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Hypergraph colouring

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We discuss the colouring theory of finite set systems. This is not merely an extension of results from collections of 2*-element sets* (*graphs*) *to larger sets. The wider structure* (*hypergraphs*) *offers many interesting new kinds of problems, which either have no analogues in graph theory or become trivial when we restrict them to graphs.*

1. Introduction

In this introductory section we give the most important definitions required to study hypergraph colouring, and briefly survey the half-century history of this topic. For more details on the material of Sections 1 and 2 we refer to Berge [8], Zykov [76] and Duchet [27].

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a finite set of elements called *vertices*, and let $\mathcal{E} = \{E_1, E_2, \ldots, E_m\}$ be a family of subsets of V called *edges* or *hyperedges*. The pair $\mathcal{H} = (V, \mathcal{E})$ is called a *hypergraph* with vertex-set $V = V(\mathcal{H})$ and edge-set $\mathcal{E} = \mathcal{E}(\mathcal{H})$. The hypergraph $\mathcal{H} = (V, \mathcal{E})$ is sometimes called a *set system*. If each edge of a hypergraph contains precisely two vertices, then it is a graph. As in graph theory, the number $|V| = n$ is called the *order* of the hypergraph. Edges with fewer than two elements are usually allowed, but will be disregarded here. Thus, throughout this chapter we assume that each edge $E \in \mathcal{E}$ contains at least two vertices, unless stated explicitly otherwise. Edges that coincide are called *multiple edges*.

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In a hypergraph, two vertices are said to be *adjacent* if there is an edge containing both of these vertices. The adjacent vertices are sometimes called *neighbours* of each other, and the set of neighbours of a given vertex v is called the (open) *neighbourhood* $N(v)$ of v. If $v \in E$, then the vertex v and the edge E are *incident* with each other. For an edge E, the number $|E|$ is called the *size* or *cardinality* of E.

If every edge of $\mathcal{H}$ is of size r, then $\mathcal{H}$ is called an *r-uniform hypergraph;* evidently, a simple graph is a 2-uniform hypergraph. For $2 \leq r \leq n$, we define the *complete r-uniform hypergraph* to be the hypergraph $\mathcal{K}_n^r = (V, \mathcal{E})$ for which $|V| = n$ and $\mathcal{E}$ is the family of all subsets of V of size r. Thus, the complete 2-uniform hypergraph $\mathcal{K}_n^2$ is the complete graph K_n. Traditionally, for a set X and an integer $k \geq 1$, the family of all k-element subsets of X is denoted by $\binom{X}{k}$. With this notation we have $\mathcal{K}_n^r = (V, \binom{V}{r})$.

A subset of vertices $S \subseteq V$ is called *stable* (or *independent*) if no edge of $\mathcal{H}$ is a subset of S. The maximum cardinality of a stable set is termed the *stability number* (or *independence number*) $\alpha(\mathcal{H})$ of $\mathcal{H}$.

A *proper* λ*-colouring* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a mapping $c : V \to \{1, 2, \ldots, \lambda\}$ for which every edge $E \in \mathcal{E}$ has at least two vertices of different colours. The number of proper λ-colourings of $\mathcal{H}$ is a polynomial in λ; it is denoted by $P(\mathcal{H}, \lambda)$ and is called the *chromatic polynomial*. The minimum value of λ for which there exists a proper λ-colouring of a hypergraph $\mathcal{H}$ is called the *chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$. A hypergraph $\mathcal{H}$ is *k-colourable* if $\chi(\mathcal{H}) \leq k$, and is *k-chromatic* if $\chi(\mathcal{H}) = k$. When $\chi(\mathcal{H}) \leq 2$, the hypergraph $\mathcal{H}$ is called *bicolourable*; in parts of the literature the term 'bipartite' is also used.

As for graphs, proper colourings generate partitions of the vertex-set into a number of stable (monochromatic) non-empty subsets called *colour classes*, with as many classes as the number of colours actually used.

A graph $G = (V, E)$ is called a *host graph* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ if each edge of $\mathcal{H}$ induces a connected subgraph in G. Some structural subclasses of hypergraphs are identified referring to this notion. In particular, *interval hypergraphs, hyperstars, hypertrees* and *circular hypergraphs* are defined as hypergraphs with a host graph that is a path, star, tree or cycle. Since 1-element edges are excluded, a 'hyperstar' means that the intersection of all edges is non-empty.

Historical background

The above definitions represent natural generalizations of the respective concepts from graph theory. It was realized in the early 1960s that many graph-theoretical methods can be successfully extended to the more general structure of set systems. The term 'hypergraph' was first suggested at the seminar series of Claude Berge [9].

The colouring of hypergraphs started in 1966 when Erdős and Hajnal [31] introduced the notions of colouring and the chromatic number of a hypergraph, and obtained the first important results. In particular, they defined the *colouring number*,

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whose graph analogue is also called the 'Szekeres–Wilf number' and is equal to the 'degeneracy' + 1; this became an important tool in studying the chromatic number of a hypergraph. From the definition, in a proper colouring *no edge is allowed to be monochromatic*. In the literature these colourings are sometimes called *weak colourings*.

This generalization of graph colourings initiated a wide area of further research. First, some old problems in set systems were formulated as colouring problems and many results in graph colouring were extended to hypergraphs. For example, using a natural generalization of the degree of a vertex, the classic theorem of Brooks (see Chapter 2) was shown to hold for hypergraphs (see [8], [76]). Another generalization of the degree of a vertex allowed Tomescu [61] to extend to hypergraphs some advanced upper bounds on the chromatic number from graph results of Welsh and Powell [71]. Berge [8] introduced the β-degree of a vertex and showed that by using it many previous bounds follow, including those related to the chromatic number. In addition, it provided an algorithm for finding a bound on the chromatic number of a hypergraph.

It is well known that bipartite graphs play a significant role in graph theory. A simple but very important characterization of them is the one by König, that a graph is bipartite if and only if it contains no odd cycles. For hypergraphs, there is no such simple condition for bicolourability, although some criteria in terms of transversal hypergraphs were found by Berge [8]. In the next section we present some further important sufficient conditions on bicolourability. Also, the bicolourable hypergraphs were the first application of the widely used Lovász local lemma [32]. Many more classes of hypergraphs generalizing bipartite graphs have been investigated; see [8] for details.

It is generally accepted that graph colouring started in 1852 with the question of whether any map can be coloured using just four colours (see Chapter 1). It eventually led to the concept of a planar graph. In 1974 Zykov [76] introduced the notion of a planar hypergraph and obtained the first results on them. He also generalized the notions of connection–contraction and of the Hadwiger number of a graph (see Chapter 4). The latter makes it possible to consider hypergraph versions of problems related to Hadwiger's conjecture.

Sometimes a hypergraph approach permits problems to be solved more easily than a graph approach. A celebrated example is Berge's weak perfect graph conjecture, which was first proved by Lovász [46] using so-called normal hypergraphs.

Many generalizations of graph colourings to hypergraphs are more specific and more restrictive than the Erdős–Hajnal approach. We discuss a few of these in the following sections. However, there is one thing that they have in common: they all have *graph analogues*, because they all emerged from a single concept – graph colourings: this means that they all exclude monochromatic edges. As we see next, a new approach that is even more general than the Erdős–Hajnal generalization allows us to introduce colourings that have *no analogues for graph colouring*. In part, this is because, by assumption, the possibility of monochromatic edges is not excluded.

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We may observe further that the Erdős–Hajnal classic concept of hypergraph colouring is asymmetric: with the chromatic number as its central notion, this theory focuses on the *minimum* number of colours, while the *maximum* number of colours has no mathematical interest since a totally multicoloured vertex-set is always feasible. This asymmetry of classical hypergraph colouring was inherited from graph colouring.

A different direction for research concerns the maximum number of colours, excluding polychromatic edges. In hypergraph terms this was proposed by Berge and was first investigated in detail by Sterboul [59] in the early 1970s. We may say that the so-called 'sub-Ramsey numbers' and 'rainbow numbers' also fit here, but those studies went along a different track and were not investigated as widely. We discuss the hypergraph version under the name '$\mathcal{C}$-colouring'. Although the first published paper on this is more than forty years old, it gained its importance only two decades ago when a more general model was born.

