Ergod. Th. & Dynam. Sys. (2004), **24**, 347–365
DOI: 10.1017/S0143385703000324
© *2004 Cambridge University Press*
Printed in the United Kingdom

Spectral characterization of Wiener–Wintner dynamical systems

I. ASSANI

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599, USA
(e-mail: assani@math.unc.edu)

(*Received* 26 *March* 2001 *and accepted in final form* 4 *July* 2003)

Abstract. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system on the finite measure space $(X, \mathcal{B}, \mu, T)$ and $\mathcal{K}$ its Kronecker factor. We denote by U the restriction of T onto $\mathcal{K}^{\perp}$ the orthocomplement of $\mathcal{K}$. We give a spectral characterization in L^2 of Wiener–Wintner functions in terms of the capacity of the support of the maximal spectral type of U and the almost everywhere continuity of the fractional rotated ergodic Hilbert transform. The study of the L^2 case leads to new classes of dynamical systems.

1. *Introduction*

In [**A1**], we introduced and studied ergodic dynamical systems that we called Wiener–Wintner dynamical systems of power type in L^p. We defined a function f to be a Wiener–Wintner function in the following way.

Definition 1. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. A function f is a Wiener–Wintner function of power type α in L^p if there exist finite positive constants C_f and α such that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_p \leq \frac{C_f}{N^{\alpha}} \quad \text{for all positive integers } N.$$

We also said that an ergodic dynamical system is a Wiener–Wintner dynamical system of power type α in L^p if we could find in the orthocomplement of its Kronecker factor $\mathcal{K}$ a dense set (for the L^2 norm) of Wiener–Wintner functions of power type α in L^p. It was shown that a difficult theorem such as almost everywhere double recurrence [**B**] is much easier to prove for such systems. Many known systems have this property (see [**A1**] for examples and some of their structural properties and [**AN**] and [**L**] for examples with singular spectrum on $\mathcal{K}^{\perp}$). At the present time it is not known if all ergodic dynamical systems are Wiener–Wintner dynamical systems of power type α in L^1.

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

In this paper we focus on the spectral properties of Wiener–Wintner dynamical systems in L^2. We characterize Wiener–Wintner functions in L^2 by the help of the almost everywhere continuity of the random Fourier series

$$H_\gamma^\varepsilon(f)(x) = \sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x)\,\operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k\varepsilon}$$

that we call the fractional rotated ergodic Hilbert transform (the notation $\sum_{k=-\infty}^{\infty}{}'$ is used to delete the term corresponding to $k = 0$ in the sum). In §1 we show that an L^2 function is a Wiener–Wintner function of power type α in L^2 if and only if for almost every x $H_\gamma^\varepsilon(f)(x)$ is a continuous function of ε. This answers one of the questions in [**A1**].

We study the cases where the L^p norm is used instead of the L^1 norm as well as a logarithmic growth of convergence. In this paper we will show that study of the L^2 case leads to apparently new classes of dynamical systems.

We will get these new classes by considering the apparently weaker conditions

$$\sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right\|_p \le \frac{C_f}{N^\alpha} \quad (1 \le p \le \infty)$$

and slower rates of convergence of the form $C_f/[\ln(N+1)]^{1+\alpha}$. To be more precise we introduce the following definitions (see also [**A4**]).

Definition 2. Consider $(X, \mathcal{B}, \mu, T)$ to be an ergodic dynamical system. We say that a function f is a weak Wiener–Wintner (WWW) function of logarithmic type α in L^p if there exist finite positive constants C_f and α such that

$$\sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right\|_p \le \frac{C_f}{[\ln(N+1)]^{1+\alpha}}$$

for all positive integers N.

Definition 3. An ergodic dynamical system $(X, \mathcal{B}, \mu, T)$ is said to be a WWW dynamical system of logarithmic type α in L^p if there exists in $\mathcal{K}^\perp$ a dense set of WWW functions of logarithmic type α in L^p.

Similar definitions can be given if we consider instead rates of the form C_f/N^α (WWW functions of power type α, WWW dynamical systems of power type α). It is clear that a WWW function of power type α is also of logarithmic type α. We can note that the range of α is $(-1, \infty)$ for the logarithmic type.

The two rates—power or logarithmic type—are, in fact, linked to two well-known capacities: the α and logarithmic capacities, C_α and $C_{\ln^\gamma}$. For more on these notions we may consult [**Z, KS, K, SZ**]. Let us denote by U the restriction of T onto the orthocomplement of the Kronecker factor of T, namely $\mathcal{K}^\perp$, and by S the support of the maximal spectral type of U on $\mathbb{T}$. (For references on the maximal spectral type, see [**CFS**] for instance.) We prove in §3 that if T is a WWW dynamical system of logarithmic type α in L^2, then the γ-logarithmic capacity of S is positive for all $0 < \gamma < 2(1+\alpha)$. An analogous result will be given for WWW dynamical systems of power type α in L^2.

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

Related to these notions we introduce the γ-logarithmic (respectively α) capacity of U, $\mathcal{C}_{\ln^\gamma}(U)$ (respectively $\mathcal{C}_\alpha(U)$):

$$\mathcal{C}_{\ln^\gamma}(U) = \left(\inf_{\nu\sim\sigma} \sup_{x\in[0,1]} \int_0^1 \left(\log\left(\frac{1}{|t-x|} \right) \right)^\gamma d\nu \right)^{-1}$$

$$\mathcal{C}_\alpha(U) = \left(\inf_{\nu\sim\sigma} \sup_{x\in[0,1]} \int_0^1 \left| \frac{1}{|t-x|^\alpha} \right| d\nu \right)^{-1},$$

where σ denotes the maximal spectral type of U and $\nu(\mathbb{T}) = \sigma(\mathbb{T}) = 1$.

We show that $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system in L^2 of logarithmic type α if $\mathcal{C}_{\ln^{2(1+\alpha)}}(U) > 0$. We also study the structural properties of such systems. Using this characterization we provide an example of a WWW dynamical system which is of logarithmic type $-\frac{1}{2}$ in L^2 and is not of power type β for all $0 < \beta < 1$. This example shows at least that Wiener–Wintner dynamical systems of logarithmic type α in L^2 form are a strictly larger class of dynamical systems than the class of Wiener–Wintner dynamical system of power type α. While this paper was being written, an example was given in [**L**] of a WWW dynamical system which is not of power class α in L^2 for any $0 < \alpha < 1$. These two examples were obtained independently.

Since the results of this paper were originally announced in [**A3**], we have shown in [**A5**] that, for each ergodic dynamical system, there exists a continuous positive function g satisfying $\lim_N g(N) = \infty$ and a dense set of functions $f \in \mathcal{K}^\perp$ such that

$$\sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n\varepsilon} \right\|_2 \leq \frac{C_f}{g(N)}$$

for all positive integers. Thus, there always exists a growth in the convergence.

