6x),$$
$$\cos 2x \cos 6x = \tfrac{1}{2}(\cos 4x + \cos 8x) \text{ and } \cos^2 6x = \tfrac{1}{2}(1 + \cos 12x).$$

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Using these we get

$$\begin{aligned}\sin^2 3x \cos^3 2x &= \tfrac{1}{8}(1 - \cos 6x)(3\cos 2x + \cos 6x)\\ &= \tfrac{1}{8}\left(3\cos 2x - 3\cos 2x \cos 6x + \cos 6x - \cos^2 6x\right)\\ &= \tfrac{1}{8}\left(3\cos 2x - \tfrac{3}{2}(\cos 8x + \cos 4x) + \cos 6x - \tfrac{1}{2}(1 + \cos 12x)\right)\\ &= -\tfrac{1}{16} - \tfrac{1}{16}\cos 12x - \tfrac{3}{16}\cos 8x + \tfrac{1}{8}\cos 6x - \tfrac{3}{16}\cos 4x + \tfrac{3}{8}\cos 2x,\end{aligned}$$

which is what we wanted.

Aisha: That's a good sign we've got the evaluation right. Perhaps we can work on our joint presentation tomorrow after the physics class?

Rebecca: That's fine with me. See you then.

On her way across the campus to go home, Aisha goes into the college library and looks at several books on combinatorial identities including *Combinatorial Identities* by H. W. Gould [1]. She does not find an evaluation of the sum in question. Having looked at so many identities involving sums of binomial coefficients she leaves the library wondering if she somehow has missed the one she was looking for or that it does not have a simple evaluation.

Two days later Rebecca and Aisha remain behind in class to hear Dr Domath's assessment of their presentation.

Dr Domath: Aisha and Rebecca you gave a very clear and well-organised presentation and you obtained exactly the kind of result that I was hoping you would find. Somewhat surprisingly the integral I assigned you does not appear in Gradshteyn and Ryzhik's *Table of integrals, series, and products* [2]. Did your evaluation suggest any other integrals that might be determined in the same way?

Aisha: Dr Domath, we just concentrated on the integral you gave us and didn't think about any others.

Dr Domath: When you have solved a mathematics problem, it is always a good idea to think about whether your solution can be applied to other problems.

Rebecca: Yes, on thinking about it, I don't see why our method wouldn't apply to the integrals

$$\int \sin^m rx \sin^n sx\, dx \qquad \text{and} \qquad \int \cos^m rx \cos^n sx\, dx.$$

Dr Domath: (who liked to have the last word) Yes, you are quite right. Your method works in exactly the same way on those integrals.

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Dr Domath pauses and the two students move towards the door to go to their physics class.

Dr Domath: (warming to her subject) There is quite a lot more you can do with your evaluation. For example you can try and obtain an explicit formula for the integral between 0 and $\pi/2$ or you could equate your evaluations with known ones to obtain some trigonometric identities, or ... .

But with a polite 'Sorry, Dr Domath but we have to go to our physics class now' the students, like Elvis, have left the building.

References

1. H. W. Gould, *Combinatorial Identities*, published by the author (1972).
2. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, (5th edn), Academic Press, (1994).

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The answers to the *Nemo* page from July 2025 on zero were:

1.	Martin Amis	The Rachel Papers Twenty-five of eleven: the Low	
2.	Emily Dickinson	The Snake	
3.	Jack London	To Build a Fire	
4.	George Lord Byron	Don Juan	Canto IX
5.	Thomas Hardy	A Pair of Blue Eyes	Chapter XIV
6.	Fyodor Dostoevsky	The Gambler	Chapter 12

Congratulations to Martin Lukarewski on identifying all of these apart from 1.

This will be the last *Nemo* page. Since the idea of finding connections between mathematics and literature struck me as interesting in March 2005, I have enjoyed the challenge of finding a topic and compiling a set of six quotations. It is interesting, in passing, how many authors appear regularly in the puzzles. Some of these: HG Wells, Jules Verne, Hermann Melville, are not that surprising, since their subject matter involves ideas relating to science calculation or engineering. It is the other, unexpected, writers who are interesting: Emily Dickinson, Thomas Hardy, William Blake, AE Housman, James Joyce (but then he covered everything). However, the real surprise was Henry James, whose interest on repetitive behaviour allowed him a Nemo of his own on iteration in March 2023.

I am in the process of compiling a Nemo Archive to cover all 23 challenges, and readers who are interested in having a copy of this can email me on g.leversha@btinternet.com

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