In 1993 Voloshin [66], [67] introduced the concept of a mixed hypergraph colouring, which eliminated the above asymmetry and opened up an entirely new direction of research. Instead of $\mathcal{H} = (V, \mathcal{E})$, the basic idea is to consider a structure $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$, termed a *mixed hypergraph*, with *two* families of subsets called $\mathcal{C}$-edges and $\mathcal{D}$-edges. By definition, a *proper* λ*-colouring* of a mixed hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is a mapping $c : V \to \{1, 2, \ldots, \lambda\}$ for which two conditions hold:

- every $C \in \mathcal{C}$ has at least two vertices of a $\mathcal{C}$ommon colour
- every $D \in \mathcal{D}$ has at least two vertices of $\mathcal{D}$ifferent colours.

Formally, this is a combination of classical proper colourings and $\mathcal{C}$-colourings, but mixed hypergraphs were introduced independently.

The concept of a mixed hypergraph colouring has led to the discovery of new principal properties of colourings that do not exist in classical graph and hypergraph colourings: uncolourability in its most general setting, chromatic polynomials of degree less than n, phantom vertices, gaps in the chromatic spectrum, and hypergraph perfection, to name just a few. Moreover, it led to the introduction and study of new classes of hypergraphs, and further types of constraints that are imposed on the edges of a hypergraph when we colour the vertices. It also brings a new look at some classical graph and hypergraph colouring problems, such as colouring planar hypergraphs, chromatic polynomials, edge-colourings, colouring and probability, colouring algorithms, and so on. Finally, it has generated, and continues to generate, a wide variety of combinatorial problems and applications that formerly had no analogues. (The newest, and perhaps most unexpected, application of mixed hypergraphs is the modelling of problems arising in distributed computing and cybersecurity – see [37].) A significant number of subsequent new ideas, results and publications have led to a situation where colouring theory as a whole is taking a new shape.

In the next section we describe a collection of significant results in classical hypergraph colouring, where monochromatic edges are excluded. We then consider $\mathcal{C}$-colourings, which may be viewed as the counterpart on excluding multicoloured

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edges. In later sections we survey some of the new trends which, in their turn, give rise to new challenges in this fast-developing area. In particular, Section 4 is devoted to mixed hypergraphs, which opened up this new dimension of hypergraph colouring. For further information on these, we refer to the research monograph [68] and the regularly updated website [69]. Some results and many open problems are collected in the surveys [64] and [2]. Finally, we discuss an even more general colouring model of hypergraphs, introduced in 2007 by Bujtás and Tuza [15], [13]. In that setting, lower and upper bounds on the number of different colours, and those on the size of the largest monochromatic vertex-subset, can be prescribed for each edge independently. A hypergraph with these colouring restrictions is called a *stably bounded hypergraph*.

2. Proper vertex- and edge-colourings

In this section, we survey the most significant results in classical hypergraph colouring, their relationship with graph colouring, and some further general hypergraph concepts and parameters.

We mention first that, for each $k \geq 2$, it is NP-complete to decide in general whether a hypergraph is k-colourable. It remains NP-complete for $k = 2$ on 3-uniform hypergraphs (see Lovász [47]). For all $k \geq 2$ and $r \geq 2$, it is NP-complete also on r-uniform hypergraphs in which any two edges share at most one vertex, except for bipartite graphs when $k = r = 2$ (see Phelps and Rödl [55]).

In fact, the decision problem for bicolourability had already been proved to be NP-hard in 1972 by the following result of Woodall [73].

Theorem 2.1 *Let G be a graph, and construct a hypergraph $\mathcal{H}$ as follows: each vertex of $\mathcal{H}$ corresponds to an edge of G, and each odd cycle of G forms an edge of $\mathcal{H}$. Then G is 4-colourable if and only if $\mathcal{H}$ is 2-colourable.*

For *uniform* hypergraphs, a sufficient condition for bicolourability can be given in terms of the number of edges. Erdős [29] observed that every r-uniform hypergraph with at most 2^{r-1} edges is bicolourable: in fact, assigning one of two colours to each vertex randomly and independently, we obtain a proper 2-colouring with positive probability. This bound on the number of edges is not tight and has been improved a couple of times. The current record is held by Radhakrishnan and Srinivasan [56].

Theorem 2.2 *For sufficiently large values of r, every r-uniform hypergraph with at most $0.7 \times 2^r \sqrt{r/\ln r}$ edges is bicolourable, and can be properly 2-coloured by a polynomial-time algorithm.*

On the other hand, for every $r \geq 2$, non-bicolourable r-uniform hypergraphs exist with fewer than $r^2\, 2^{r+1}$ edges (see Erdős [30]).

Hypergraphs that are r-uniform and r-regular are interesting combinatorial objects – for instance, they describe the neighbourhood structures (both open and closed) of regular graphs. Concerning their bicolourability, Alon and Bregman [1]

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proved the sufficient condition $r \geq 8$, and it was recently proved by Henning and Yeo [36] that $r \geq 4$ is sufficient. (The Fano plane shows that the assertion is not valid for $r = 3$.)

A *cycle of length s* ($s \geq 2$) in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is a sequence $v_0E_0v_1E_1v_2 \dots v_{s-1}E_{s-1}v_s$ with $v_0 = v_s$, consisting of s distinct vertices and s distinct edges, such that $v_i, v_{i+1} \in E_i$ for all $i = 0, 1, \dots, s-1$. Depending on the parity of s, the cycle is called *odd* or *even*. Moreover, the odd cycle $v_0E_0v_1E_1v_2 \dots v_{s-1}E_{s-1}v_s$ is an *anti-Sterboul cycle* if any two non-consecutive edges are disjoint and if $|E_i \cap E_{i+1}| = 1$ for every $i = 0, 1, \dots, s-2$ (but not necessarily for $E_{s-1} \cap E_0$). A *Sterboul hypergraph* is a hypergraph containing no anti-Sterboul cycle.

The next two theorems give sufficient conditions for bicolourability in terms of conditions imposed on cycles. The first one, due to Fournier and Las Vergnas [33], was the deepest theorem of this kind for several decades. Its formulation with 3-critical hypergraphs is taken from Zykov [76]. (As for graphs, a k-chromatic hypergraph is *critical* if the removal of any edge decreases the chromatic number.)

Theorem 2.3 *If every cycle of odd length in a hypergraph $\mathcal{H}$ contains three edges with a vertex in common, then $\mathcal{H}$ is bicolourable. Equivalently, every 3-chromatic critical hypergraph contains an odd cycle in which no three edges have a vertex in common.*

This theorem was recently strengthened by Défossez [24], who proved an old conjecture of Sterboul.

Theorem 2.4 *Every Sterboul hypergraph is bicolourable.*

There are also lower bounds on the number of edges in critical hypergraphs. The first one was proved by Seymour [58] in 1974; the second is more recent, by Kostochka and Stiebitz [40].

Theorem 2.5 *If a hypergraph on n vertices is 3-chromatic critical, then it has at least n edges.*

Theorem 2.6 *If a hypergraph on n vertices is $(k+1)$-chromatic critical and contains no 2-element edges, then the number of its edges is at least $(k - 3/\sqrt[3]{k})\, n$.*

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, and let T be a subset of V. Then the hypergraph $\mathcal{H}/T = (T, \mathcal{E}')$ is the *restriction of $\mathcal{H}$ to T*, where $\mathcal{E}'$ consists of all non-empty intersections of edges of $\mathcal{E}$ with T. A hypergraph is said to be *balanced* if every odd cycle has an edge that contains three vertices of the cycle. Berge [7] proved the following characterization.

Theorem 2.7 *A hypergraph is balanced if and only if every restriction is 2-colourable.*

Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. The *chromatic index* $\chi'(\mathcal{H})$ of $\mathcal{H}$ is the least number of colours necessary to colour the edges of $\mathcal{H}$ in such a way that any two intersecting edges have distinct colours.

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For a vertex v, the number of all edges containing v is the *degree of* v, denoted by $d(v)$. The maximum vertex-degree in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is $\Delta(\mathcal{H}) = \max_{v \in V} d(v)$. For any hypergraph $\mathcal{H}$, it is clear that $\chi'(\mathcal{H}) \geq \Delta(\mathcal{H})$. A hypergraph $\mathcal{H}$ has the *edge-colouring property* if $\chi'(\mathcal{H}) = \Delta(\mathcal{H})$. In this connection, the following famous result was proved by Baranyai [6].