In §4 we show that for any ergodic dynamical system, f a function in L^p and $g \in L^\infty$, the L^1 norm of the averages

$$\frac{1}{N} \sum_{n=1}^N f \circ T^n g \circ T^{2n}$$

can be controlled by

$$\sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n\varepsilon} \right\|_p.$$

This motivates in part the study of WWW dynamical systems in L^p.

2. *Spectral characterization of Wiener–Wintner dynamical systems in L^2*

We use several known facts on the spectral measure of a unitary operator. The reader may consult [**Q**] or [**N**] for more details on spectral measures. If we denote by E the spectral measure, then a family of measures $\sigma_{f,g}$ can be defined by the formula $\sigma_{f,g}(B) = \langle f, E(B)g\rangle$ for any Borelian set B of $\mathbb{T}$. If U is the unitary operator on H (the Hilbert space), then the n Fourier coefficients of this measure are given by $\hat{\sigma}_{f,g}(n) = \langle U^n f, g\rangle$. When $f = g$ the measure $\sigma_f = \sigma_{f,f}$ is non-negative. If σ_f and σ_g are absolutely

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

continuous with respect to the same measure ν, then the Radon–Nykodim derivative h of $\sigma_{f,g}$ satisfies the pointwise inequality

$$|h(x)| \leq \frac{d\sigma_f^{\frac{1}{2}}}{d\nu}\frac{d\sigma_g^{\frac{1}{2}}}{d\nu}.$$

We have the equation $\sigma_{f+g} = \sigma_f + \sigma_g + \sigma_{f,g} + \sigma_{g,f}$. If σ_f and σ_g are mutually singular, then $\sigma_{f+g} = \sigma_f + \sigma_g$. We assume that T is invertible.

One way to directly access the spectral measure is through the helical transform or rotated ergodic Hilbert transform. By using the functional calculus starting with the equation $T = \int_{\mathbb{T}} e^{2\pi it}\, dE(t)$, as in [**C**], we get the following equation:

$$H_\varepsilon f = \sum_{n=-\infty}^{\infty}{}' \frac{f \circ T^n}{n} e^{2\pi in\varepsilon} = i\varepsilon f + 2i\,E(-\varepsilon, 0)f + H_0 f \tag{1}$$

for a function f in $\mathcal{K}^\perp$. (The notation $\sum_{n=-\infty}^{\infty}{}'$ is used to eliminate the term corresponding to $n = 0$ in the sum.) This can be seen by writing $H_\varepsilon f - H_0 f = \int_{\mathbb{T}} S(t+\varepsilon) - S(t)\, dE(t)$, where S is the odd function which on $[0, \frac{1}{2})$ takes the value $1-2t$ and $S(0) = 0$. One of the equivalent conditions for a function f to be in $\mathcal{K}^\perp$ is to have a continuous spectral measure (i.e. no atoms) (see [**A2**], for instance). Hence, this eliminates the terms $E(\lambda)f$ in the difference $H_\varepsilon f - H_0 f$. An easy consequence of this difference equation is the following result which we use later:

$$2i\,E(-b, -a)f = \sum_{n=-\infty}^{\infty}{}' \frac{f(T^n)}{n}(e^{2\pi inb} - e^{2\pi ina}) - i(b-a)f. \tag{2}$$

To prove one of the implications of the next theorem we need the following proposition.

PROPOSITION 1. *Given* $0 < \gamma < 1$, *there exist* C_γ *and* $0 < r < 1$ *such that*

$$\int_\delta^{1-\delta} \left| \sum_{k=1}^{\infty} \frac{\cos 2\pi k(\varepsilon - \delta)}{k^{1-\gamma}} - \sum_{k=1}^{\infty} \frac{\cos 2\pi k(\varepsilon + \delta)}{k^{1-\gamma}} \right| d\varepsilon \leq C_\gamma(\delta^r)$$

for all $0 < \delta < \frac{1}{2}$.

Proof. We use the known formula

$$\int_0^\infty t^{-\gamma} e^{-it}\, dt = \Gamma(1-\gamma) e^{\frac{1}{2}\pi i(1-\gamma)}$$

in order to compute the Fourier coefficients of the function $t^{-\gamma}$. We have

$$c_n = \int_0^1 t^{-\gamma} e^{-2\pi int}\, dt = \frac{1}{(2\pi n)^{1-\gamma}}\Gamma(1-\gamma)e^{\frac{1}{2}\pi i(1-\gamma)} - \frac{1}{(2\pi n)^{1-\gamma}}\int_{2\pi n}^\infty \frac{e^{-it}}{t^\gamma}\, dt$$

for $n > 0$ and $c_0 = \frac{1}{2}1/(1-\gamma)$. We have

$$\int_{2\pi n}^\infty \frac{e^{-it}}{t^\gamma}\, dt = \lim_L \int_{2\pi n}^{2\pi L} \frac{e^{-it}}{t^\gamma}\, dt.$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

Using two integrations by parts and taking the limit with L, we obtain

$$-\frac{1}{(2\pi n)^{1-\gamma}}\int_{2\pi n}^{\infty}\frac{e^{-it}}{t^{\gamma}}\,dt=-\frac{1}{2\pi in}-\frac{\gamma}{(2\pi n)^2}+\frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}}\int_{2\pi n}^{\infty}\frac{e^{-it}}{t^{(\gamma+2)}}\,dt.$$

Hence,

$$c_n=\frac{1}{(2\pi n)^{1-\gamma}}\Gamma(1-\gamma)e^{\frac{1}{2}\pi i(1-\gamma)}+\frac{1}{2\pi in}-\frac{\gamma}{(2\pi n)^2}+\frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}}\int_{2\pi n}^{\infty}\frac{e^{-it}}{t^{(\gamma+2)}}\,dt.$$

We can write c_n as a_n+ib_n where

$$c_0+\sum_{n=1}^{\infty}a_n\cos(2\pi nx)+b_n\sin(2\pi nx)$$

converges almost everywhere to the function $x^{-\gamma}$. Using the change of variable $x\to(1-x)$, we obtain

$$c_0+\sum_{n=1}^{\infty}a_n\cos(2\pi nx)=\frac{1}{2}(x^{-\gamma}+(1-x)^{-\gamma})$$

for almost every x in $[0,1)$. As a_n is the real part of c_n, it is equal to

$$\frac{1}{(2\pi n)^{1-\gamma}}\Gamma(1-\gamma)\cos\left(\frac{1}{2}\pi(1-\gamma)\right)-\frac{\gamma}{(2\pi n)^2}+\frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}}\int_{2\pi n}^{\infty}\frac{\cos(t)}{t^{(\gamma+2)}}\,dt.$$