Theorem 2.8 *If n is a multiple of r, then $\chi'(\mathcal{K}_n^r) = \Delta(\mathcal{K}_n^r) = \binom{n-1}{r-1}$.*

In a hypergraph $\mathcal{H} = (V, \mathcal{E})$, for any subfamily $\mathcal{F} \subseteq \mathcal{E}$ we call the hypergraph $\mathcal{H}_\mathcal{F} = (V, \mathcal{F})$ the *partial subhypergraph* of $\mathcal{H}$. We say that a hypergraph $\mathcal{H}$ is *normal* if every partial subhypergraph of $\mathcal{H}$ satisfies the edge-colouring property. The following result was proved by Fournier and Las Vergnas [33].

Theorem 2.9 *Every normal hypergraph is bicolourable.*

A subset $T \subseteq V$ is called a *transversal* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ if $|T \cap E| \geq 1$ for each edge $E \in \mathcal{E}$. The cardinality of a minimum transversal is denoted by $\tau(\mathcal{H})$. In a hypergraph $\mathcal{H}$, a set of edges that pairwise have no vertices in common is called a *matching*. The maximum size of a matching (over all matchings) is denoted by $\nu(\mathcal{H})$.

Since any matching is a set of pairwise non-intersecting edges, any transversal must have at least one vertex from each edge of the matching. This implies that $\tau(\mathcal{H}) \geq \nu(\mathcal{H})$ for any hypergraph $\mathcal{H}$. We say that $\mathcal{H}$ satisfies the *König property* if $\tau(\mathcal{H}) = \nu(\mathcal{H})$. The next theorem, due to Lovász [46], is sometimes called the *perfect graph theorem* because it implies that the complement of a perfect graph is perfect. The latter statement was formerly known as the *weak perfect graph conjecture*, formulated by Claude Berge in the early 1960s (see Chapter 7).

Theorem 2.10 *A hypergraph $\mathcal{H}$ is normal if and only if every partial subhypergraph $\mathcal{H}'$ has the König property.*

Besides normal and balanced hypergraphs, there are many other classes of hypergraphs that are either bicolourable or generalize the bipartite graphs: hypergraphs without odd cycles, unimodular hypergraphs, hypertrees (also called arboreal hypergraphs), Mengerian hypergraphs, paranormal hypergraphs, and others. The hierarchy of these and further related classes is exhibited in [8, p.163]. Some of them have min-max properties arising from bipartite graphs and are used in polyhedral optimization (see [8], [27]).

It may appear that increasing both the size of edges and the length of the shortest cycle decreases the chromatic number of a hypergraph and eventually makes it bicolourable. This is not true: there exist high-chromatic hypergraphs for any values of these parameters. The following theorem was first proved by Erdős and Hajnal [31], and constructions were given later by Lovász [45] and by Nešetřil and V. Rödl [52].

Theorem 2.11 *For any integers $r, s, t \geq 2$, there exists an r-uniform hypergraph $\mathcal{H}$ with no cycles shorter than s, for which $\chi(\mathcal{H}) \geq t$.*

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, we define the *bipartite* (*König*) *representation* of $\mathcal{H}$ to be the bipartite graph $B(\mathcal{H})$ with partite sets V and $\mathcal{E}$, where a vertex $v \in V$ is

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adjacent to a vertex $E \in \mathcal{E}$ in $B(\mathcal{H})$ if and only if the vertex $v \in V$ is incident with the edge $E \in \mathcal{E}$ in $\mathcal{H}$.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is said to be *planar* if $B(\mathcal{H})$ is a planar graph (see [68], [76]). This means that any planar hypergraph can be drawn in the plane in the following way: vertices are points; each edge is a closed curve whose interior region contains the points of the edge, and any two edges intersect only at small neighbourhoods of their common vertices. The remaining connected regions of the plane form the faces of such a plane embedding. Planar graphs are special cases of planar hypergraphs: we can replace the curves corresponding to graph edges with closed curves encircling the adjacent vertices.

We next cite two results, the first by Bulitko and the second by Burshtein and Kostochka [76, p. 138].

Theorem 2.12 *The four-colour theorems for planar graphs and for planar hypergraphs are equivalent.*

Theorem 2.13 *If a planar hypergraph contains at most one edge of size* 2*, then* $\chi(\mathcal{H}) \leq 2$.

There are several types of more restrictive hypergraph colourings that are regularly encountered in the literature (see [8], [76], [68]). Here we list some of them.

Strong colouring

A *strong* λ*-colouring* of $\mathcal{H}$ is a partition of V into λ stable sets S_i $(i = 1, 2, \ldots, \lambda)$ such that the inequality $|E_j \cap S_i| \leq 1$ holds for each edge E_j and for every i. The *strong chromatic number* $\gamma(\mathcal{H})$ is the smallest number λ for which there exists a strong λ-colouring of $\mathcal{H}$. It follows that $\gamma(\mathcal{H}) \geq \chi(\mathcal{H})$, because every strong colouring is also a weak colouring. Note that the strong and weak colourings coincide when $\mathcal{H}$ is a graph. Also, $\gamma(\mathcal{H})$ is the chromatic number of the 2*-section graph* of $\mathcal{H}$ – that is, the graph obtained from $\mathcal{H}$ on replacing each edge E_j by the complete graph with vertex-set E_j.

Equitable colouring

An *equitable* λ*-colouring* of $\mathcal{H}$ is a partition of V into λ stable sets S_i $(i = 1, 2, \ldots, \lambda)$ such that the following inequalities hold for each edge E_j and for each i:

$$\left\lfloor |E_j|/\lambda \right\rfloor \leq |E_j \cap S_i| \leq \left\lceil |E_j|/\lambda \right\rceil .$$

Good colouring

A *good* λ*-colouring* of $\mathcal{H}$ is a partition of V into λ stable sets S_i $(i = 1, 2, \ldots, \lambda)$ such that each edge E_j has $\min\{|E_j|, \lambda\}$ colours. If $\lambda \geq \max_{j \in J} |E_j|$, then a good

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λ-colouring is a strong λ-colouring. For each λ, any equitable λ-colouring is also a good λ-colouring.

I-regular colouring

For each edge E_j in $\mathcal{H}$, let a_j and b_j be integers with $0 \le a_j \le b_j < |E_j|$. An *I-regular λ-colouring* of $\mathcal{H}$ is a partition of V into λ stable sets S_i $(i = 1, 2, \ldots, \lambda)$, in such a way that $a_j \le |E_j \cap S_i| \le b_j$ for each edge E_j and each $i = 1, 2, \ldots, \lambda$. Notice that each weak colouring is an I-regular colouring if $a_j = 0$ and $b_j = |E_j| - 1$, that every strong colouring is an I-regular colouring with $a_j = 0$ and $b_j = 1$, and that every equitable λ-colouring is an I-regular colouring with $a_j = \lfloor |E_j|/\lambda \rfloor$ and $b_j = \lceil |E_j|/\lambda \rceil$.

List-colouring

As for graphs, list-colouring and related notions (see Chapter 6) can also be defined for hypergraphs; for example, it is also true that the choice number of any hypergraph is equal to its fractional chromatic number (see [48]). Since a hypergraph can represent a set system of any generality, this theorem implies the corresponding equality for many kinds of colourings, including edge- and total-colourings, generalized colourings defined in terms of hereditary properties, and so on.

There are further connections between hypergraph colourings and extremal problems, Kneser's problem, Ramsey-type problems, etc., and many applications. These issues are not discussed here; for information we refer to [8] and [27].

3. $\mathcal{C}$-colourings

The classical proper colouring of hypergraphs discussed in the previous section requires that two vertices have different colours inside each edge. A *$\mathcal{C}$-colouring* of a hypergraph applies an opposite colouring constraint, prescribing the presence of two vertices with a $\mathcal{C}$ommon colour inside each edge. Assigning the same colour to all vertices always yields a $\mathcal{C}$-colouring. Thus, the essential parameter here is not the minimum possible number of colours, but the maximum. This is called the *upper chromatic number* of a hypergraph $\mathcal{H}$ and is denoted by $\bar{\chi}(\mathcal{H})$.

Historically, the upper chromatic number appears in works of Sterboul [59], Berge [8] (who called $\bar{\chi}(\mathcal{H})+1$ the 'cochromatic number'), Voloshin [67] and other authors. Recent results are surveyed in [20].

The following are some basic facts concerning $\mathcal{C}$-colourings.

- The inequality $\bar{\chi}(\mathcal{H}) \le \alpha(\mathcal{H})$ holds for every hypergraph: indeed, select one vertex from each colour class of a $\bar{\chi}$-colouring of $\mathcal{H}$. The $\bar{\chi}$-element vertex-set obtained in this way is independent, as otherwise we would have a polychromatic edge.

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- Denote by $\tau_2(\mathcal{H})$ the minimum cardinality of a '2-*transversal*', a vertex-set $T \subseteq V$ for which $|T \cap E| \geq 2$ for each $E \in \mathcal{E}$. We obtain a $\mathcal{C}$-colouring of $\mathcal{H}$ by assigning colour 1 to each element of T and colouring the remaining vertices with colours $2, 3, \ldots, n - \tau_2 + 1$ pairwise differently. This proves that $\bar{\chi}(\mathcal{H}) \geq n - \tau_2(\mathcal{H}) + 1$.
- Let $\mathcal{S} = (V, \mathcal{E})$ be a hyperstar, so the intersection $\bigcap_{E\in\mathcal{E}} E$ is not empty. In addition, assume that $|E| \geq 3$ for all $E \in \mathcal{E}$, and consider a vertex z contained in all edges. Then on deleting z from each $E \in \mathcal{E}$ we obtain a hypergraph $\mathcal{S}^-$ for which $\bar{\chi}(\mathcal{S}) = \alpha(\mathcal{S}^-) + 1$. Moreover, if $\mathcal{S}$ is 3-uniform, then $\mathcal{S}^-$ is simply a graph. It is thus easy to see that the determination of the upper chromatic number is an $\mathcal{NP}$-hard problem, even for the class of 3-uniform hypertrees. Another consequence of this simple construction is that the difference $\alpha(\mathcal{S}) - \bar{\chi}(\mathcal{S})$ can be arbitrarily large, since $\alpha(\mathcal{S}) = |V| - 1$.

In connection with the upper chromatic number, Voloshin [67] introduced the notion of a *$\mathcal{C}$-perfect hypergraph*. Analogously to the well-known definition of a perfect graph, we start with an inequality that is valid for all objects and prescribe that it hereditarily holds with equality. More precisely, let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, and let T be a subset of V. Then the hypergraph $\mathcal{H}' = (T, \mathcal{E}')$ is called the *induced subhypergraph* if $\mathcal{E}'$ consists of all edges of $\mathcal{E}$ that lie entirely in T. A hypergraph $\mathcal{H}$ is called *$\mathcal{C}$-perfect* if $\bar{\chi}(\mathcal{H}') = \alpha(\mathcal{H}')$ for every induced subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$; otherwise, $\mathcal{H}$ is *$\mathcal{C}$-imperfect*. Finally, $\mathcal{C}$-imperfect hypergraphs, all of whose proper induced subhypergraphs are $\mathcal{C}$-perfect, are called *minimally $\mathcal{C}$-imperfect*.

The following are some basic facts concerning $\mathcal{C}$-perfect and $\mathcal{C}$-imperfect hypergraphs.

- A typical example of a $\mathcal{C}$-perfect hypergraph is a hyperstar $\mathcal{H}$ with $|\bigcap_{E\in\mathcal{E}} E| \geq 2$. Indeed, if $\mathcal{H}$ is of order n, then $\alpha(\mathcal{H}) = \bar{\chi}(\mathcal{H}) = n - 1$, and every induced subhypergraph of it is either edgeless or has at least two vertices contained in the common intersection of the edges.
- A *monostar* $\mathcal{H}$ is a hyperstar with $|\bigcap_{E\in\mathcal{E}} E| = 1$. It is always $\mathcal{C}$-imperfect, since $\bar{\chi}(\mathcal{H}) < \alpha(\mathcal{H}) = n - 1$. (For example, the hypergraphs in Fig. 1 are monostars.)

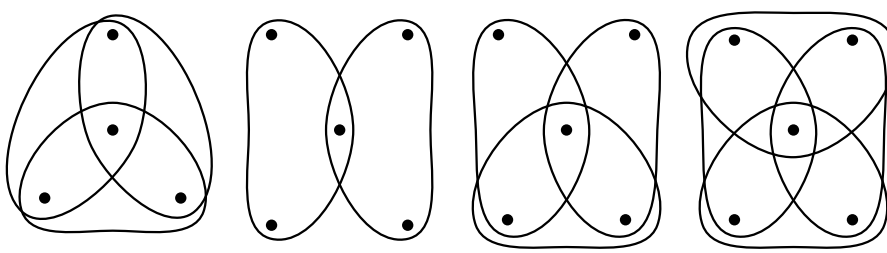


Fig. 1. The four minimal $\mathcal{C}$-imperfect 3-uniform monostars

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- A *cycloid* $\mathcal{C}_n^r$ is an r-uniform circular hypergraph with n vertices and n edges (where $n > r \geq 3$) for which each edge contains r consecutive vertices of the host cycle. (See Fig. 2, which is the cycloid $\mathcal{C}_5^3$.) Every cycloid of the form $\mathcal{C}_{2r-1}^r$ (with $r \geq 3$) is minimally $\mathcal{C}$-imperfect, since $\alpha(\mathcal{C}_{2r-1}^r) = 2r-3$ and $\bar{\chi}(\mathcal{C}_{2r-1}^r) = 2r-4$; moreover, each of its non-edgeless proper induced subhypergraphs is a $\mathcal{C}$-perfect hyperstar. Otherwise, $\mathcal{C}_n^r$ is $\mathcal{C}$-perfect if $n \leq 2r-2$, and $\mathcal{C}$-imperfect (but not minimally) if $n \geq 2r$.

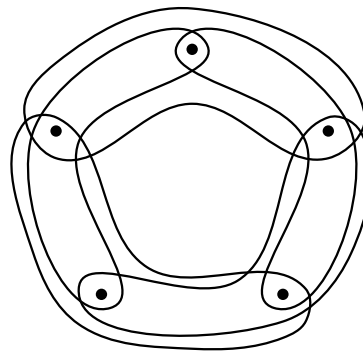


Fig. 2. The cycloid $\mathcal{C}_5^3$

Clearly, a hypergraph is $\mathcal{C}$-perfect if and only if it has no induced subhypergraph that is minimally $\mathcal{C}$-imperfect. For some structural subclasses of hypergraphs, forbidden subhypergraph characterizations have been proved for $\mathcal{C}$-perfectness. The following statements appear in [23], [18] and [19], respectively.

Theorem 3.1

- *An interval hypergraph is $\mathcal{C}$-perfect if and only if it contains no induced monostar.*
- *Let $\mathcal{H}$ be a circular hypergraph in which no edge contains any other edge as a subset. Then $\mathcal{H}$ is $\mathcal{C}$-perfect if and only if it contains no induced monostar and is not isomorphic to the cycloid $\mathcal{C}_{2r-1}^r$ for any $r \geq 3$.*
- *A hypertree is $\mathcal{C}$-perfect if and only if it contains no induced monostar.*

It has been conjectured that the number of r-uniform minimally $\mathcal{C}$-imperfect hypergraphs is finite for each r, but in fact we do not even know whether there is a 3-uniform minimally $\mathcal{C}$-imperfect hypergraph different from the six examples shown in Figs. 1, 2 and 3. Here, $C_6(1,2,5) \cup C_6(1,3,5)$ is Král's example [41], which consists of six edges obtained by rotating clockwise the edge on the left plus two more edges on the right (they should be in the same vertex-set).

On the number of r-uniform minimally $\mathcal{C}$-imperfect hypergraphs, finiteness has been verified in the cases where also the transversal number is bounded. More precisely, the following result was proved by Bujtás and Tuza [18].