From this we can now estimate what we wanted, i.e. $\sum_{n=1}^{\infty}\cos(2\pi nx)/n^{1-\gamma}$. We have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\cos(2\pi nx)}{n^{1-\gamma}}=C_{\gamma}\Bigg(&\frac{1}{2}\left(\frac{1}{x^{\gamma}}+\frac{1}{(1-x)^{\gamma}}\right)-\frac{1}{1-\gamma}\\&+\sum_{n=1}^{\infty}\cos(2\pi nx)\left(\frac{\gamma}{(2\pi n)^2}+\frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}}\int_{2\pi n}^{\infty}\frac{\cos(t)}{t^{(\gamma+2)}}\,dt\right)\Bigg)\end{aligned}$$

where C_{γ} is an absolute constant depending only on γ. From this we can finish the proof of this proposition. We have

$$\begin{aligned}&\int_{\delta}^{1-\delta}\left|\sum_{k=1}^{\infty}\frac{\cos 2\pi k(\varepsilon-\delta)}{k^{1-\gamma}}-\sum_{k=1}^{\infty}\frac{\cos 2\pi k(\varepsilon+\delta)}{k^{1-\gamma}}\right|d\varepsilon\\&\quad\le\int_{\delta}^{1-\delta}C_{\gamma}\left|\frac{1}{2}\left(\frac{1}{(\varepsilon-\delta)^{\gamma}}-\frac{1}{(\varepsilon+\delta)^{\gamma}}+\frac{1}{(1-\delta+\varepsilon)^{\gamma}}-\frac{1}{(1-\delta-\varepsilon)^{\gamma}}\right)\right|d\varepsilon\\&\qquad+\int_{\delta}^{1-\delta}C_{\gamma}'\left|\sum_{n=1}^{\infty}\left|\frac{\sin(2\pi n\delta)}{n^{1+d}}\right|\right|d\varepsilon\end{aligned}$$

where C_{γ}' and d are positive constant depending only on γ. The second integral satisfies the estimate of the form $C(\delta^r)$ for some $r>0$ as $|\sin(2\pi k\delta)|\le(2\pi k\delta)^b$ for $0<b<d$. The first integral also satisfies the same kind of estimate as we can split the integral into two where we group the first two terms then the next two. The nature of the functions allows to compare them, to eliminate the absolute values and to integrate them. □

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

THEOREM 2. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a L^2 function in $\mathcal{K}^\perp$. The following statements are equivalent.*

(1) *There exists $0 < r < 1$ and a finite constant C_r such that for all positive integer N we have*

$$\left\| \sup_{1\le j\le N} \left| E\left(-\delta - \frac{j}{N}, \delta - \frac{j}{N}\right) f \right| \right\|_2 \le C_r(\delta^r)$$

for all $0 < \delta < \frac{1}{2}$.

(2) *The function f is a Wiener–Wintner function of power type α in L^2 for some $0 < \alpha < 1$.*

(3) *There exists $0 < \gamma < 1$ such that the random Fourier series*

$$\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k\varepsilon}$$

is almost everywhere a continuous function of ε and

$$\sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k\varepsilon} \right| \in L^2.$$

Proof. First, we prove that (1) implies (2). We note that, by Bernstein's theorem, there exists an absolute constant K such that for all N' positive integer, we have

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N'} \sum_{n=1}^{N'} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_2^2 \le C \left\| \sup_{1\le j\le N} \left| \frac{1}{N'} \sum_{n=1}^{N'} f \circ T^n e^{2\pi i n j/N} \right| \right\|_2^2$$

for $N = KN'$. This can be seen by applying Bernstein's theorem pointwise to the trigonometric polynomial $(1/N)\sum_{n=1}^{N} f(T^n x)e^{2\pi i n\varepsilon}$ and by using the mean value theorem on each interval of the form $[j/N, (j+1)/N]$. If we denote by $U_{j/N}$ the unitary operator defined by $U_{j/N} f = f \circ T e^{2\pi i j/N}$ we have, for each j, $1 \le j \le N$,

$$\frac{1}{N} \sum_{n=1}^{N} f(T^n x) e^{2\pi i n j/N} = \frac{1}{N} \sum_{n=1}^{N} {U_{j/N}}^n f(x).$$

We can write f in the following way:

$$f = E_{j/N}(-\delta, \delta) f + f - E_{j/N}(-\delta, \delta) f,$$

where $E_{j/N}$ denotes the spectral measure of the operator $U_{j/N}$ restricted to $\mathcal{K}^\perp$. We will use the following three properties.

(a) We have $E_{j/N}(-\delta, \delta) f = E(-\delta - j/N, \delta - j/N) f$, where E is the spectral measure of T on $\mathcal{K}^\perp$.

(b) There exists a function $g_{\delta,j}$ such that $f - E_{j/N}(-\delta, \delta) f = g_{\delta,j} - U_{j/N} g_{\delta,j}$.

(c) On $(I - E_{j/N}(-\delta, \delta))(\mathcal{K}^\perp)$, the operator $I - U_{j/N}$ is invertible and we have $\|(I - U_{j/N})^{-1}\| \le C(1/|\sin \pi\delta|)$.

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

The properties (b) and (c) were observed in [**K**]. Let us see quickly why (b) and (c) are true. To show (b) first, we note that

$$\sup_M \left\| \sum_{n=0}^{M} [U_{j/N}]^n (f - E_{j/N}(-\delta, \delta) f) \right\|_2 < \infty$$

and this follows easily by using the spectral measure of f with respect to $U_{j/N}$. Such an evaluation gives us a bound in the order of $(C/|\sin(\pi\delta))|\|f\|_2$. Now we can find a function $g_{\delta,j}$ and a subsequence of integers M_k that we denote by K for convenience such that the averages

$$\frac{1}{K} \sum_{k=1}^{K} \sum_{n=0}^{k} [U_{j/N}]^n (f - E_{j/N}(-\delta, \delta) f)$$

converge weakly to the function $g_{\delta,j}$.

Applying $U_{j/N}$ to these averages, we see that $g_{\delta,j} - U_{j/N} g_{\delta,j}$ is the weak limit of

$$f - \frac{1}{K} \sum_{k=1}^{K} [U_{j/N}]^{k+1} (f - E_{j/N}(-\delta, \delta) f).$$

As the second term converges to zero in norm we have (b). Furthermore, $\|g_{\delta,j}\|_2 \leq (C/|\sin(\pi\delta)|)\|f\|_2$.

To prove (c), we only have to show that $I - U_{j/N}$ is one to one. If this were not true, then we would have on $\mathcal{K}^\perp$ an eigenfunction for T for the eigenvalue $e^{2\pi i j/N}$ and this is impossible on $\mathcal{K}^\perp$ by definition. Thus (b) and (c) are true.

Now we proceed with the proof. We have

$$\left\| \sup_{1 \leq j \leq N'} \frac{1}{N'} \left| \sum_{n=1}^{N'} (U_{j/N})^n f \right| \right\|_2^2$$
$$\leq C \frac{1}{N'^2} \sum_{j=1}^{N'} \|g_{\delta,j}\|_2^2 + \left\| \frac{1}{N'} \sum_{n=1}^{N'} \left| \sup_{1 \leq j \leq N} E(-\delta - j/N, \delta - j/N) f \circ T^n \right| \right\|_2^2.$$

Hence, using (1) and (c), we can conclude that

$$\left\| \sup_{1 \leq j \leq N} \frac{1}{N'} \left| \sum_{n=1}^{N'} (U_{j/N})^n f \right| \right\|_2^2 \leq C \left(\delta^{2r} + \frac{1}{N\delta^2} \right).$$

Now we can pick δ (as a function of N) appropriately to make f a Wiener–Wintner function of power type α. This proves the implication (1) $\implies$ (2).

The implication (2) $\implies$ (3) can be shown by using the first part of the proof of Theorem 3 in [**AN**]. Note that, as stated, (3) does not assume the convergence of the series for each ε off a null set of x. The continuity of the limit is obtained by the uniform convergence of the subsequence of the partial sums

$$\sum_{k=-N^\delta}^{N^\delta}{}' \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k\varepsilon}$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

for some $\delta > 0$. The same argument shows that

$$\sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k\varepsilon} \right| \in L^2.$$

It remains to prove (3) $\Longrightarrow$ (1). We use the proposition and equation (2) linking the ergodic Hilbert transform to the spectral measure of T on $\mathcal{K}^\perp$ for this. We have

$$2i\,E(-\delta + j/N, \delta + j/N) f = \sum_{n=-\infty}^{\infty}{}' \frac{f \circ T^n}{n} (e^{2\pi i n j/N} 2i \sin(2\pi k\delta)) - 2i\delta f,$$

so we only have to estimate the series. We need an estimate for almost every x. By the pointwise convergence of the ergodic Hilbert transform in L^p for an invertible measure-preserving transformation (see [**Co**]) we can note that, for each δ and each N, the series

$$\sum_{n=-\infty}^{\infty}{}' \frac{f(T^n x)}{n} (e^{2\pi i n j/N} 2i \sin(2\pi n\delta))$$

converges almost everywhere. This follows by applying the pointwise convergence result to the ergodic Hilbert transform of the transformation $(x, y) \to (Tx, y + j/N + \delta)$ and to the function $f \otimes e^{2\pi i y}$. Hence, to estimate the series pointwise, we can write

$$\begin{aligned}
&\left| \sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x)}{k} (e^{2\pi i k j/N} 2i \sin(2\pi k\delta)) \right| \\
&= \left| \int_0^1 \left(\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} (e^{2\pi i k j/N} e^{-2\pi i k t}) \right) \left(\sum_{k=-\infty}^{\infty}{}' \sin(2\pi k\delta) \frac{e^{2\pi i k t}}{|k|^{1-\gamma}} \right) dt \right| \\
&\le \sup_\theta \left| \left(\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k\theta} \right) \right| \int_0^1 \left| \left(\sum_{k=-\infty}^{\infty}{}' \sin(2\pi k\delta) \frac{e^{2\pi i k t}}{|k|^{1-\gamma}} \right) dt \right| \\
&= \sup_\theta \left| \left(\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k\theta} \right) \right| \int_0^1 \left| \sum_{k=1}^{\infty} \frac{\cos 2\pi k(t-\delta)}{|k|^{1-\gamma}} - \sum_{k=1}^{\infty} \frac{\cos 2\pi k(t+\delta)}{|k|^{1-\gamma}} \right| dt.
\end{aligned}$$

The first term in the last expression is in L^2 and independent of j. For the second term we can split the integral from 0 to 1 into three integrals: $\int_0^\delta$, $\int_\delta^{1-\delta}$ and $\int_{1-\delta}^1$. Proposition 1 tells us that the second integral is less than $C_\gamma(\delta^r)$ for some $0 < r < 1$ for all $\delta > 0$. The first and third terms can be handled similarly by using the pointwise estimate

$$\left| \sum_{k=0}^{\infty} \frac{\cos(2\pi k x)}{k^{1-\gamma}} \right| \le C_\gamma x^{-\gamma}.$$

This estimate can be found in [**Z**, p. 191]. This shows that there exists a constant s depending on γ such that

$$E(-\delta + j/N, \delta + j/N) f(x) \le \sup_\theta \left| \left(\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k\theta} \right) \right| C_\gamma \delta^s.$$

As

$$\sup_\theta \left| \left(\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k\theta} \right) \right|$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

is in L^2 by assumption we can conclude that

$$\left\| \sup_{1\le j\le N} E\left(-\delta+\frac{j}{N},\delta+\frac{j}{N}\right) f \right\|_2 \le C_\gamma \delta^s$$

for some $s>0$ and this ends the proof of the theorem. □

Remark. If one assumes that $f\in L^\infty$, then one can add to (3) in Theorem 2 the almost everywhere convergence for all ε of the series

$$\sum_{k=-\infty}^{\infty}{}' \frac{f(T^k x)}{|k|^\gamma}\operatorname{sgn} k e^{2\pi i k\varepsilon}.$$

We do not know at the present time whether one can reach the same conclusion for $f\in L^2$. Our method of proof uses Proposition 1 in [**AN**] which seems to restrict the domain of f.

3. *WWW dynamical systems*

In the first part of this section we study some structural properties of WWW dynamical systems. The second part deals with links between capacities and WWW dynamical systems. The third part provides an example of a WWW dynamical system not of power type but of logarithmic type. The fourth part deals with estimates for the double recurrence.

3.1. *Structural properties of WWW dynamical systems.*

We begin with a simple proposition on spectral measures, showing the link between Bochner's theorem characterizing the continuity of a measure on $\mathbb{T}$ and the concept of WWW dynamical systems.

PROPOSITION 3. (1) *If f is a WWW function in L^2 of power type α, then for all N*

$$\sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)|\le\frac{C}{N^{\alpha/2}}.$$

(2) *If*

$$\sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)|\le\frac{C}{N^{\alpha}}$$

for all N, then f is a WWW function in L^2 of power type α.

(3) *If f is a WWW function in L^2 of logarithmic type α, then*

$$\sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)|\le\frac{C}{(\ln(N+1))^{(1+\alpha)/2}}$$

for all N.

(4) *If*

$$\sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)|\le\frac{C}{(\ln(N+1))^{1+\alpha}}$$

for all N, then f is a WWW function in L^2 of logarithmic type α.