Theorem 3.2 *For each fixed $r \geq 3$ and $t \geq 1$, there are only finitely many r-uniform minimally $\mathcal{C}$-imperfect hypergraphs with transversal number t.*

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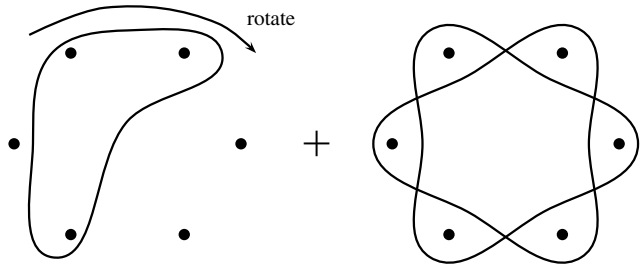


Fig. 3. A minimally $\mathcal{C}$-imperfect hypergraph $C_6(1,2,5) \cup C_6(1,3,5)$ that is neither a monostar nor a cycloid

In the same paper, Bujtás and Tuza characterized minimally $\mathcal{C}$-imperfect hypergraphs with transversal numbers $\tau = 2$ and $\tau = 3$ and constructed r-uniform examples with $\tau = 2$ and 3 for which the number of minimally $\mathcal{C}$-imperfect constructions increases without bound as $r \to \infty$. We do not know any minimally $\mathcal{C}$-imperfect hypergraphs with $\tau \geq 4$.

The $\mathcal{C}$-perfection of a hypergraph may lead to efficient algorithms for solving some problems that are NP-complete in general. For example, although the determination of the upper chromatic number is an NP-complete problem, even for just the class of 3-uniform monostars, the $\mathcal{C}$-perfect hypertrees (even without any restrictions on edge sizes) can still be $\bar{\chi}$-coloured in polynomial time (see [19]). On the other hand, the recognition problem for $\mathcal{C}$-perfect hypertrees is co-NP-complete, but it becomes polynomial-time solvable if edge-sizes are bounded from above. Further results on time complexity can be found in [43] and [19], which show that the situation is more complicated than in the case of graph perfectness.

The minimum number of r-edges, for $\bar{\chi} < r$

By the pigeonhole principle, for every r-uniform hypergraph $\mathcal{H}$ any vertex colouring with $r-1$ colours yields a $\mathcal{C}$-colouring; thus, $\bar{\chi}(\mathcal{H}) \geq r-1$ always holds. This observation leads to the extremal problem of determining the minimum number of r-element edges on n vertices for which $\bar{\chi} = r-1$. We denote this minimum number by $f(n,r)$, where $n \geq r \geq 2$. Alternatively, we can define $f(n,r)$ to be the minimum number of r-element subsets of an n-element underlying set V, selected so that, for each r-partition $(V_1, V_2, \ldots, V_r)$ of V, there exists a selected r-element set S that intersects each V_i in precisely one element.

For $r = 2$, $f(n,2)$ is the minimum number of edges in a connected graph of order n, so $f(n,2) = n-1$. Taking an n-element vertex-set V and fixing a vertex $z \in V$, we obtain a general upper bound for $f(n,r)$ from the hyperstar $\mathcal{H} = (V, \mathcal{E})$ with edge-set

$$\mathcal{E} = \left\{ E \in \binom{V}{r} : z \in E \right\}.$$

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Observe that $\bar{\chi}(\mathcal{H}) = r - 1$ and $|\mathcal{E}| = \binom{n-1}{r-1}$. For a lower bound, consider an r-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\bar{\chi}(\mathcal{H}) = r - 1$, and take a set U of $r - 2$ vertices. Since the graph with vertex-set $V \setminus U$ and edge-set $\{E \setminus U : U \subseteq E \in \mathcal{E}\}$ must be connected for every choice of U, a lower bound on $f(n, r)$ can be calculated. These estimates appeared first in a paper of Sterboul [60].

Theorem 3.3 *For every $n \geq r \geq 3$,*

$$\frac{2}{n-r+2}\binom{n}{r} \leq f(n,r) \leq \binom{n-1}{r-1}.$$

The lower bound in this theorem is tight for 3-uniform hypergraphs – that is, $f(n, 3) = \lceil \frac{1}{3}n(n-2) \rceil$, for each $n \geq 3$. The simplest proof appeared in [25], and further references to a series of papers proving the same theorem can be found in the survey [20]. For $r \geq 4$ and n large, however, the lower bound is no longer tight (see [17]).

Let us mention a further equality from [60]:

$$f(n, n-2) = \binom{n}{2} - \mathrm{ex}(n, \{C_3, C_4\}),$$

where the last term is the Turán number for graphs of order n and girth 5. Its determination (exact or asymptotic) has been an open problem for several decades. The ratio of known upper and lower bounds is still about $\sqrt{2}$ for any large n. This indicates that finding an exact general formula for $f(n, k)$ is a hopeless task.

Recently, Bujtás and Tuza [17] gave an asymptotic solution:

Theorem 3.4 *If $n > k > 2$, then*

- *for all fixed k and all n,*

$$f(n,k) \leq \frac{2}{n-1}\binom{n-1}{k} + \frac{n-1}{k-1}\left(\binom{n-2}{k-2} - \binom{n-k-1}{k-2}\right).$$

- *for all $k = o(n^{1/3})$ as $n \to \infty$,*

$$f(n,k) = (1 + o(1))\,\frac{2}{k}\binom{n-2}{k-1}.$$

$\mathcal{C}$-type colourings of graphs

For graphs (that is, 2-uniform hypergraphs), $\mathcal{C}$-colouring imposes the simple restriction that each connected component of the graph must be monochromatic, so there is little to explore in this direction. But the following defective version of $\mathcal{C}$-colouring introduced by Bujtás *et al.* [11] raises interesting questions, even for graphs.

For each integer $k \geq 1$, a *k-improper $\mathcal{C}$-colouring* is a colouring of vertices of a graph G such that, for each vertex v, at most k vertices in the neighbourhood $N(v)$ receive colours different from that of v; the *k-improper upper chromatic number*

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$\bar{\chi}_{k\text{-imp}}(G)$ is the maximum number of colours that can be achieved in such a colouring. Equivalently, $\bar{\chi}_{k\text{-imp}}(G)$ is precisely the maximum number of components that can be obtained from G by deleting at most k edges at each vertex. As proved in [11], the Nordhaus–Gaddum-type inequality

$$\bar{\chi}_{k\text{-imp}}(G) + \bar{\chi}_{k\text{-imp}}(\overline{G}) \leq n + 1$$

holds for every $k > 0$ and for every graph G of order $n \geq 8k + 1$. Moreover, if both G and $\overline{G}$ have colourings with more than one colour, then the upper bound is $4k + 2$, independently of n.

Some further versions of graph colouring that can be described by $\mathcal{C}$-type colouring constraints are surveyed in [20].

4. Colourings of mixed hypergraphs

We recall from the end of Section 1 that a *mixed hypergraph* is a triple $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$, where both $\mathcal{C}$ and $\mathcal{D}$ are set-systems over the vertex-set V; their members $C \in \mathcal{C}$ and $D \in \mathcal{D}$ are termed *$\mathcal{C}$-edges* and *$\mathcal{D}$-edges*, respectively. A vertex λ-colouring $c : V \to \{1, 2, \ldots, \lambda\}$ is *proper* if

$$|c(C)| < |C| \text{ for all } C \in \mathcal{C}, \quad \text{and} \quad |c(D)| > 1 \text{ for all } D \in \mathcal{D},$$

where $c(Y)$ denotes, for any $Y \subseteq V$, the set of colours assigned by the mapping c to the vertices in Y. We say that $\mathcal{H}$ is a *bi-hypergraph* if $\mathcal{C} = \mathcal{D}$, and for any mixed hypergraph a set in $\mathcal{C} \cap \mathcal{D}$ is called a *bi-edge*. Note that the case $\mathcal{C} = \varnothing$ (termed a *$\mathcal{D}$-hypergraph*) corresponds to a proper vertex-colouring in the classical sense, whereas $\mathcal{D} = \varnothing$ (termed a *$\mathcal{C}$-hypergraph*) was discussed in the previous section with respect to $\mathcal{C}$-colouring.

In our historical introduction we have already indicated briefly that mixed hypergraphs opened up a new dimension in hypergraph colouring theory. In this section we give more details.

Example. Consider the complete r-uniform hypergraph $\mathcal{K}_n^r$ of order n, and consider all r-tuples as bi-edges. Then we may use no more than $r - 1$ colours (because $\mathcal{C} = \binom{V}{r}$) and no colour may occur on more than $r - 1$ vertices (because $\mathcal{D} = \binom{V}{r}$). Thus, for $n > (r-1)^2$, the hypergraph does not admit any proper colouring. In particular, for $r = 2$ and $n = 2$, the vertex pair, which would simultaneously be a $\mathcal{C}$-edge and a $\mathcal{D}$-edge, leads to a contradiction of colouring requirements.