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

Proof. We only prove (1) and (2). The proofs for (3) and (4) are similar. Assume that f is a WWW function in L^2 of power type α. By using Fubini and Holder's inequality, we have

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)|^2 &= \frac{1}{N}\sum_{n=1}^{N}\widehat{\sigma_{f,g}}(n)\overline{\widehat{\sigma_{f,g}}(n)}\\
&= \left|\frac{1}{N}\sum_{n=1}^{N}\iint \overline{g}(x)f(T^n x)\overline{g}(y)f(T^n y)\,d(\mu\otimes\mu)(x,y)\right|\\
&= \left|\iint \overline{g}(x)\overline{g}(y)\frac{1}{N}\sum_{n=1}^{N}f(T^n x)f(T^n y)\,d(\mu\otimes\mu)(x,y)\right|\\
&= \left|\int \overline{g}(x)\left(\int \overline{g}(y)\frac{1}{N}\sum_{n=1}^{N}f(T^n x)f(T^n y)\,d\mu(y)\right)d\mu(x)\right|\\
&\le \int\left[|\overline{g}(x)|\|g\|_2\left\|\frac{1}{N}\sum_{n=1}^{N}f(T^n x)f(T^n y)\right\|_2\right]d\mu(x)\\
&= \|g\|_2\int\left[\overline{g}(x)\left(\int\left|\frac{1}{N}\sum_{n=1}^{N}f(T^n x)e^{2\pi in\varepsilon}\right|^2 d\sigma_f(\varepsilon)\right)^{1/2}\right]d\mu(x)\\
&\le \|g\|_2^2\left(\iint\left|\frac{1}{N}\sum_{n=1}^{N}f(T^n x)e^{2\pi in\varepsilon}\right|^2 d\sigma_f(\varepsilon)\,d\mu(x)\right)^{1/2}.
\end{aligned}$$

This last term is less than or equal to

$$\begin{aligned}
&\|g\|_2^2\left(\int\left(\int\left|\frac{1}{N}\sum_{n=1}^{N}f(T^n x)e^{2\pi in\varepsilon}\right|^2 d\mu(x)\right)d\sigma_f(\varepsilon)\right)^{1/2}\\
&\le \|g\|_2^2\|f\|_2\sup_{\varepsilon}\left\|\frac{1}{N}\sum_{n=1}^{N}f\circ T^n e^{2\pi in\varepsilon}\right\|_2\\
&\le \frac{C}{N^{\alpha}},
\end{aligned}$$

where the last inequality follows by assumption. Hence,

$$\begin{aligned}
\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)| &\le \left(\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)|^2\right)^{1/2}\\
&\le \frac{C}{N^{\alpha/2}}.
\end{aligned}$$

Thus (2) $\Longrightarrow$ (1).

Assume that

$$\sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}|\widehat{\sigma_{f,g}}(n)| \le \frac{C}{N^{\alpha}}.$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

Then, by duality, we get that

$$\begin{aligned}
\sup_{\varepsilon}\left\|\frac{1}{N}\sum_{n=1}^{N} f\circ T^n e^{2\pi in\varepsilon}\right\|_2 &= \sup_{\varepsilon}\sup_{\|g\|_2\le 1}\left|\int g(x)\left(\frac{1}{N}\sum_{n=1}^{N} f(T^n x)e^{2\pi in\varepsilon}\right)d\mu(x)\right| \\
&= \sup_{\varepsilon}\sup_{\|g\|_2\le 1}\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi in\varepsilon}\int g(x)f(T^n x)\,d\mu(x)\right| \\
&\le \sup_{\varepsilon}\sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}\left|\int g(x)f(T^n x)\,d\mu(x)\right| \\
&= \sup_{\|g\|_2\le 1}\frac{1}{N}\sum_{n=1}^{N}\left|\widehat{\sigma_{f,g}}(n)\right| \\
&\le \frac{C}{N^{\alpha}},
\end{aligned}$$

where the last inequality follows by assumption. □

The following theorem lists some stability properties.

THEOREM 4. *Let $(X,\mathcal{B},\mu,T)$ be an ergodic dynamical system. Then the following hold.*

(1) *There exists in $\mathcal{P}^{\perp}$, the orthocomplement of the Pinsker σ algebra of T, a dense set of WWW functions of logarithmic type α in L^2.*

(2) *If $(X,\mathcal{B},\mu,T)$ is a WWW dynamical system of logarithmic type α in L^2 then any one of its factors is a WWW dynamical system of logarithmic class α. Furthermore any closed T invariant vector subspace of $\mathcal{K}^{\perp}$ contains a dense set of WWW dynamical systems of logarithmic type α.*

(3) *If we decompose $L^2(X)$ into the direct sum $H_s + H_l$, where H_s represents the functions with spectral measure singular with respect to Lebesgue measure and H_l those with spectral measure absolutely continuous with respect to Lebesgue measure, then H_l contains a dense set of WWW functions of power type $\frac{1}{2}$ in L^2.*

(4) *Assume $(X,\mathcal{B},\mu,T)$ is a WWW dynamical system of logarithmic type α in L^2 and its product with the WWW dynamical system of logarithmic type α, $(Y,\mathcal{C},\nu,S)$, is ergodic. Then this product is a WWW dynamical system of logarithmic type α in L^2.*

Proof. The proof of (1) and the first part of (2) follows in the same way as in [**A1**], so we omit it. The second part of (2) follows by using the projection of $\mathcal{K}^{\perp}$ onto the closed T invariant vector space.

Property (3) follows immediately by considering the dense set of functions f whose spectral measure σ_f is of the form *hdm* where $h \in L^{\infty}(m)$, m being Lebesgue measure. For the proof of (4) we can remark that the orthocomplement of the Kronecker factor of $T\times S$ is spanned by the finite linear combinations of product of functions $f\otimes g$ where $f \in \mathcal{K}_T^{\perp}$ or $g \in \mathcal{K}_S{}^{\perp}$. It is enough then to show that the product $f\otimes g$ is a WWW function of logarithmic class α in $L^2(\mu\otimes\nu)$ if one of the functions is a WWW function of

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

logarithmic type α in L^2. We have

$$\begin{aligned}
&\sup_{\varepsilon}\int\int\left|\frac{1}{N}\sum_{n=1}^{N}f(T^nx)g(S^ny)e^{2\pi in\varepsilon}\right|^2 d\mu(x)\otimes\nu(y)\\
&\quad=\sup_{\varepsilon}\int\int\left|\frac{1}{N}\sum_{n=1}^{N}f(T^nx)e^{2\pi in(t+\varepsilon)}\right|^2 d\sigma_g(\varepsilon)\,d\mu(x)\\
&\quad\le\int\sup_{\theta}\int\left|\frac{1}{N}\sum_{n=1}^{N}f(T^nx)e^{2\pi in\theta}\right|^2 d\mu(x)\,d\sigma_g(t)\\
&\quad\le\frac{C}{(\ln(N+1))^{2(1+\alpha)}}.
\end{aligned}$$

Remark. Similar results can be obtained for WWW dynamical systems of power type α.

3.2. *Capacity and WWW dynamical systems.* We recall a few points about capacities that we can find, for instance, in [**Z**]. Let λ be a real-valued, periodic, integrable, even continuous function on the torus, $\mathbb{T}$, such that $\lim_{x\to 0}\lambda(x)=\infty$. Let μ be a positive measure on the Borel σ-algebra, $\mathcal{B}(\mathbb{T})$, on $\mathbb{T}$, and F a Borel set such that $\mu(F)=1$. Define a new function $l:\mathbb{T}\to\mathbb{R}$ by $l(x)=l(x,\mu)=\int_0^1\lambda(x-t)\,d\mu(t)$. Let $L(\mu)=\sup_{x\in[0,1]}l(x,\mu)$.

Definition 4. The λ capacity of F is

$$C(F)=\Big(\inf_{\{\mu:\mu(F)=1\}}L(\mu)\Big)^{-1}.$$

Definition 5. For $\gamma>0$, the γ-logarithmic capacity of F, denoted by $C_{\ln^\gamma}(F)$, is the λ capacity of F when $\lambda(x)=(\ln(1/|x|))^\gamma$.

Definition 6. For $0<\alpha<1$, the α-capacity of F, denoted by C_α, is the λ capacity of F when $\lambda(x)=1/|x|^\alpha$.

Connected to these notions, we introduce the following definition.

Definition 7. Let $(X,\mathcal{B},\mu,T)$ be an ergodic dynamical system and U its restriction to the orthocomplement of the Kronecker factor. The γ-logarithmic capacity of U (respectively the α-capacity of U) is defined as $\mathcal{C}_{\ln^\gamma}(U)$ (respectively $\mathcal{C}_\alpha(U)$):

$$\mathcal{C}_{\ln^\gamma}(U)=\left(\inf_{\nu\sim\sigma}\sup_{x\in[0,1]}\int_0^1\left(\ln\left(\frac{1}{|t-x|}\right)\right)^\gamma d\nu\right)^{-1},$$

respectively

$$\mathcal{C}_\alpha(U)=\left(\inf_{\nu\sim\sigma}\sup_{x\in[0,1]}\int_0^1\left(\frac{1}{|t-x|^\alpha}\right)d\nu\right)^{-1},$$

where σ is the maximal spectral type of U and ν and σ are probability measures.

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

The following lemma gives a characterization in L^2 of a WWW function.

LEMMA 1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a function in $\mathcal{K}^{\perp}$. The function f is a WWW function of logarithmic type α (respectively of power type α) if and only if*

$$\sup_{x\in[0,1]} \sigma_f \star \delta_x([-\delta, \delta]) \leq \frac{C}{|\ln(\delta)|^{2(1+\alpha)}}$$

respectively

$$\sup_{x\in[0,1]} \sigma_f \star \delta_x([-\delta, \delta]) \leq \frac{C}{\delta^{2\alpha}}$$

for all $\delta > 0$.

Proof. The proof uses part of the method of Theorem 3 in [**K**]. We sketch it in the logarithmic case. By periodicity of the functions on the torus, we can use the interval $(-1/2, 1/2]$ instead of $(0, 1]$. If f is a WWW function of logarithmic type α in L^2, then

$$\sup_x \int_{-1/2}^{1/2} \left| \sum_{n=1}^{N} e^{2\pi i n t} \right|^2 d\sigma_f \star \delta_x(t) \leq \frac{C_f N^2}{(\ln N)^{2(1+\alpha)}}$$

and this shows that

$$\sup_x \sigma_f \star \delta_x([-\delta, \delta]) \leq \frac{C}{|\ln \delta|^{2(1+\alpha)}}.$$

For the reverse implication assuming that

$$\sup_x \sigma_f \star \delta_x([-\delta, \delta]) \leq \frac{C}{|\ln \delta|^{2(1+\alpha)}}$$

and following the decomposition made in [**K**, p. 664], we just need to dominate

$$\int_{|t|\geq \pi/n} \frac{1}{x^2}\, d\sigma_f \star \delta_x(t).$$

Writing $\sigma_f \star \delta_x\{\pi/(k+1) \leq |t| \leq \pi/k\}$ as $s_k - s_{k+1}$, where $s_k = \sigma_f \star \delta_x\{0 < |t| \leq \pi/k\}$ and using partial summation, we are left with controlling

$$\sum_{k=2}^{n} \frac{k}{|\ln(k)|^{2(1+\alpha)}}.$$

We dominate this term by

$$C'n + \sum_{k=[\sqrt{n}]}^{n} \frac{k}{|\ln(k)|^{2(1+\alpha)}}.$$

This quantity is less than

$$C'' \frac{n^2}{|\ln(n)|^{2(1+\alpha)}}. \qquad \square$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

PROPOSITION 5. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a function in $\mathcal{K}^{\perp}$.*

(1) *If f is a WWW function of logarithmic type α in L^2, then*

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^{\beta} d\sigma_f(t) < \infty$$

for all $0 < \beta < 2(1+\alpha)$.

(2) *If*

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^{2(1+\alpha)} d\sigma_f(t) < \infty,$$

then f is a Wiener–Wintner function of logarithmic type α in L^2.

(3) *If the function f is a WWW function of power type α in L^2, then*

$$\sup_{x\in[0,1]} \int_0^1 \left(\frac{1}{|x-t|^{\beta}}\right) d\sigma_f(t) < \infty$$

for all $0 < \beta < 2\alpha$.

(4) *If*

$$\sup_{x\in[0,1]} \int_0^1 \left(\frac{1}{|x-t|^{2\alpha}}\right) d\sigma_f(t) < \infty,$$

then f is a WWW function of power type α.

Proof. We only prove the logarithmic case, the power case using a similar path. If f is a WWW function of logarithmic type α in L^2 then, by the lemma, we have

$$\sup_x \sigma_f \star \delta_x[\delta, \delta] \leq \frac{C}{|\ln(\delta)|^{2(1+\alpha)}}.$$

Elementary computations shows that this implies

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^{\beta} d\sigma_f(t) < \infty$$

for all $0 < \beta < 2(1+\alpha)$. This shows (1). Part (2) follows without difficulty from the lemma. □

THEOREM 6. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Let U be its restriction to $\mathcal{K}^{\perp}$. Then the following hold.*

(1) *If $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system of logarithmic type α in L^2, then $\mathcal{C}_{(\ln)^{\beta}}(U) > 0$, for all $0 < \beta < 2(1+\alpha)$.*

(2) *If $\mathcal{C}_{(\ln)^{2(1+\alpha)}}(U) > 0$, then $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system of logarithmic type α in L^2.*

(3) *If $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system of power type α in L^2, then $\mathcal{C}_{\beta}(U) > 0$ for all $0 < \beta < 2\alpha$.*

(4) *If $\mathcal{C}_{2\alpha}(U) > 0$, then $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system of power type α in L^2.*

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

Proof. Again we only show the logarithmic part. The notion of support that we use in the proof (and later) is meant in the measurable sense with respect to the maximal spectral type and not as the topological support. We assume that we have in $\mathcal{K}^\perp$ a dense set of WWW functions of logarithmic type α in L^2. We want to show that there exists a WWW function of the same logarithmic type in L^2 that realizes the maximal spectral type of U. We know that there exists a function $g \in \mathcal{K}^\perp$ that realizes the maximal spectral type. We can assume that $\|g\|_2 = 1$. Let us consider a sequence of WWW functions f_k of logarithmic type α in L^2 which converge to g. The union of the support of σ_{f_k} is equal to S the support of σ_g. Furthermore, by Proposition 5, for each k we have a constant C_k that we can assume to be greater than one, such that