Arising from this example, we call a mixed hypergraph *colourable* if it admits at least one proper colouring, and call it *uncolourable* otherwise. Uncolourable mixed hypergraphs were studied in detail by Tuza and Voloshin [63], who presented the following construction for uncolourable hypergraphs, based on a principle substantially different from the one above.

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Example. Let $G = (V, E)$ be a graph with chromatic number k, and define $\mathcal{D} = E$. Moreover, let $\mathcal{C} \subseteq \binom{V}{k}$ be the k-uniform set system whose members are the vertex-sets of P_k subpaths in G. Then $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is uncolourable, by a theorem of Gallai and Roy (see [34], [57], [62]).

A further construction from the same paper can be thought of as 'list-colouring without lists'. For any instance of the list-colouring problem with a graph $G = (V, E)$ and lists L_v on its vertices, we can extend V with the set of colours to obtain a mixed hypergraph $\mathcal{H}$ on the vertex-set $V \cup \bigcup_{v \in V} L_v$. The $\mathcal{D}$-edges of $\mathcal{H}$ are the edges of G, and the $\mathcal{C}$-edges are the sets $L_v \cup \{v\}$, for all $v \in V$. Then $\mathcal{H}$ is colourable if and only if G admits a list-colouring.

Algorithmically it is an NP-complete problem to recognize colourable mixed hypergraphs, and so there is no hope of finding easily identifiable obstacles against colourability. Quantitatively, only the very strong requirement $\bar{\chi}(\mathcal{C}) - \chi(\mathcal{D}) \geq n - 3$ is sufficient to imply colourability; even replacing $n - 3$ by $n - 4$ yields some uncolourable hypergraphs (see Tuza and Voloshin [63]).

The *chromatic inversion* of a mixed hypergraph $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ is the mixed hypergraph $\mathcal{H}^c = (V, \mathcal{C}^c, \mathcal{D}^c)$, with $\mathcal{C}^c = \mathcal{D}$ and $\mathcal{D}^c = \mathcal{C}$. Nearly two decades ago, Voloshin [67] asked whether there is a connection between the colourability of $\mathcal{H}$ and that of $\mathcal{H}^c$. A negative answer was recently given by Hegyháti and Tuza [35], who proved that it is NP-complete to test whether the chromatic inversion of a colourable 3-uniform mixed hypergraph is colourable, and co-NP-complete to test whether the chromatic inversion of an uncolourable 3-uniform mixed hypergraph is uncolourable. This negative result is derived from the positive theorem that if $\mathcal{D} = \binom{V}{3}$ and $\mathcal{H}$ is 3-uniform, then the colourability of $\mathcal{H}$ is decidable in polynomial time.

Properties of colourable mixed hypergraphs

For the rest of this section we restrict our attention to mixed hypergraphs $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ that are colourable. Under this assumption it is interesting to study the *feasible set* $\Phi(\mathcal{H})$ of $\mathcal{H}$, defined as the set of those integers k for which $\mathcal{H}$ admits a proper k-colouring with exactly k-colours.

Among the elements of $\Phi(\mathcal{H})$, the minimum $\chi(\mathcal{H})$ and the maximum $\bar{\chi}(\mathcal{H})$ are termed the *lower chromatic number* and *upper chromatic number*, respectively. In some classes of hypergraphs, where $\bar{\chi}$ is near to n, the *decrement*, defined as $\mathrm{dec}(\mathcal{H}) = n - \bar{\chi}(\mathcal{H})$, may be informative.

Further information, more detailed than $\Phi(\mathcal{H})$, about the colourability properties of a mixed hypergraph, is provided by the *chromatic spectrum*. It is defined as the sequence $(r_1, r_2, \ldots, r_n)$, where each r_k is the number of *feasible partitions* – that is, the partitions of the vertex-set into colour classes induced by proper colourings using precisely k colours. A *gap* in the chromatic spectrum and in the feasible set is an integer k for which $\chi(\mathcal{H}) < k < \bar{\chi}(\mathcal{H})$ and $r_k = 0$.

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If $\mathcal{H}$ is a $\mathcal{D}$-hypergraph, then its feasible set is an interval of integers: $\Phi(\mathcal{H}) = \{\chi(\mathcal{H}), \chi(\mathcal{H}) + 1, \ldots, n\}$. Similarly, if $\mathcal{H}$ is a $\mathcal{C}$-hypergraph, then its feasible set is the interval $\Phi(\mathcal{H}) = \{1, 2, \ldots, \bar{\chi}(\mathcal{H})\}$. Somewhat unexpectedly, the feasible sets of non-1-colourable mixed hypergraphs are rather unrestricted – namely, for any finite set $S \subset \mathbb{N} \setminus \{1\}$ of integers, there exists a mixed hypergraph $\mathcal{H}$ for which $\Phi(\mathcal{H}) = S$ (see Jiang *et al.* [38]). Even more generally, for any finite sequence $(r_k)_{k=a}^{b}$ of non-negative integers with $2 \leq a \leq b$ and $r_a \neq 0 \neq r_b$, there exists a mixed hypergraph $\mathcal{H}$ for which $\chi(\mathcal{H}) = a$, $\bar{\chi}(\mathcal{H}) = b$, and for each $a \leq k \leq b$, the number of feasible partitions is exactly r_k (see Král' [42]). A similar statement is valid for feasible sets of r-uniform mixed hypergraphs with only one additional restriction: if an integer $k < r - 1$ belongs to Φ, then all integers between k and $r - 1$ also have to be contained in Φ (see Bujtás and Tuza [14]). The case of chromatic spectra with $r_1 = 0$ and $r_k = 0$ or 1 for all $k \geq 2$ was recently studied by Zhao, Diao and Wang [74], [75], especially concerning the minimum number of vertices and edges.

Hypertrees are more restrictive: colourability implies that $\chi = 2$ (unless $\mathcal{D} = \varnothing$, in which case $\chi = 1$) and that the feasible set is gap-free. More generally, no gap occurs if the hypergraph admits a host graph in which any two cycles are vertex-disjoint (see Král *et al.* [43]). A proper 2-colouring of a hypertree is also easy to find: as long as there exist $\mathcal{C}$-edges of size 2, contract them to single vertices, and then colour the contracted host tree properly in the usual sense. A hypertree is uncolourable if and only if this procedure yields a contracted $\mathcal{D}$-edge of size 1 (see Tuza and Voloshin [63]).

The upper chromatic number turns out to be a harder issue. Although Bulgaru and Voloshin [23] proved that it can be expressed with a nice formula on mixed *interval* hypergraphs, the determination of $\bar{\chi}$ on hypertrees is NP-hard (see [43]). Moreover, various results on its non-approximability are known (see [22]).

For the feasible set of a planar mixed hypergraph $\mathcal{H}$, Kobler and Kündgen [39] proved that either $\Phi(\mathcal{H})$ is a gap-free interval with minimum value 4 or less, or the lower chromatic number is $\chi = 2$ and the only gap occurs at 3.

On the boundary of colourability and uncolourability, those mixed hypergraphs have been located that admit just one feasible partition – that is, those with just one $r_k = 1$ in their chromatic spectrum, and zeros otherwise. The systematic study of these *uniquely colourable* mixed hypergraphs was initiated by Tuza, Voloshin and Zhou [65]. This class is NP-hard to recognize (it is co-NP-complete when the input hypergraph is given, together with a proper colouring) and it does not admit a characterization in terms of forbidden induced subhypergraphs; in fact, every mixed hypergraph can be embedded as an induced subhypergraph into a uniquely colourable one. Moreover, Bujtás and Tuza [12] proved that it is also NP-complete to decide whether $\mathcal{H}$ admits a vertex ordering $v_1, v_2, \ldots, v_n$ such that, for all $1 \leq i \leq n$, the subhypergraph induced by $\{v_1, v_2 \ldots, v_i\}$ is uniquely colourable.

Some uniquely colourable subclasses admit a good characterization and efficient recognition algorithm – for instance, those with $\chi = n - 1$ and $\chi = n - 2$ (see [54]), uniquely colourable mixed hypertrees (see [53]) and circular mixed hypergraphs

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(see [70]). Moreover, the possible size distributions of colour classes in uniquely colourable *r-uniform bi-hypergraphs* have also been characterized (see [5]).