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^\beta d\sigma_{f_k}(t) < C_k < \infty$$

for all $0 < \beta < 2(1+\alpha)$. The function

$$F = f_1 + \sum_{k=2}^{\infty} \frac{1}{C_k 2^k} E\left(\operatorname{supp}(\sigma_{f_k}) \setminus \left(\bigcup_{j=1}^{k-1} (\operatorname{supp}(\sigma_{f_j}))\right)\right)[f_k]$$

has a spectral measure σ_F whose support is equal to the union of the support of the measures σ_{f_k}. (Note that the spectral measures in this sum have disjoint support.) To conclude we just have to observe that for a WWW function g of logarithmic type α, for each Borelian set B on $\mathbb{T}$, the function $E(B)g$ is also a WWW function of the same type. The spectral measure of the function $G = F/\|F\|_2$ is a probability measure realizing the maximal spectral type of U. We easily verify that

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^\beta d\sigma_G(t) < \infty.$$

This proves that the β-logarithmic capacity of U is finite. For the reverse implication, let us assume that $\mathcal{C}_{\ln^{2(1+\alpha)}}(U)$ is positive. Then, by definition, there exists a positive measure ρ supported on a set S such that

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^{2(1+\alpha)} d\rho(t) < \infty.$$

This measure ρ, by definition, is equivalent to the maximal spectral type σ of U. The functions f with spectral measure of the form $hd\rho$ where $h \in L^\infty(\rho)$ are dense in $\mathcal{K}^\perp$. For each one of these functions, we have

$$\sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^{2(1+\alpha)} d\sigma_f(t) \le \|h\|_\infty \sup_{x\in[0,1]} \int_0^1 \left(\ln\left(\frac{1}{|x-t|}\right)\right)^{2(1+\alpha)} d\rho.$$

By Proposition 5, these functions f are WWW functions of logarithmic type α in L^2. This ends the proof of the theorem. □

An immediate consequence of this theorem is the following corollary.

COROLLARY 1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Let S be support of the maximal spectral type of $U = T|_{\mathcal{K}}$. Then the following hold.*

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

(1) *If $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system of logarithmic type α in L^2, then the β-logarithmic capacity of S, $C_{\ln^\beta}(S)$ is positive for all $0 < \beta < 2(1+\alpha)$.*

(2) *If $(X, \mathcal{B}, \mu, T)$ is a WWW dynamical system of power type α in L^2, then the β capacity of S, $C_\beta(S)$ is positive for all $0 < \beta < 2\alpha$.*

3.3. *An ergodic dynamical system which is a WWW dynamical system of logarithmic type $-\frac{1}{2}$ in L^2 but is not of power type β for all $0 < \beta < 1$.*

THEOREM 7. *There exists a weakly mixing system which is a Wiener–Wintner dynamical system of logarithmic type $-\frac{1}{2}$ in L^2 which is not of power type β for all $0 < \beta < 1$.*

Proof. The example is a Gauss dynamical system. We start by exhibiting a set P of Haussdorf measure zero but of positive logarithmic capacity. The starting point for the construction of such a set is standard (see [**SZ**] or [**KS**]). It consists of a symmetric Cantor-type subset of [0, 1) formed of points of the form

$$x = (t_1(1-a_1) + t_2a_1(1-a_2) + \cdots + t_ka_1\cdots a_{k-1}(1-a_k) + \cdots$$

where the $0 < a_k < \frac{1}{2}$ are the successive ratios of dissection of the intervals in three parts proportional to $(a_k, 1-2a_k, a_k)$ and the t_k are 0 or 1. In [**SZ**] the following properties are shown:

(a) The logarithmic capacity of P is positive if and only if the series

$$\sum_1^\infty \frac{1}{2^k} \log\left(\frac{1}{a_k}\right)$$

converges.

(b) The α capacity of P is positive if and only if the series

$$\sum_{k=1}^\infty \frac{1}{2^k (a_1\cdots a_k)^\alpha}$$

converges.

Following these properties if we pick $a_k = 1/(k+2)$, then simple calculations show that the series

$$\sum_1^\infty \frac{1}{2^k} \log\left(\frac{1}{a_k}\right)$$

converges while the series

$$\sum_{k=1}^\infty \frac{1}{2^k (a_1\cdots a_k)^\alpha}$$

has a generic term equivalent to $[(k+2)!]^\alpha/2^k$ and, thus, diverges for all α. Hence, we have $C_\alpha(P) = 0$ for all $0 < \alpha < 1$ and $C_{\ln}(P) > 0$.

By Frostman's theorem (see [**KS**] for instance), the Hausdorff dimension of P is also equal to zero as it is equal to the sup of those β for which $C_\beta(P) > 0$.

As the logarithmic capacity of P is positive, there exists a positive continuous probability measure m concentrated on P (i.e. with support M a subset of P) such that

$$\sup_{x\in[0,1]} \int_0^1 \ln\left(\frac{1}{|t-x|}\right) dm < \infty.$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

We can symmetrize the measure m to obtain a measure σ with a symmetric support Q. We normalize σ to make it a probability measure. As Q is a subset of P its α-capacity is equal to zero for all $0 < \alpha < \frac{1}{2}$ while its logarithmic capacity is positive.

The construction of the system $(X, \mathcal{B}, \mu, T)$ is classical. One considers the Gaussian stationary process generated by the measure σ (see [**CFS**] for more details). The maximal spectral type of this Gaussian system on the orthocomplement of the constant functions is given by

$$e^{\sigma} = \sum_{n=1}^{\infty} \sigma^{(n)}/n!.$$

(The Kronecker factor is trivial in this case.) Its support is $S = \bigcup_{n=1}^{\infty} Q_n$ where $Q_n = Q + \cdots + Q$, n times. We consider the closed subset H_σ of functions f whose spectral measures is absolutely continuous with respect to σ. None of these functions can be of power type α in L^2: if not we could find a positive measure σ_f with support included in Q for which

$$\sup_{x\in[0,1]} \int_0^1 \left(\frac{1}{|x-t|^{\beta}}\right) d\sigma_f(t) < \infty$$

for all $0 < \beta < 2\alpha$. In view of Definition 4, this would yield a contradiction with the zero capacity of the set Q. Thus, for each $0 < \beta < 1$, we cannot find in $\mathcal{K}^{\perp}$ a dense set of WWW functions of power type β in L^2. If not the projection of such functions onto H_σ would create WWW functions of the same power type. Thus, the system is not a WWW dynamical system of power type β for all β.