An analogy with complete graphs is that uniquely colourable separators (vertex subsets that induce a uniquely colourable subhypergraph and whose removal makes $\mathcal{H}$ disconnected) can be applied to derive a recursive formula to compute the chromatic polynomial (see [65]). On the other hand, colourable *graphs* do not satisfy the requirement put on uniquely colourable mixed hypergraphs; instead, they form a subclass of *weakly uniquely colourable* mixed hypergraphs; this means that $r_\chi = r_{\bar{\chi}} = 1$. They have also been studied to some extent in [65].

$\mathcal{C}$-perfect mixed hypergraphs

$\mathcal{C}$-perfectness can be defined for mixed hypergraphs $\mathcal{H} = (V, \mathcal{C}, \mathcal{D})$ in the same way as for $\mathcal{C}$-colouring, by requiring that $\bar{\chi}(\mathcal{H}') = \alpha_{\mathcal{C}}(\mathcal{H}')$ for all *induced* subhypergraphs $\mathcal{H}' = (V', \mathcal{C}', \mathcal{D}')$ of $\mathcal{H}$, where $\alpha_{\mathcal{C}}(\mathcal{H}')$ is the independence number of the hypergraph $(V, \mathcal{C}')$. Generalizing the class of monostars, one can obtain $\mathcal{C}$-imperfect mixed hypergraphs by taking some $\mathcal{C}$-edges with non-empty intersection, and inserting each 2-element subset of their common intersection as a $\mathcal{D}$-edge. (These constructs are called *polystars*.) But the characterization problem for $\mathcal{C}$-perfect mixed hypergraphs looks even harder than that for $\mathcal{C}$-perfect hypergraphs. Some examples given in [19] indicate that the situation already becomes quite complicated for mixed hypertrees.

Steiner systems and finite projective planes

A *Steiner system* $\mathrm{S}(t, k, v)$ is a k-uniform hypergraph of order v, for which each t-tuple of vertices is contained in precisely one edge. To mention some celebrated examples, a system $\mathrm{S}(2, 3, v)$ is a *Steiner triple system* $\mathrm{STS}(v)$, an $\mathrm{S}(3, 4, v)$ is a *Steiner quadruple system* $\mathrm{SQS}(v)$, and an $\mathrm{S}(2, q+1, q^2+q+1)$ is a *finite projective plane* of order q.

The edges are traditionally called *blocks*. We may view each block as a $\mathcal{C}$-edge (where an $\mathrm{STS}(v)$ is denoted by $\mathrm{CSTS}(v)$) or as a bi-edge – that is, a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time (where an $\mathrm{STS}(v)$ is denoted by $\mathrm{BSTS}(v)$). The notations $\mathrm{CS}(t, k, v)$, $\mathrm{BS}(t, k, v)$, $\mathrm{CSQS}(v)$ and $\mathrm{BSQS}(v)$ are derived for the respective systems in a similar way.

The study of the upper chromatic number in Steiner triple systems started with a paper by Milazzo and Tuza [49], where the authors proved that

$$\bar{\chi}(BSTS(v)) \le \bar{\chi}(CSTS(v)) \le s,$$

for all $v \le 2^s - 1$. This upper bound on $\bar{\chi}$ is tight for all $s \ge 2$, and the systems attaining equality were also characterized. More generally,

$$\bar{\chi}(BS(t, t+1, v)) \le \bar{\chi}(CS(t, t+1, v)) = O(\ln v)$$

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holds for any fixed $t \geq 2$ as $v \to \infty$ (see [50]). Interestingly, from below it is not even known whether $\bar{\chi}(BSQS(v))$ can tend to infinity with v. Moreover, no uncolourable *BSQS* has so far been found.

Concerning the decrements of projective planes $\Pi(q)$ of order q, Bacsó and Tuza [4] proved that $\mathrm{dec}(\Pi(q)) \geq 2q + \sqrt{q}/2 - o(\sqrt{q})$. For an infinite sequence of values q this is provably optimal, even in the order $\Theta(\sqrt{q})$ of its second term: if q is a square, then the union of two disjoint Baer subplanes in the Galois plane $PG(2,q)$ meets each line in more than one point, and this allows us to construct a colouring; we deduce that $\mathrm{dec}(PG(2,q)) \leq 2q + 2\sqrt{q} + 1$. Recently, Bacsó, Héger and Szőnyi [3] gave sufficient conditions to ensure that $\mathrm{dec}(PG(2,q)) = \tau_2((PG(2,q))) - 1$. (The right-hand side is always an upper bound if there exists a 2-transversal of minimum cardinality that is independent – see the section on $\mathcal{C}$-colouring.)

For projective planes $\Pi(q)$ without an underlying algebraic structure, a general upper bound for $\mathrm{dec}(\Pi(q))$ is $3q - 2$. Indeed, one can pick three lines L, L', L'' in general position (that is, with empty intersection), and assign colour 1 to $L' \cap L''$ and to all points of $L \setminus (L' \cup L'')$, and colour 2 to all points of $(L' \cup L'') \setminus (L' \cap L'')$. Then every line contains two points of the same colour, and so each remaining point can have its distinct colour.

More details about colourings of block designs can be found in the survey [51].

5. Colour-bounded and stably bounded hypergraphs

In this section we consider generalizations of mixed hypergraphs; we shall view them as hypergraphs $\mathcal{H} = (V, \mathcal{E})$ with additional constraints prescribed for the edges. For convenience, we assume that $\mathcal{E} = \{E_1, E_2 \ldots, E_m\}$. These structures include colour-bounded hypergraphs, stably bounded hypergraphs and pattern hypergraphs.

The most general class is the third one, introduced by Dvořák *et al.* [28]. For each edge E_i, a family $\mathcal{P}_i$ of partitions is specified, and a vertex-colouring is considered to be *proper* if its colour classes induce a partition of each E_i which is a member of $\mathcal{P}_i$. Even in this very general model, the authors managed to develop a theory concerning feasible sets with or without gaps.

The other two classes are closer to the flavour of mixed hypergraphs; their conditions on the edges are given quantitatively. For the more general one, a *stably bounded hypergraph* required four functions $s, t, a, b : \mathcal{E} \to \mathbb{N}$ to be given: we write $s_i = s(E_i)$, $t_i = t(E_i)$, $a_i = a(E_i)$, $b_i = b(E_i)$, for all $1 \leq i \leq m$. Given these parameters, a colouring is *proper* if, for each $E_i \in \mathcal{E}$:

- the number of different colours assigned to the vertices of E_i is at least s_i
- the number of different colours assigned to the vertices of E_i is at most t_i
- there exists a colour assigned to at least a_i vertices of E_i
- no colour occurs on more than b_i vertices of E_i.

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So the functions s and t impose bounds on the largest polychromatic subsets of the edges, while the functions a and b impose bounds on their largest monochromatic subsets.

These four colour-bounding functions can capture many kinds of partitioning problems, allowing a concise model description. A practical example of this is resource allocation where, loosely speaking, the lower bounds (ensuring several kinds of resources or multiple copies of one resource) increase the security and stability of a system, while the upper bounds correspond to constructing the system in an economical way (since more pieces of resource increase the cost).

Functional subclasses

Colour-bounding functions of types $s_i = 1$, $t_i = |E_i|$, $a_i = 1$, and $b_i = |E_i|$ are called *non-restrictive* on the edge E_i, because they express no real restrictions on proper colourings. A colour-bounding function is *non-restrictive* in a hypergraph $\mathcal{H}$ if it is so on all edges of $\mathcal{H}$; otherwise, it is called *restrictive*. In notation, we write capital letters to indicate the functions that are allowed to be restrictive in a hypergraph. For instance, an (S,T)-hypergraph means that only s and t are given – that is, only the numbers of different colours occurring in the edges are bounded; an (S,T)-hypergraph is also called a *colour-bounded hypergraph*. Similarly, in S-hypergraphs only the minimum number of required colours is given for each edge. Note that every S-hypergraph is an (S,T)-hypergraph at the same time, because t is not required to be restrictive for the latter.