It remains to show that the logarithmic capacity of this system is positive. What we know so far is that $C_{\ln}(P)$ is positive. This implies that $C_{\ln}(S)$ is also positive as S contains P. However, this does not say that the logarithmic capacity of T is positive. To be able to make such a statement, we need to find a function F whose spectral measure σ_F realizes the maximal spectral type and such that

$$\sup_{x\in[0,1)} \int_0^1 \ln\left(\frac{1}{|t-x|}\right) d\sigma_F < \infty.$$

In the definitions of capacities, we note that we can also take the periodic distance

$$|x-t|^* = \inf_{s\in\mathbb{Z}} |x-t-s|.$$

The definition of convolution gives immediately for f non-negative

$$\int_0^1 f(|x-t|^*)\, d\sigma^{(n)}(t) \le \sup_y \int_0^1 f(|y-t|^*)\, d\sigma(t)$$

as σ is a probability measure. We conclude that

$$\int_0^1 \ln\left(\frac{1}{|x-t|^*}\right) de^{\sigma}(t) \le e \sup_y \int_0^1 \ln\left(\frac{1}{|y-t|^*}\right) d\sigma(t),$$

and

$$\sup_x \int_0^1 \ln\left(\frac{1}{|x-t|^*}\right) de^{\sigma}(t) \le e \sup_y \int_0^1 \ln\left(\frac{1}{|y-t|^*}\right) d\sigma(t).$$

This shows that the logarithmic capacity of T is positive as $e^{\sigma} = \sigma_F$ for some function F. □

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

Remark. Frostman's theorem was used in [**L**] to produce an example of a weakly mixing dynamical system (a Gauss system) which is not of power type α in L^2 for any $0 < \alpha < 1$. Our example is not of power type β in L^2 for any $0 < \beta < 1$, but it is of logarithmic class α in L^2. These two examples were obtained independently.

3.4. *WWW dynamical systems of power type α in L^p and almost everywhere double recurrence.* In this section we want to give an estimate of the double recurrence averages in terms of the quantity

$$\sup_{\varepsilon} \left\| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right\|_p.$$

THEOREM 8. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and p a real number, $\frac{1}{2} \leq p < \infty$ and*

$$a_N = \sup_{\varepsilon} \left\| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) e^{2\pi i n\varepsilon} \right\|_{2p}.$$

Then, for all $f_1 \in \mathcal{K}^{\perp} \cap L^{2p}$ and $f_2 \in L^{\infty}$, $\|f_2\|_{\infty} \leq 1$, we have

$$\left\| \frac{1}{N} \sum_{n=1}^{N} f_1 \circ T^n f_2 \circ T^{2n} \right\|_1 \leq C(N^{1/2p} a_N)^{2/3}$$

for each positive integer N. In particular, if f_1 is a WWW function in L^{2p} of power type α such that $2p > 1/\alpha$, then there exists $\theta > 0$ such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} f_1 \circ T^n f_2 \circ T^{2n} \right\|_1 \leq \frac{C}{N^{\theta}}$$

for all positive integers N.

Proof. We can derive from the arguments used in the first lines of the proof of Theorem 2 the following inequality:

$$\left\| \sup_{\varepsilon} \left| \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_p^p \leq CN \sup_{\varepsilon} \left\| \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right\|_p^p.$$

Combining it with formula (7) in [**A1**], we easily obtain a proof of Theorem 8. □

Remark.

(1) The inequality

$$\left\| \sup_{\varepsilon} \left| \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right| \right\|_p^p \leq CN \sup_{\varepsilon} \left\| \sum_{n=1}^{N} f \circ T^n e^{2\pi i n\varepsilon} \right\|_p^p$$

for any function $f \in L^p$ is valid for $p > 0$. The estimate in Theorem 8 seems only relevant when $p > 1$: this is because already for $p = 1$ there are no WWW functions in L^2 of power-type $\alpha > \frac{1}{2}$. This can be seen by the following inequalities:

$$\sup_{t} \left\| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n t} \right\|_2 \geq \left(\iint \left| \frac{1}{N} \sum_{n=1}^{N} f \circ T^n e^{2\pi i n t} \right|^2 d\mu\, dt \right)^{1/2} = \frac{\|f\|_2}{N^{\frac{1}{2}}}.$$

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press

(2) It is easy to check that most examples already known of Wiener–Wintner dynamical systems are WWW dynamical systems of power type α in L^{2p} with $2p > 1/\alpha$. In fact, all WWW functions of power type α in L^∞ trivially satisfy the condition $2p > 1/\alpha$. Examples of such systems are a family of skew products in [**A1**] and Morse dynamical system [**L**]. These systems are, in fact, Wiener–Wintner dynamical systems of power α in L^∞.

Acknowledgement. We would like to thank the referee for his remarks.

REFERENCES

[A1] I. Assani. Wiener–Wintner dynamical systems. *Preprint*, 1998. *Ergod. Th. & Dynam. Sys.* to appear.
[A2] I. Assani. *Wiener Wintner Ergodic Theorems*. World Scientific, Singapore, 2003.
[A3] I. Assani. Spectral characterization of Wiener Wintner dynamical systems. *C. R. Acad. Sci. Paris* Ser. I **332** (2001), 321–324.
[A4] I. Assani. Corrections to 'A Wiener–Wintner property for the helical transform'. *Ergod. Th. & Dynam. Sys.* **18** (1998), 1331–1333.
[A5] I. Assani. Spectral characterization of ergodic dynamical systems. *Contemp. Math.* **284** (2001), 13–22.
[AN] I. Assani and K. Nicolaou. Properties of Wiener Wintner dynamical systems. *Bull. Soc. Math. France* **129**(3) (2001), 361–377.
[B] J. Bourgain. Double recurrence and almost sure convergence. *J. Reine Angew. Math.* **404** (1990), 140–161.
[Co] M. Cotlar. A unified theory of Hilbert transforms and ergodic theorems. *Rev. Mat. Cuyana* **1** (1955), 105–167.
[C] J. Campbell. Spectral analysis of the ergodic Hilbert transform. *Ind. Univ. Math. J.* **35** (1986), 379–390.
[CFS] I. Cornfeld, S. V. Fomin and Y. Sinai. *Ergodic Theory*. Springer, Berlin, 1982.
[K] A. Kachurovski. The rate of convergence in ergodic theorems. *Russian Math. Surveys* **51**(4) (1996), 73–124.
[KS] J. P. Kahane and R. Salem. 'Ensembles parfaits et series trigonometriques'. Hermann, Paris, 1963.
[L] E. Lesigne. Private communication, 2000.
[N] M. G. Nadkarni. *Spectral Theory of Dynamical Systems*. Birkhäuser, Basel, 1999.
[Q] M. Queffelec. *Substitution Dynamical Systems—Spectral Analysis (Lecture Notes in Mathematics, 1294)*. Springer, Berlin.
[SZ] R. Salem and A. Zygmund. Capacity of sets and Fourier series. *Trans. Amer. Math. Soc.* **59** (1946).
[Z] A. Zygmund. *Trigonometric Series*. Cambridge University Press, Cambridge, 1959.

https://doi.org/10.1017/S0143385703000324 Published online by Cambridge University Press