Chromatic polynomials

As in the classical theory of graph colourings, for a hypergraph $\mathcal{H} = (V, \mathcal{E})$ and an integer $\lambda \in \mathbb{N}$, we denote by $P(\mathcal{H}, \lambda)$ the number of proper colourings $c : V \to \{1, 2, \ldots, \lambda\}$. This $P(\mathcal{H}, \lambda)$ is a polynomial in λ for every $\mathcal{H}$ (in the most general class, pattern hypergraphs, too), and is thus termed the *chromatic polynomial* of $\mathcal{H}$. Voloshin [66] extended the concept of $P(\mathcal{H}, \lambda)$ from the classical one to mixed hypergraphs, and discovered some new properties of $P(\mathcal{H}, \lambda)$. For example, he showed that the degree of a chromatic polynomial is not determined by the number of vertices in general, as it is equal to the upper chromatic number $\bar{\chi}(\mathcal{H})$; only in classical graph and hypergraph colourings do the two invariants coincide. Another unusual property is that some mixed hypergraphs contain vertices whose removal from $\mathcal{H}$ changes nothing with respect to colourability: all colourings and $P(\mathcal{H}, \lambda)$ remain the same. Such vertices are called *phantom vertices*; they are invisible for $P(\mathcal{H}, \lambda)$, and $\mathcal{H}$ may contain any number of them.

The famous connection–contraction algorithm for finding the chromatic polynomial, as originated by Birkhoff [10] in 1912, formalized by Whitney [72] in 1932, and generalized to hypergraphs by Zykov [76] in 1974, was extended to the

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splitting–contraction algorithm for mixed hypergraphs by Voloshin [66] in 1993. Further, some properties of $P(\mathcal{H},\lambda)$ in S-hypergraphs have been described in [26].

It is not known which polynomials occur as chromatic polynomials of mixed hypergraphs. For non-1-colourable structures, however, Bujtás and Tuza [15] gave the following necessary and sufficient conditions. Here $S(n,k)$ is the Stirling number of the second kind, the number of partitions of n elements into precisely k non-empty sets.

Theorem 5.1 *Let $P(\lambda)=\sum_{k=0}^{\ell} a_k\lambda^k$ be a non-zero polynomial for which $P(1)=0$ (so $\sum_{k=0}^{\ell} a_k=0$). Then the following properties are equivalent.*

- *$P(\lambda)$ is the chromatic polynomial of a colour-bounded hypergraph*
- *$P(\lambda)$ is the chromatic polynomial of a mixed hypergraph*
- *$P(\lambda)$ satisfies the following conditions:*
  - *all coefficients a_k of $P(\lambda)$ are integers*
  - *the leading coefficient a_ℓ is positive*
  - *the constant term a_0 is 0*
  - *for each positive integer $j\le\ell$, $\sum_{k=j}^{\ell} a_k S(k,j)\ge 0$.*

These characterizations remain the same if we replace the pair (S,T) (colour-bounded hypergraphs) by any of $(S,A),(T,B)$ and (A,B) (those that admit uncolourable hypergraphs), or by (S,T,A,B) itself, or by pattern hypergraphs.

One way to describe relations between functional classes (defined in terms of the subsets of $\{S,T,A,B\}$) is to consider the classes of chromatic polynomials that occur in them. In Fig. 4 we exhibit their Hasse diagram; we indicate the positions of the classes of C-, D-, and mixed hypergraphs by $\mathcal{P}_{\mathcal{C}}$, $\mathcal{P}_{\mathcal{D}}$ and $\mathcal{P}_M$, respectively. The sets of chromatic polynomials belonging to non-1-colourable (S,T,A,B)-hypergraphs and to non-1-colourable (S,T)-hypergraphs are the same; the difference shown in Fig. 4 arises from the fact that there are 1-colourable (S,T,A,B)-hypergraphs whose chromatic spectra do not occur for any (S,T)-hypergraphs. Note that there are examples of problems (such as algorithmic problems), and of classes, where the problem is provably harder on the class located lower in the hierarchy of chromatic polynomials.

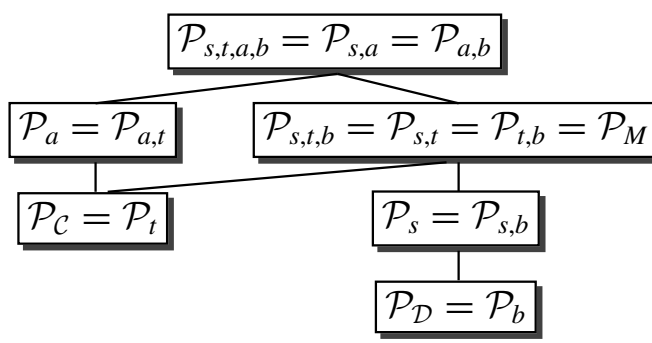


Fig. 4. The Hasse diagram for classes of chromatic polynomials

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Interval hypergraphs and hypertrees

The colouring properties of some well-structured hypergraph classes have been analysed in detail. Many results are known about the computational complexity of the problems of deciding colourability and of determining χ and $\bar{\chi}$ for hypertrees, interval hypergraphs and some of their subclasses. For a summary of what is known and what is still open, we refer to the three handy tables in Bujtás and Tuza [21]. Below we mention a few of these results.

The cases of just one restrictive function are quite well understood, except in interval A-hypergraphs, where the algorithmic complexity of determining $\bar{\chi}$ is still an open problem. The interval S- and B-hypergraphs admit optimal *periodic* colourings, with $\chi = \max s_i$ and $\chi = \max\lceil |E_i|/b_i \rceil$ colours, respectively. We observe that $\bar{\chi}$ for interval T-hypergraphs can be computed efficiently.

Concerning two restrictive functions, the simplest case is (S, B), for which $\chi = \max(s_i, \lceil |E_i|/b_i \rceil)$. For each of the pairs (T, B), (A, B) and (S, A), however, it is $\mathcal{NP}$-complete to decide whether a given interval hypergraph is colourable. (This implies intractability for any three restrictive functions, while colourability and the values of χ and $\bar{\chi}$ can be determined efficiently, even for (S, T, A, B) on classes of hypergraphs with bounded edge size.) The important unsolved case is to decide colourability and to determine $\bar{\chi}$ for an interval (S, T)-hypergraph. But interestingly enough, as proved in [16], once we know that it is colourable, we immediately obtain $\chi = \max s_i$, and if a proper colouring is given also, then it can be transformed efficiently to an optimal one; moreover, the feasible set is gap-free. For other pairs of restrictive functions, only a few upper bounds for χ have been found for interval hypergraphs (for example, $\chi \le \max |E_i|$, even for (S, T, A, B)), and it is not known whether some interval (S, A)-hypergraphs, (T, B)-hypergraphs or (A, B)-hypergraphs can have any gaps between $\chi + 1$ and $\max |E_i| - 1$.

Colour-bounded and stably bounded *hypertrees* turn out to be much more complex than mixed hypertrees. The decision problem for colourability is $\mathcal{NP}$-complete for any non-trivial pair of restrictive functions (that is, other than (S, B) and (T, A)), even in the 3-uniform case. Recalling that every mixed hypertree has a gap-free feasible set, we report that a further dramatic change is illustrated by the following theorem from [16].

Theorem 5.2 *Let F be a finite set of positive integers. Then there exists an (S, T)-hypertree with feasible set F if and only if* $\min(F) = 1$ *or* $\min(F) = 2$ *and F contains all integers between* $\min(F)$ *and* $\max(F)$*, or* $\min(F) \ge 3$.

Finally, we mention that the results on interval (S, T)-hypergraphs can be partially extended to *circular* (S, T)-hypergraphs (see [16]); namely, if a hypergraph from this class is colourable, then it satisfies the inequality $\chi \le 2(\max s_i) - 1$ and no gaps can occur between $2(\max s_i) - 1$ and $\bar{\chi}$. For $\max s_i \ge 3$ we do not know, however, whether there can exist any gaps between χ and $2(\max s_i) - 1$.

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6. Conclusion

In this chapter we described the current state of hypergraph colourings as a separate subject. With so many relations to graph colouring and new fundamental ideas, results and applications, this area is developing fast. The impact of every result may be to give rise to a set of new problems, often without any analogues from the past. While the number of publications continues to grow, we observe that some possible subdirections remain untouched. We predict that hypergraph generalizations of such areas as colouring graphs on surfaces, edge-colourings, colouring and probability, Hadwiger's conjecture, colouring games, orientations and flows (to name just a few) have great research potential in the observable future.

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